Triangle groups and Jacobians of CM type

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Dédié à la mémoire de Claude Itzykson

In the last years, two new tools have been developed for an approach to the question whether a given nonsingular projective algebraic curve over a number field has a Jacobian of *CM type*. First, such curves can be characterized by the existence of *Belyi functions* or Grothendieck's *dessins d'enfants* ([Be], [Gr], [VS], [CIW], [JS3]) — for detailed definitions see Section 1. There is some reasonable hope that dessins also encode deeper properties of curves which has already been proved for places of bad reduction ([Fu], [Bec]), so one should try to read information about the endomorphism algebra of the Jacobian also from combinatorial properties of dessins. We will mainly consider regular dessins or equivalently, Riemann surfaces with many automorphisms. The reduction to this particular case is described in Section 2.
Second, there has been considerable progress in transcendence. A classical criterion due to Th. Schneider says that an elliptic curve defined over $\bar{\mathbb{Q}}$ has complex multiplication if and only if the period quotient is algebraic. From Wüstholz' analytic subgroup theorem follows a generalization to abelian varieties and their *period quotients* in the Siegel upper half space ([Coh], [SW], see Theorem 3) and has applications in particular to curves with many automorphisms (Sections 4 and 7).
The key in joining both tools is the use of the canonical representation of the automorphism group G of the curve X on the space of holomorphic differentials. The Jacobian of X is isogenous to a direct product

$$A_1^{k_1} \times \ldots \times A_m^{k_m}$$

of simple, pairwise non–isogenous abelian varieties A_ν, $\nu = 1, \ldots, m$, and among others it will be shown that

1. irreducible subspaces U of this representation belong to isotypic components of $\operatorname{Jac} X$, i.e. consist of pullbacks of differentials on the factors $A_\nu^{k_\nu}$ (they can even built up from $\operatorname{End} A_\nu$–invariant subspaces of $H^0(\Omega, A_\nu)$, see Section 7),

2. $\dim U = 1$ implies that A_ν has complex multiplication (Theorem 4, Section 4),

3. large representation degrees $\dim U$ indicate high multiplicity k_ν of A_ν and/or that $\operatorname{End} A_\nu$ has large degree over its center (Theorems 8 and 9).

These canonical representations — which can be effectively constructed as subrepresentations of the regular representation of G ([St1], [St2]) — can also be used to determine the period quotient, hence to apply Theorem 3, but this will be an object of further research ([St3]). In the present paper, the simpler criteria for $\mathrm{Jac}\, X$ to be of CM type (Theorem 4, Remark 3) turn out to give only another look to the well–known fact that Fermat curves and their quotients have Jacobians of CM type ([KR], [Ao]). More interesting is the fact that 'many automorphisms' do *not* imply 'CM', explained in Section 5 using certain subvarieties of the Siegel upper half space. Surprisingly, even Hurwitz curves (for which the maximal number $84(g-1)$ of automorphisms is attained) do not always have Jacobians of CM type, as is shown in Section 6.5 for Macbeath's curve in genus 7. Since in every genus there is only a finite number of curves with many automorphisms, Section 6 includes a detailed discussion of all such curves for genera < 5. It turns out that only one (hyperelliptic) curve in genus 3 and two curves in genus 4 have Jacobians not of CM type.

The main new results of the present paper are contained in Sections 4, 5 and 7. The other Sections collect material of which a good part is known to experts but often not presented in the literature in a form we need here. I hope that e.g. Theorem 2 about the regularization of dessins or the examples of Section 6 may be useful in other contexts also.

The first attempt to the question treated in this paper emerged from discussions with Claude Itzykson in 1993. On the financial side, it has been generously supported by a PROCOPE grant, then profited by a stay at ETH Zürich in 1995. Later, I learned many things by discussions with Manfred Streit, Gareth Jones, David Singerman and Antoine Coste.— I am indebted to Thomas A. Schmidt for the elimination of a long series of linguistic errors.

1 Triangle groups and dessins

Let Y denote a compact Riemann surface or equivalently a complex nonsingular projective algebraic curve. We are looking for properties of $\mathrm{Jac}\, Y$ and may therefore suppose that the genus of Y is $g > 0$. A simple complex polarized abelian variety A has **complex multiplication** if its endomorphism algebra

$$\mathrm{End}_0\, A \ := \ \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}\, A$$

is a number field $\mathbb{K}$ of degree

$$[\mathbb{K} : \mathbb{Q}] \ = \ 2 \dim A \ .$$

Then, $\mathbb{K}$ is necessarily a *CM field*, i.e. a totally imaginary quadratic extension of a totally real field of degree $\dim A$. If the polarized complex abelian variety A is not simple, it is isogenous to a direct product of simple abelian varieties. Then, A is said to be of **CM type** if the simple factors have complex multiplication. Abelian varieties with complex multiplication and hence abelian varieties of CM type are $\mathbb{C}$–isomorphic to abelian varieties defined over number fields [ST]. In short: they *may be defined over number fields.* Since curves and their Jacobians may be defined over the same field [Mi], we can restrict to curves which may be defined over number fields. The corresponding Riemann surfaces given by their complex points can be characterized in a reformulation of Belyĭ's theorem as quotients $\Gamma\backslash\mathcal{H}$ of the upper half plane $\mathcal{H}$ by a subgroup Γ of finite index in some cocompact Fuchsian triangle group Δ ([Wo1], [CIW]). Therefore one has

Theorem 1 *Let Y be a compact Riemann surface of genus $g > 0$ with a Jacobian $\mathrm{Jac}\,Y$ of CM type. Then Y is isomorphic to $\Gamma\backslash\mathcal{H}$ for some subgroup Γ of finite index in a cocompact Fuchsian triangle group Δ.*

In this theorem, Δ and Γ are not uniquely defined. They are constructed by means of a Belyi function $\beta : Y \to \mathbb{P}_1(\mathbb{C})$ ramified at most above $0, 1$ and $\infty \in \mathbb{P}_1$. If we identify Y with $\Gamma\backslash\mathcal{H}$, the Belyi function β is the canonical projection

$$\beta \;:\; \Gamma\backslash\mathcal{H} \;\to\; \Delta\backslash\mathcal{H} \;\cong\; \mathbb{P}_1(\mathbb{C}) \tag{1}$$

ramified only over the orbits of the three fixed points of Δ of order p, q and r. The Δ–orbits of these fixed points may be identified with $0, 1$ and $\infty \in \mathbb{P}_1$, and p, q, r must be multiples of the respective ramification orders of β over $0, 1, \infty$. But this is the only condition imposed upon the signature of Δ; we may and will normalize Δ by a minimal choice of the signature, i.e. by assuming that p, q, r are the *least* common multiples of all ramification orders of β above $0, 1, \infty$ respectively. Under this normalization, the groups Δ and Γ are uniquely determined by Y and β up to conjugation in $PSL_2\mathbb{R}$ (assuming they are Fuchsian groups, see below). Such a pair (Γ, Δ) will be called **minimal**. In very special cases these **minimal pairs** are not pairs of Fuchsian, but of euclidean groups acting on $\mathbb{C}$ instead of $\mathcal{H}$. The euclidean triangle groups have signature $< 2, 3, 6 >$, $< 3, 3, 3 >$ or $< 2, 4, 4 >$, therefore the Riemann surfaces Y are elliptic curves $\mathbb{C}/\Gamma$, in these cases isogenous to elliptic curves with fixed points of order 3 or 4 induced by the action of Δ, hence with complex multiplication. They are treated in detail in [SSy]; in most cases, we will take no notice of them in the present paper. The restriction to minimal pairs is useful by the following reason.

Lemma 1 *Let Γ be a cocompact Fuchsian group, contained with finite index in a triangle group Δ. Suppose that the genus of $\Gamma\backslash\mathcal{H}$ is > 0 and that the pair (Γ, Δ) is minimal. There is a maximal subgroup N of Γ which is normal and of finite index in Δ. This group N is torsion–free, hence the universal covering group of $X := N\backslash\mathcal{H}$.*

The subgroup N can be constructed by taking the intersection of all Δ–conjugates of Γ. That the index $[\Delta : N]$ is finite, may be seen either by group–theoretic arguments or by the fact that $N\backslash\mathcal{H} \to \Delta\backslash\mathcal{H}$ is the normalization of the covering $\Gamma\backslash\mathcal{H} \to \Delta\backslash\mathcal{H}$. Since we have a normal covering, all ramification orders above the respective Δ–fixed point are the same; on the other hand, they must be common multiples of the respective ramification orders of the covering $\Gamma\backslash\mathcal{H} \to \Delta\backslash\mathcal{H}$. (It will turn out that they are even the least common multiples, but we do not need this fact.) Now, by minimality, these ramification orders are multiples of the orders occurring in the signature of Δ. If N had an elliptic element γ, its fixed point would be a Δ–fixed point of order p, say; but in this fixed point — or rather in its image on X —, the normal covering map would be ramified with order $p/\operatorname{ord}\gamma$, a proper divisor instead of a multiple of p in contradiction to our assumption.

The Riemann surfaces X found in this Lemma are of special interest: a Riemann surface of genus $g > 1$ is said to have **many automorphisms** ([Rau], [Po]) if it corresponds to a point x of the moduli space M_g of all compact Riemann surfaces of genus g with the following property: There is a neighbourhood $U = U(x)$ in the complex topology of M_g such that to any $z \in U$, $z \neq x$, corresponds a Riemann surface Z with strictly fewer automorphisms then X. Using rigidity properties of triangle groups one proves easily [Wo2] the

Lemma 2 *The compact Riemann surface X of genus $g > 1$ has many automorphisms if and only if it is isomorphic to a quotient $N\backslash\mathcal{H}$ of the upper half plane by a torsion–free normal subgroup N of a cocompact Fuchsian triangle group Δ.*

Remark 1. The notion **many automorphisms** extends to genus $g = 1$ if one divides out the obvious automorphism subgroup induced by the translations on $\mathbb{C}$. Then, elliptic curves with many automorphisms are precisely those with fixed points of order 3 or 4 already mentioned above.

Remark 2. Shabat and Voevodsky were the first having made more explicit Grothendieck's ideas about the use of Belyĭ's theorem for algebraic curves, see [Gr] and [SV]. In [SV] the reader can find a lot of nice low genus examples of Riemann surfaces with many automorphisms and their dessins. In their paper, dessins are the inverse images of the real interval $[0, 1]$ under *clean* Belyi functions, i.e. Belyi functions ramifying with order 2 in all preimages of 1. In the language of the present paper, this means a restriction to triangle groups of signature $< s, 2, t >$. For some purposes, this is no loss of generality because we can pass from an arbitrary Belyi function β to a clean one taking

$$\beta' \;:=\; 4\,\beta\,(1\,-\,\beta)\;.$$

In other words, smooth projective algebraic curves defined over number fields can always be written as $\Gamma'\backslash\mathcal{H}$ with subgroups Γ' of finite index of triangle groups $\Delta' =< s, 2, t >$, see Theorem 2.2.1 of [SV] from which one can derive our Theorem 1 as well. Also, a special case of our Lemma 1 can be found in [SV] as Theorem 2.2.2. (*Balanced dessins* in [SV] lead to torsion–free subgroups Γ' of Δ' and *Galois dessins* to normal subgroups). Moreover, in section 2.3.4 [SV] states without proof a result comparable to our Lemma 2. However, this statement should be handled with care: On the one hand, curves with many automorphisms in genus 2 do not always correspond to isolated singularities of the moduli space (classified by Igusa), see also [Po]. On the other hand, Theorem 2.3.4 of [SV] fails for higher genera because it omits those curves with many automorphisms whose universal covering groups Γ are normal torsion–free subgroups of triangle groups of more general signature than $< s, 2, t >$; see the examples (11), (16), (19), (20), (21) in Section 6. The passage from arbitrary to clean Belyi functions described above does in general not replace normal torsion–free subgroups Γ of Δ by normal torsion–free subgroups Γ' of Δ'. Therefore, curves with many automorphisms (equivalently: isolated fixed points of the Teichmüller modular group) cannot always be represented by Galois dessins as defined in [SV].

One can overcome this difficulty for not necessarily clean Belyi functions β by using a more general definition of **dessins** as (oriented) *hypermaps* or *bipartite graphs* with white vertices (the points of $\beta^{-1}\{0\}$) and black vertices (the points of $\beta^{-1}\{1\}$); the (open) edges are the connected components of $\beta^{-1}]0,1[$. For other definitions of hypermaps better reflecting the triality between $\beta^{-1}\{0\}$, $\beta^{-1}\{1\}$ and $\beta^{-1}\{\infty\}$, see [JS3]. Finite index subgroups Γ of arbitrary triangle groups Δ correspond via (1) to such more general dessins, and normal subgroups N correspond to **regular dessins**, i.e. whose automorphism group ($\cong \Delta/N \cong$ the covering group of β — a normal covering in that case) acts transitively on the edges.

2 From dessins to regular dessins

We will restrict the considerations of this article mainly to these regular dessins or equivalently, their Riemann surfaces with many automorphisms. Often, a solution of the problem in this special case gives an answer to the question for arbitrary Riemann surfaces by the following reason.

Lemma 3 *Let Δ be a cocompact Fuchsian triangle group, Γ a subgroup of finite index and genus $g > 0$, N the maximal subgroup of Γ which is normal in Δ. For the Riemann surfaces*

$$Y := \Gamma\backslash\mathcal{H} \qquad \textit{and} \qquad X := N\backslash\mathcal{H}$$

we have: $\mathrm{Jac}\, Y$ *is of CM type if* $\mathrm{Jac}\, X$ *is of CM type, and* $\mathrm{Jac}\, X$ *has CM factors if* $\mathrm{Jac}\, Y$ *is of CM type.*

Proof. Since the covering map $X \to Y$ induces a surjective morphism of Jacobians, $\mathrm{Jac}\, Y$ is a homomorphic image, hence isogenous to a factor of $\mathrm{Jac}\, X$.
In many cases one can prove even more. Consider $G = \Delta/N$ as an automorphism group of X (in fact, it is *the* automorphism group of X if Δ is the normalizer of N in $PSL_2\mathbb{R}$, which is true at least if Δ is a maximal triangle group; the possible inclusions among triangle groups are well known by [Si2]). Then $Y \cong H\backslash X$ for the subgroup $H = \Gamma/N$ of G. Now the vector space $H^0(X,\Omega)$ of holomorphic differentials on X contains a subspace of differentials lifted from Y. This subspace may be identified with $H^0(Y,\Omega)$ and consists of the H–invariant differentials under the canonical representation

$$\Phi : G \to GL(H^0(X,\Omega)) \quad ; \quad \Phi(\alpha) : \omega \mapsto \omega \circ \alpha^{-1} \tag{2}$$

for all $\alpha \in G$, $\omega \in H^0(X,\Omega)$. We can identify $H^0(X,\Omega)$ with $H^0(\mathrm{Jac}\, X,\Omega)$, $H^0(Y,\Omega)$ with $H^0(\mathrm{Jac}\, Y,\Omega)$ and G with the group of automorphisms induced on $\mathrm{Jac}\, X$. Since every $\alpha \in G$ maps Y onto an isomorphic Riemann surface $\alpha(Y) = \alpha H \alpha^{-1}\backslash X$, it sends the factor $\mathrm{Jac}\, Y$ of $\mathrm{Jac}\, X$ onto an isomorphic factor. Now recall that factors of abelian varieties are isogenous to abelian subvarieties. Taking the sum B of all these abelian subvarieties isogenous to $\mathrm{Jac}\, Y$ we obtain

Lemma 4 *Under the hypotheses of Lemma 3, let V be the smallest $\Phi(G)$–invariant subspace of $H^0(\mathrm{Jac}\, X,\Omega)$ containing $H^0(\mathrm{Jac}\, Y,\Omega)$. Then*

1. *V is generated by $\Phi(G)H^0(\mathrm{Jac}\, Y,\Omega)$.*
2. *An irreducible $\Phi(G)$–invariant subspace U is contained in V if and only if*
$$U \cap H^0(\mathrm{Jac}\, Y,\Omega) \neq 0 \,,$$
i.e. if its corresponding irreducible subrepresentation Ψ has the property that $\Psi|_H$ contains the trivial representation 1_H.
3. *There is an abelian subvariety B of $\mathrm{Jac}\, X$ with $V \cong H^0(B,\Omega)$.*
4. *Every simple factor of B is a simple factor of $\mathrm{Jac}\, Y$.*
5. *If $\mathrm{Jac}\, Y$ is of CM type, so is B.*
6. *In particular, we can take $B = \mathrm{Jac}\, X$ if $H^0(X,\Omega)$ is generated by $\Phi(G)H^0(Y,\Omega)$, i.e. if for every irreducible subrepresentation Ψ of Φ, the restriction $\Psi|_H$ contains the trivial representation 1_H.*

The hypothesis of the last point is not always satisfied what can be seen by examples given in [StW] (Thm. 3.5 and Cor. 4.2 with $Y = U^m_{n,t,v}$, $X = X_{n,t,v}$). In the last section we will give some more material on the interplay between the decomposition of Φ and the decomposition of $\mathrm{Jac}\, X$, making possible an explicit construction of B by means of point 2 of the Lemma.

For the convenience of the reader, we will describe how X or N can be explicitely constructed from the data $\Gamma \subset \Delta$ for Y or equivalently from β or its corresponding dessin. To do so, the following inclusion criteria for Fuchsian groups developed by Singerman [Si1] turn out to be crucial. We restrict our attention to cocompact Fuchsian groups Γ_0 and recall that these are of signature
$$< g\,;\, m_1, \ldots , m_r > \,,$$
where g denotes the genus of the quotient space $\Gamma_0\backslash\mathcal{H}$ and the m_j denote the orders of elliptic elements γ_j which generate a maximal system of elliptic elements in the following sense: every

elliptic $\gamma \in \Gamma_0$ is Γ_0–conjugate to a power of precisely one γ_j. The positivity of the normalized covolume

$$M(\Gamma_0) \;:=\; 2g - 2 + \sum_{j=1}^{r} (1 - \frac{1}{m_j}) \;>\; 0$$

is the only restriction on the possible signatures. For triangle groups, we continue to write simply $< p, q, r >$ instead of $< 0\,; p, q, r >$. Subgroups Γ_1 of finite index s in Γ_0 are again cocompact. The elliptic generators of Γ_1 are powers of conjugates of the γ_j, therefore the signature of Γ_1 may be arranged in the form

$$< g_1\,; n_{11}, \ldots, n_{1\rho_1}, \;\ldots\;, n_{r1}, \ldots, n_{r\rho_r} > ,$$

where the orders n_{ji} belong to generators of Γ_1 conjugated in Γ_0 to powers of γ_j. Clearly, $n_{ji} \mid m_j$ for all i and j. If Γ_1 contains no element $\neq \mathrm{id}$ being Γ_0–conjugated to a power of γ_j, we put $\rho_j = s/m_j$ and $n_{ji} = 1$ for all i. Now [Si1] tells us

Lemma 5 *The cocompact Fuchsian group Γ_0 of signature $< g\,; m_1, \ldots, m_r >$ contains a subgroup Γ_1 of index s with Γ_1 of signature*

$$< g_1\,; n_{11}, \ldots, n_{1\rho_1}, \;\ldots\;, n_{r1}, \ldots, n_{r\rho_r} >$$

if and only if 1)

$$s \;=\; \frac{M(\Gamma_1)}{M(\Gamma_0)}$$

and 2) there is a transitive permutation group G of s objects and an epimorphism $\Theta : \Gamma_0 \to G$ sending each elliptic generator γ_j of Γ_0 onto a product of ρ_j disjoint cycles of length

$$\frac{m_j}{n_{j1}}, \ldots, \frac{m_j}{n_{j\rho_j}}$$

(the cases $n_{ji} = m_j$ are counted as cycles of length 1), and for each $j = 1, \ldots, r$ the sum of all these lengths must give s. The subgroup Γ_1 can be taken as a normal subgroup of Γ_0 if one can take G as a group of order s and if

$$n_{11} = \ldots = n_{1\rho_1}\,, \;\ldots\;, n_{r1} = \ldots = n_{r\rho_r}\,,$$

$$s \;=\; \rho_j \cdot \frac{m_j}{n_{ji}} \quad \textit{for all} \quad j = 1, \ldots, r\,,\; i = 1, \ldots, \rho_j\,.$$

In this case, Γ_1 is the kernel of Θ and we can identify G with Γ_0/Γ_1. In particular, Γ_1 can be taken as normal and torsion–free subgroup of Γ_0 if $G \cong \Theta(\Gamma_0) \cong \Gamma_0/\Gamma_1$ and Θ preserves the orders of the elliptic generators of Γ_0.

The last conclusions on normal torsion–free subgroups follow from the general case by the action of G on itself by left multiplication. Then G acts on s objects as a transitive permutation group of order $s = [\Gamma_0 : \Gamma_1]$ and all $\Theta(\gamma_j)$ are products of s/m_j cycles of length m_j .

The permutation group G is the **monodromy group** of the covering

$$\Gamma_1\backslash\mathcal{H} \to \Gamma_0\backslash\mathcal{H} .$$

One may imagine that it permutes the sheets of the covering. In the special case of a triangle group $\Delta = \Gamma_0$ and a finite index subgroup $\Gamma = \Gamma_1$, the permutation group G and the homomorphism Θ have nice combinatorial interpretations in terms of the dessin belonging to the Belyi function β as given in (1). By [JS1] or [BI], G is the well–known *hypermap group* or (in the special case of triangle groups $< p, 2, r >$) the *cartographic group* and can be defined as follows. Represent the edges of the dessin $\beta^{-1}[0,1]$ by the numbers $1, \ldots, s = [\Delta : \Gamma]$ and recall that Δ is presented by

$$\Delta = < \gamma_0, \gamma_1, \gamma_\infty \mid \gamma_0^p = \gamma_1^q = \gamma_\infty^r = 1 = \gamma_0\gamma_1\gamma_\infty > .$$

We may suppose that the white vertices of the dessin, i.e. the points of $\beta^{-1}\{0\}$, are the Δ–images of the fixpoint of γ_0 (mod Γ of course) and the black points ($\beta^{-1}\{1\}$) the Δ–images of the fixed point of γ_1 . Let g_0 permute the edges joining any white vertex in cyclic conter–clockwise order; note that this is well–defined since every edge joins a unique white and a unique black vertex. In the same way, define g_1 as permuting the edges joining any black vertex in cyclic counterclockwise order. Then g_0 and g_1 generate a permutation group $G \subseteq S_s$ of the s edges — transitive because the dessin is connected. Note that $g_\infty := (g_0 g_1)^{-1}$ decomposes into cyclic counterclockwise permutations of (half of the) edges around each *face* of the dessin, i.e. the connected components of $Y - \beta^{-1}[0,1]$. By construction, the valencies of the dessin in the white and black vertices divide p and q respectively, and the number of edges bounding any face divides $2r$: note that each face contains precisely one point of $\beta^{-1}\{\infty\}$. Therefore,

$$\gamma_0 \mapsto g_0 , \quad \gamma_1 \mapsto g_1 , \quad \gamma_\infty \mapsto g_\infty$$

defines a homomorphism $\Theta : \Delta \to G$ as required in Lemma 5.

The subgroup B_x of the monodromy group G stabilizing one edge x of this dessin on $Y = \Gamma\backslash\mathcal{H}$ is of particular interest: The group of (orientation preserving) automorphisms of the dessin is the centralizer of G in the full permutation group S_s of all edges. It is isomorphic to $N(B_x)/B_x$ where $N(B_x)$ denotes the normalizer of B_x in G (see e.g. [JS1] or Prop. 3 of [BI] for the special case of cartographic groups, i.e. for clean Belyi functions or $\operatorname{ord}\gamma_1 = 2$; the proof for monodromy groups of general Belyi functions is almost the same, using a bijection between edges and residue classes of G/B_x). Note that for any other edge y the stabilizer group B_y is conjugate to B_x , and that $\{(1)\}$ is the common intersection of all B_y . The automorphism group of the dessin is isomorphic to a group of (conformal) automorphisms of the Riemann surface, and the dessin is regular if and only if $B_x = \{(1)\}$. This case occurs precisely if $\Gamma = N$ is a normal subgroup of Δ , and then $G \cong \Delta/N$ is itself isomorphic to the automorphism group of the dessin. Then the Belyi function (1) is a normal covering $Y \to \mathbb{P}^1(\mathbb{C})$ and, following Section 1, we can replace (Γ, Δ) by a minimal pair in which (by Lemma 1) $\Gamma = N$ is torsion–free, whence Y has many automorphisms.

In the general case, the monodromy group G of the Belyi function (1) allows one to construct a regular dessin on the covering surface X of Lemma 1 as in [JS1] (or Prop. 4 of [BI] for the special case of cartographic groups):

Theorem 2 *Let Δ be a cocompact Fuchsian triangle group, $\Gamma \subset \Delta$ a subgroup of finite index, let x be an edge of the corresponding dessin on $Y := \Gamma\backslash\mathcal{H}$. Let N be the kernel of the homomorphism Θ of Δ onto the monodromy group G of*

$$\beta \,:\, \Gamma\backslash\mathcal{H} \,=\, Y \,\to\, \Delta\backslash\mathcal{H} \,\cong\, \mathbb{P}^1(\mathbb{C}) \,.$$

Then

1. *N is the maximal normal subgroup of Δ contained in Γ ,*
2. *N is torsion–free if (Γ, Δ) is a minimal pair,*
3. *$G \cong \Delta/N$ acts as group of automorphisms of the covering surface $X = N\backslash\mathcal{H}$,*
4. *and also as automorphism group of the regular dessin obtained by lifting the dessin from Y to X .*
5. *Further, G is isomorphic to the monodromy group and to the covering group of the (normal) covering Belyi function*
$$X \,\to\, \Delta\backslash\mathcal{H} \,=\, \mathbb{P}_1(\mathbb{C}) \,.$$
6. *The covering group $H \subseteq G$ of the covering*
$$X \,\to\, Y \,\cong\, H\backslash X$$
is isomorphic to the stabilizer B_x .
7. *The permutations $\Theta(\gamma)$, $\gamma \in G$, of the sheets of β correspond to the action of γ on G/H by left multiplication.*

For the *proof* consider G not only as permutation group of the sheets of $Y \to \mathbb{P}^1(\mathbb{C})$ but also as permutation group operating on itself by left translation, i.e. on $|G| = \operatorname{ord} G$ objects. Then the same $\Theta : \Delta \to G$ serves as homomorphism of Lemma 5 for the inclusion $\Delta \supset \Gamma$ and for the inclusion $\Delta \supset N$: If G is seen as permutation group operating on itself by left translation, every $\Theta(\gamma_j)$ is a product of $|G|/\operatorname{ord}\gamma_j$ cycles of length $m_j = \operatorname{ord}\gamma_j$. Then for $X := N\backslash\mathcal{H}$ parts 3 and 5 of the Theorem are obvious. For part 4 observe that there is a bijection

$$\gamma\Gamma \,\mapsto\, \Theta(\gamma)B_x$$

of the set of residue classes Δ/Γ with the sheets of the covering β, with the edges of the corresponding dessin, and with the residue classes G/B_x. By construction, the Δ–action on Δ/Γ is mapped via Θ to the G–action on the edges. This shows in particular that $N \subseteq \Gamma \subset \Delta$ whence the dessin on X is a (regular) covering of that on Y. The same argument shows parts 6 and 7 by $H \cong \Gamma/N \cong B_x$. Part 1 is a consequence of $\bigcap_{g\in G} g^{-1}B_x g = \{(1)\}$ mentioned above, now giving $\bigcap_{\gamma\in\Delta} \gamma^{-1}\Gamma\gamma = N$. With Lemma 1, this also implies part 2.

3 Transcendence

The endomorphism algebra of an abelian variety A defined over $\bar{\mathbb{Q}}$ acts on the vector space $H^0(A,\Omega_{\bar{\mathbb{Q}}})$ of holomorphic differentials defined over $\bar{\mathbb{Q}}$ and on the homology $H_1(A,\mathbb{Z})$ as well. These actions imply $\bar{\mathbb{Q}}$–linear relations between the **periods (of first kind)** $\int_\gamma \omega$, $\omega \in H^0(A,\Omega_{\bar{\mathbb{Q}}})$, $\gamma \in H_1(A,\mathbb{Z})$. By Wüstholz' analytic subgroup theorem it is in fact known that *all* $\bar{\mathbb{Q}}$–linear relations between such periods are induced by these actions of $\mathrm{End}\,A$. This section gives a report on this principle and shows how the endomorphism algebra $\mathrm{End}_0\,A$ is visible on the $\bar{\mathbb{Q}}$–vector space generated in $\mathbb{C}$ by the periods of the holomorphic differentials defined over $\bar{\mathbb{Q}}$. Since we are ultimately interested in Jacobians, we will assume that all abelian varieties are principally polarized, but most results given here do not depend on the specific polarization, so we do not introduce an extra notation for it. The proofs of the Lemmas 6 to 8 and some remarks on the history can be found in [SW] (Prop. 1, 2, 3 and their Corollaries); Lemma 9 is a very special case of the Main Theorem of [SW] giving the corresponding result for all analytic families of polarized abelian varieties. For a more modern version, see [Coh].

Lemma 6 *Let A be an abelian variety defined over $\bar{\mathbb{Q}}$, isogenous to the direct product*

$$A_1^{k_1} \times \ldots \times A_m^{k_m}$$

of simple, pairwise non–isogenous abelian varieties A_ν, $\nu = 1,\ldots,m$ of complex dimension n_ν, and let V_A be the $\bar{\mathbb{Q}}$–vector subspace of $\mathbb{C}$ generated by all periods of the first kind

$$\int_\gamma \omega\,, \quad \omega \in H^0(A,\Omega_{\bar{\mathbb{Q}}})\,, \quad \gamma \in H_1(A,\mathbb{Z})\,.$$

Then

$$\dim_{\bar{\mathbb{Q}}} V_A \;=\; \sum_{\nu=1}^{m} \frac{2n_\nu^2}{\dim_{\mathbb{Q}}\,\mathrm{End}_0\,A_\nu}\quad.$$

If in particular A is simple and U is a $\bar{\mathbb{Q}}$–vector subspace of $H^0(A,\Omega_{\bar{\mathbb{Q}}})$ invariant under the action of $\mathrm{End}_0\,A$, the periods $\int_\gamma \omega$, $\omega \in U$, $\gamma \in H_1(A,\mathbb{Z})$ generate a $\bar{\mathbb{Q}}$–vector space of dimension

$$\frac{2\,(\dim_{\bar{\mathbb{Q}}} U)\,(\dim_{\mathbb{C}} A)}{\dim_{\mathbb{Q}}\,\mathrm{End}_0\,A}\,.$$

Lemma 7 *A simple abelian variety A defined over $\bar{\mathbb{Q}}$ has complex multiplication if and only if a differential $\omega \neq 0$ of the first kind defined over $\bar{\mathbb{Q}}$ exists on A such that all periods of ω are algebraic multiples of each other.*

In Prop. 3 of [SW] the 'only if' part of this Lemma is given, but the 'if' part is easily proved taking any eigendifferential of the CM field $\mathbb{K} = \mathrm{End}_0\, A$. The same remark applies to

Lemma 8 *Let A be an abelian variety defined over $\bar{\mathbb{Q}}$.*

1. *A has a factor with complex multiplication if and only if a nonzero differential $\omega \in H^0(A, \Omega_{\bar{\mathbb{Q}}})$ exists all of whose periods are algebraic multiples of one another.*
2. *A is of CM type if and only if there is a basis $\omega_1, \ldots, \omega_n$ of $H^0(A, \Omega_{\bar{\mathbb{Q}}})$ with the property that for all ω_ν, $\nu = 1, \ldots, n$, all periods*
$$\int_\gamma \omega_\nu \ , \quad \gamma \in H_1(A, \mathbb{Z}) \ ,$$
lie in a one–dimensional $\bar{\mathbb{Q}}$–vector space $V_\nu \subset \mathbb{C}$.

In principle, Lemma 8 already gives a necessary and sufficient condition for abelian varieties and hence of Jacobians to be of CM type, but the explicit use of it presents a nontrivial problem, see the next sections. The same is true for the next Lemma; for its preparation, recall that every principally polarized complex abelian variety A of dimension n is isomorphic to an abelian variety whose underlying complex torus is

$$A_Z \ := \ \mathbb{C}^n/(\mathbb{Z}^n \oplus Z\mathbb{Z}^n) \ , \quad Z \in \mathcal{H}_n \ ,$$

where $\mathcal{H}_n$ denotes the Siegel upper half space of symmetric complex $n \times n$–matrices with positive definite imaginary part. For principally polarized abelian varieties, Z is uniquely determined by A up to transformations under the Siegel modular group $\Gamma_n := Sp(2n, \mathbb{Z})$. In particular, the property whether or not Z is an **algebraic point** of $\mathcal{H}_n$, i.e. if the matrix Z has algebraic entries depends only on the complex isomorphism class of A. We call Z a **period quotient** or a **normalized period matrix** of A. In this terminology, we have

Lemma 9 *The complex abelian variety A defined over $\bar{\mathbb{Q}}$ and of dimension n is of CM type if and only if its period quotient Z is an algebraic point of the Siegel upper half space $\mathcal{H}_n$.*

The way from a smooth complex projective curve or compact Riemann surface X of genus $g = n$ to these period quotients is well known: Take a basis $\omega_1, \ldots, \omega_n$ of the holomorphic differentials and a *symplectic* basis $\gamma_1, \ldots, \gamma_{2n}$ of the homology of X, i.e. with intersection matrix

$$J \ := \ \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} , \quad E \quad \text{the } n \times n \text{ unit matrix}$$

and form the $(n \times 2n)$–period matrix

$$(\Omega_1, \Omega_2) := (\int_{\gamma_j} \omega_i) , \quad i = 1, \dots , n , \ j = 1, \dots , 2n$$

with $n \times n$–matrices Ω_1 and Ω_2. Then $Z := \Omega_2^{-1}\Omega_1$ is called **period quotient for** X since it is a period quotient for $\operatorname{Jac} X$. Note that Z is independent of the choice of the basis of the differentials and that another choice of the symplectic basis of the cycles gives another point of the orbit of Z under the action of Γ_n. By Lemma 9, an obvious criterion for our original problem is the following

Theorem 3 *Let C be a nonsingular complex projective curve defined over $\bar{\mathbb{Q}}$. Its Jacobian is of CM type if and only if its period quotient is an algebraic point of the Siegel upper half space.*

4 Integration on regular dessins

Let X be a Riemann surface with many automorphisms of genus $g > 1$, given as quotient $N \backslash \mathcal{H}$ where N denotes its universal covering group, given as normal subgroup of a Fuchsian triangle group $\Delta = < p, q, r >$. As we saw in Sections 1 and 2, this inclusion $N \lhd \Delta$ provides X with a regular dessin and a corresponding Belyi function whose monodromy group ($\cong$ covering group) is isomorphic to an automorphism group G — an automorphism group of X and of the dessin as well —; it is in fact *the* automorphism group of X if Δ is maximal among the triangle groups containing N as normal subgroup. We fix one edge δ of the dessin. Recall that the edges are the images under the canonical projection $\mathcal{H} \to X$ of the Δ–orbit of one side of some hyperbolic triangle forming one half of a fundamental domain of Δ, and that by regularity, G acts transitively on the set of all edges of the dessin. As already mentioned in Section 1, X is isomorphic to the Riemann surface consisting of the complex points of a curve C defined over a number field. We can therefore speak of the differentials ω of the first kind on X defined over $\bar{\mathbb{Q}}$. They form the $\bar{\mathbb{Q}}$–vector space $H^0(X, \Omega_{\bar{\mathbb{Q}}})$ which we identify tacitely with $H^0(\operatorname{Jac} X, \Omega_{\bar{\mathbb{Q}}})$. Recall that $\operatorname{Jac} X$ is a principally polarized abelian variety defined over $\bar{\mathbb{Q}}$. As in Section 2,

$$\Phi(\alpha) : \omega \mapsto \omega \circ \alpha^{-1} , \quad \alpha \in G ,$$

defines the canonical representation Φ of G on $H^0(X, \Omega_{\bar{\mathbb{Q}}})$: since $X \cong C(\mathbb{C})$ has only finitely many automorphisms, an easy deformation argument shows that $\alpha : C \to C$ is defined over $\bar{\mathbb{Q}}$. Let U be a G–invariant subspace of $H^0(X, \Omega_{\bar{\mathbb{Q}}})$ and define the $\bar{\mathbb{Q}}$–vector space

$$V(U) := \{ \int_\delta \omega \mid \omega \in U \} \subset \mathbb{C} .$$

Then we have

Lemma 10 *1.* $\dim_{\bar{\mathbb{Q}}} V(U) \leq \dim_{\bar{\mathbb{Q}}} U$ *;*

2. *$V(U)$ is independent of the choice of δ ;*

3. *The $\bar{\mathbb{Q}}$–vector space V_U generated by the set of periods $\{ \int_\gamma \omega \mid \omega \in U \, , \, \gamma \in H_1(X, \mathbb{Z}) \}$ is contained in $V(U)$;*

4. $\dim_{\bar{\mathbb{Q}}} V_{\mathrm{Jac}\, X} \ \leq \ g \ = \dim_{\mathbb{C}} \mathrm{Jac}\, X$.

(Notice that in the terminologies introduced in Lemma 6 and Lemma 10.3, $V_{\mathrm{Jac}\, X} = V_{H^0(X, \Omega_{\bar{\mathbb{Q}}})}$.)
The first property is evident. The second follows from the substitution rule

$$\int_{\alpha(\delta)} \omega \ = \ \int_\delta \omega \circ \alpha \, , \quad \alpha \in G \, ,$$

and the facts that G acts transitively on the set of edges and that U is G–invariant. For the third point remember that the faces of the dessin are simply connected whence any cycle γ on X has a representative composed by edges; the orientations of the cycles and edges do not matter. By the same argument as in the proof of 2., all periods of all $\omega \in U$ are $\bar{\mathbb{Q}}$–linear combinations of integrals in $V(U)$. For the fourth point we use the well known identification of the periods on X and on $\mathrm{Jac}\, X$ — always with differentials defined over $\bar{\mathbb{Q}}$ — and the points 1. and 3. for the whole space $U = H^0(X, \Omega_{\bar{\mathbb{Q}}})$ of dimension g .
If we compare the last point of Lemma 10 with the dimension formula for V_A given in Lemma 6, we see how far these Jacobians are from generic abelian varieties, i.e. simple ones with $\mathrm{End}_0\, A = \mathbb{Q}$, where one obtains $\dim_{\bar{\mathbb{Q}}} V_A = 2g^2$ with $g = \dim_{\mathbb{C}} V_A$. For our original problem we can draw immediately the following conclusion.

Theorem 4 *Let X be a compact Riemann surface of genus $g > 1$ with many automorphisms, let $G \subseteq \mathrm{Aut}\, X$ be isomorphic to the automorphism group of a regular dessin on X , and Φ its canonical representation on the space of differentials of the first kind on X .*

1. *If Φ has a one–dimensional invariant subspace,* $\mathrm{Jac}\, X$ *has a factor with complex multiplication.*

2. *If Φ splits into one–dimensional subrepresentations,* $\mathrm{Jac}\, X$ *is of CM type.*

3. *If G is abelian,* $\mathrm{Jac}\, X$ *is of CM type.*

Proof. Let ω generate a one–dimensional G–invariant subspace U of Φ . Lemma 10.1 and 10.3 show that the periods of ω on X or $\mathrm{Jac}\, X$ lie in a one–dimensional $\bar{\mathbb{Q}}$–vector space in $\mathbb{C}$ whence Lemma 8 applies and gives the first two points of the theorem. Since all representations of abelian groups split into one–dimensional subrepresentations, the last point is obvious.

Remark 3. We will see in the last section that the representation Φ plays in fact a crucial role for the structure of the endomorphism algebra of $\mathrm{Jac}\, X$, but Theorem 4 is only a new look on the fact that Fermat curves of exponent n have Jacobians of CM type what is well–known by [KR].

On the one hand, their universal covering groups N are the commutator subgroups $[\Delta, \Delta]$ of $\Delta = < n, n, n >$ ([Wo1], [CIW], [JS3]); for the exponent $n = 3$ we obtain an elliptic curve with complex multiplication as remarked already after Theorem 1. On the other hand — as pointed out to me by Gareth Jones and David Singerman — every CM factor detected by Theorem 4 is a factor of a Jacobian of some Fermat curve what can be seen as follows. Suppose that Φ has a one–dimensional subrepresentation with invariant subspace generated by some $\omega \neq 0$. It is not the unit representation since otherwise ω would be a differential lifted by the canonical Belyi function

$$\beta \,:\, X \,=\, N\backslash\mathcal{H} \,\rightarrow\, \Delta\backslash\mathcal{H} \,\cong\, \mathbb{P}^1(\mathbb{C})$$

from a differential on $\mathbb{P}^1$, hence $\omega = 0$. Therefore we have a homomorphism of $G = \Delta/N$ onto a nontrivial abelian group. Now suppose $\Delta = < p, q, r >$ and let n be the least common multiple of p, q, r. Then we can replace Δ by $\Delta_1 = < n, n, n >$ and N by some $N_1 \lhd \Delta_1$ which is no longer torsion–free but still satisfies $X \cong N_1\backslash\mathcal{H}$, see the remarks after Theorem 1 about the possible choices of the signature of Δ for a given β. On Δ_1, we have a homomorphism onto a nontrivial abelian group whose kernel K contains both N_1 and the commutator subgroup $[\Delta_1, \Delta_1]$. Therefore, the quotient $K\backslash\mathcal{H}$ is a common quotient of X and of the Fermat curve of exponent n, and ω is a lift of a differential on this quotient.

5 Shimura families and the Jacobi locus

Let X be a compact Riemann surface of genus $g > 1$ with many automorphisms and let G denote its automorphism group. Since every automorphism of X induces a unique automorphism of $\mathrm{Jac}\, X$, we can consider G as subgroup of the (polarization preserving) automorphism group G_J of $\mathrm{Jac}\, X$. It is known (see e.g. [Ba], p.375 or Lemma 1 and 2 in [Po]) that

Lemma 11 *1. $G_J = G$ if X is hyperelliptic,*

2. $[G_J : G] = 2$ if X is not hyperelliptic.

Let Z be a fixed period quotient for X and $\mathrm{Jac}\, X$ in the Siegel upper half space $\mathcal{H}_g$, and let L be the algebra $\mathrm{End}_0\, \mathrm{Jac}\, X$. Fixing a symplectic basis of the homology of X we obtain not only a fixed Z but also a rational representation of L. For an explicit version of this representation see Section 2 of [Ru]. Here we consider therefore L as subalgebra of the matrix algebra $M_{2g}(\mathbb{Q})$. The action of G_J on a symplectic basis of the homology of X gives

Lemma 12 *The automorphism group G_J is isomorphic to the stabilizer subgroup*

$$\Sigma_J \,:=\, \{\, \gamma \in \Gamma_g \mid \gamma(Z) \,=\, Z \,\}$$

of Z in the Siegel modular group $\Gamma_g = Sp_{2g}\mathbb{Z}$.

A different choice of Z induced by some other choice of the homology basis on X gives a Γ_g–conjugate stabilizer and a Γ_g–conjugate rational representation of the endomorphism algebra as well. If we fix Z, hence Σ_J, we can linearize the analytic subset

$$S_G := \{W \in \mathcal{H}_g \mid \gamma(W) = W \quad \text{for all} \quad \gamma \in \Sigma_J\} \tag{3}$$

of the Siegel upper half space by a generalized Cayley transform (see e.g. [Go]) to see that S_G is in fact a submanifold. By definition, $Z \in S_G$. Since we consider G as a subgroup of L, S_G contains a complex submanifold $\mathbb{H}(L)$ of $\mathcal{H}_g$ parametrizing a **Shimura family** of principally polarized complex abelian varieties A_W, $W \in \mathbb{H}(L)$, containing L in their endomorphism algebras — always considered as subalgebra of $M_{2g}(\mathbb{Q})$ in a fixed rational representation. For explicit equations defining $\mathbb{H}(L)$ see Sections 2 and 3 of [Ru], in particular Runge's Lemma 2 and the following definitions. The submanifold $\mathbb{H}(L)$ is a complex symmetric domain; we will call it the **Shimura domain** for Z or for X, suppressing in this notation the dependence on the chosen rational representation. The dimension of $\mathbb{H}(L)$ is well known and depends on the type of L only ([Sh], [Ru]). In particular we have

Lemma 13 $\dim \mathbb{H}(L) = 0$, *i. e.* $\mathbb{H}(L) = \{Z\}$ *if and only if* $\mathrm{Jac}\, X$ *is of CM type.*

The period quotients of all Jacobians of compact Riemann surfaces of genus g form another locally analytic set $\mathcal{J}_g \subseteq \mathcal{H}_g$ of complex dimension $3g-3$, the **Jacobi locus**. By definition of 'many automorphisms', X has no deformation preserving the automorphism group G, whence $\mathrm{Jac}\, X$ has no deformation as a Jacobian preserving its automorphism group with the following exception: it is possible that X is hyperelliptic with an automorphism group $G = G_J$ and has non–hyperelliptic deformations X' whose automorphism group G' is isomorphic to an index two subgroup of G and such that $G'_J \cong G$ (recall that the hyperelliptic involution gives the matrix $-E \in Sp_{2g}\mathbb{Z}$ hence the identity in PSp_{2g}). Then, $S_G = S_{G'}$ — for an example see Section 6.3. Since the hyperelliptic involution generates a cyclic group C_2 central in G, we have $G \cong G' \times C_2$ in these cases. In all other cases, S_G intersects $\mathcal{J}_g$ in isolated points. A fortiori, we obtain

Theorem 5 *Let X be a Riemann surface with many automorphisms. If X is hyperelliptic and $g \geq 3$, we suppose further that the hyperelliptic involution does not generate a direct factor of the automorphism group. Then its period quotient Z is an isolated point of the intersection $\mathbb{H}(L) \cap \mathcal{J}_g$ of its Shimura domain and the Jacobi locus.*

Because $\mathcal{J}_2$, $\mathcal{J}_3$ are open and dense in $\mathcal{H}_2$, $\mathcal{H}_3$ respectively, Z is in these cases an isolated point of $\mathbb{H}(L)$, hence $\dim \mathbb{H}(L) = 0$. Lemma 13 implies that $\mathrm{Jac}\, X$ is of CM type, hence

Theorem 6 *All compact Riemann surfaces with many automorphisms of genus* 2 *and all non–hyperelliptic Riemann surfaces with many automorphism of genus* 3 *have Jacobians of CM type.*

Clearly, these arguments are not extendable to higher genus, see also the examples in Section 6. We can however draw some more conclusions on $\mathrm{End}_0 \, \mathrm{Jac} \, X$ in the case of higher genus from Theorem 5 and the results of the last section. For example, in the genus 4 case $\mathcal{J}_4$ has codimension 1 in $\mathcal{H}_4$. Since the period quotient Z of a non–hyperelliptic Riemann surface X of genus 4 with many automorphisms is an isolated point of the intersection $\mathbb{H}(L) \cap \mathcal{J}_4$, its Shimura family can be at most complex 1–dimensional. Then, $\mathbb{H}(L) \cong \mathcal{H} \cong \mathcal{H}_1$, and from [Sh] or [Ru] one knows the possibilities for L. Lemma 10.4 combined with Lemma 6 and Theorem 4.1 gives further restrictions on the endomorphism algebras of the simple factors of $\mathrm{Jac} \, X$ which imply finally

Theorem 7 *Let X be a compact Riemann surface of genus 4 and with many automorphisms. If $\mathrm{Jac} \, X$ is not of CM type, then up to isogeny and up to CM factors C, there are only the possibilities*

1. $\mathrm{Jac} \, X \; = \; E^k \times C$, *E an elliptic curve without complex multiplication, with multiplicity $k \; = \; 2, 3$ or 4 , or*
2. $\mathrm{Jac} \, X \; = \; A^2$ *or* $A \times C$, *A an abelian variety of dimension 2 whose endomorphism algebra is isomorphic to an indefinite quaternion algebra $\mathbb{B}$ over $\mathbb{Q}$.*
3. *X is hyperelliptic.*

We omit the details of the proof since in Section 6, we will give a complete classification of the genus 4 Riemann surfaces with many automorphisms showing that only the first case occurs.— Ries [Ri] gives more examples of groups G with 1–dimensional fixmanifolds S_G in higher genera, in particular for some simple Hurwitz groups, see also the last example of the next section.

6 Examples

6.1 Fermat curves and their quotients

We have already seen in Remark 3 that the Fermat curves and their quotients fall under the applications of Theorem 4. That these Jacobians are of CM type can be derived from Theorem 3 as well. In fact, the period quotients of curves with many automorphisms are calculated in the literature surprisingly often, see e.g. Bolza [Bo], Schindler [Sr1], [Sr2], Kuusalo and Näätänen [KN]. Examples are given by the affine equations

$$
\begin{aligned}
y^2 &= x^{2g+2} - 1 \\
y^2 &= x^{2g+2} - x \\
y^2 &= x^{2g+1} - x \,.
\end{aligned}
$$

All these curves are quotients of Fermat curves what can be seen directly or with Remark 3. Up to finitely many exceptions, they form all hyperelliptic curves with many automorphisms: the hyperelliptic involution commutes with all automorphisms whence the automorphism group induces an automorphism group of $\mathbb{P}^1$ acting on the ramification points of the hyperelliptic covering map, and the cyclic and the dihedral groups form the only infinite families of finite automorphism groups of $\mathbb{P}^1$. For examples not belonging to these infinite families, see equations (10) and (16) below.

For most other cases, an efficient way to calculate the period quotient is not to calculate all periods but to calculate the fixed point set S_G (see (3), Section 5) by means of the symplectic representation of the automorphism group G in the Siegel modular group Γ_g — in the hope that S_G is an isolated point, hence algebraic. This symplectic representation can be effectively constructed as a subrepresentation of the regular representation [St2].

6.2 Genus 2

For genus 2 and 3 we know already by Theorem 6 that up to possible hyperelliptic exceptions all curves with many automorphisms have Jacobians of CM type. For the convenience of the reader, we include a list of isomorphism classes of such curves in genus < 5. The classification is based in part on the classification of regular maps given in [CM], [Ga] and [Sk], i.e. the classification of all normal torsion–free subgroups of triangle groups $< 2, q, r >$, and in part on a classification of canonical representations by Kuribayashi/Kuribayashi [KK] and the references given there. All informations below can be obtained by a case–by–case application of Singerman's Theorem (Lemma 5), supported by Takeuchi's inclusion relations [Ta], and in many cases the equations are easy to guess by the ramification data of the Belyi functions.— For $g = 2$ one obtains three nonisomorphic curves X with many automorphisms (hyperelliptic, of course) with affine models

$$y^2 = x^6 - x \tag{4}$$
$$y^2 = x^6 - 1 \tag{5}$$
$$y^2 = x^5 - x\,. \tag{6}$$

Their covering groups N are normal subgroups of the triangle groups $< p, q, r >$ which are listed in the following table — for $g = 2$, all these groups are arithmetically defined. The quotient $< p, q, r > /N \subseteq \mathrm{Aut}\, X$ is clearly $= \mathrm{Aut}\, X$ for the maximal choice of $< p, q, r >$.

As always, C_m denotes the cyclic group of order m. In the automorphism group of the curve (5) $G \cong < 2, 4, 6 > /N$, the generator β of the last factor C_2 acts on the generator α of C_3 by $\beta^{-1}\alpha\beta = \alpha^{-1}$ and exchanges the two other C_2–factors.
For the curve (6), the isomorphism $G \cong < 2, 3, 8 > /N \cong GL_2\mathbb{F}_3$ is provided by

$$g_0 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \; g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; g_\infty = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

equation	p	q	r	$<p,q,r>/N$	Ord $<p,q,r>/N$
4	2	5	10	C_{10}	10
	5	5	5	C_5	5
5	2	4	6	$(C_3 \times C_2 \times C_2) \rtimes C_2$	24
	2	6	6	$C_3 \times C_2 \times C_2$	12
	3	6	6	$C_3 \times C_2$	6
	3	4	4	$C_3 \rtimes C_4$	12
6	2	3	8	$GL_2\mathbb{F}_3 \cong Q \rtimes S_3$	48
	3	3	4	$SL_2\mathbb{F}_3 \cong Q \rtimes C_3$	24
	2	4	8	$Q \rtimes C_2 \cong C_8 \rtimes C_2$	16
	2	8	8	C_8	8
	4	4	4	Q	8

where as in Section 2 we denote the generators of G induced by the generators of Δ by g_0, g_1, g_∞. For the other presentation, Q denotes the quaternion group generated by i, j, k with

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ijk = 1 ,$$

and the action of S_3 on Q is defined by

$$(12) \ : \ i \mapsto j \ , \ j \mapsto i \ , \ k \mapsto -k$$

$$(23) \ : \ i \mapsto -i \ , \ j \mapsto k \ , \ k \mapsto j \ .$$

The center of G is $Z \cong C_2$ generated by the hyperelliptic involution g_∞^4, and the factor group G/Z is isomorphic to $S_4 \cong PGL_2\mathbb{F}_3$ acting on $\mathbb{P}_1 \cong X/Z$. The fixed points of g_∞ become in fact the face-centers of a regular cube or the vertices of a regular octahedron in $\mathbb{P}_1$.

6.3 Genus 3

Here we have eight nonisomorphic curves with many automorphisms which can be described by the models

$$y^2 = x^8 - x \tag{7}$$
$$y^2 = x^7 - x \tag{8}$$
$$y^2 = x^8 - 1 \tag{9}$$
$$y^2 = x^8 - 14x^4 + 1 \tag{10}$$
$$y^3 = x\,(x^3 - 1) \tag{11}$$
$$y^4 + x^3 = 1 \tag{12}$$
$$y^4 + x^4 = 1 \tag{13}$$
$$x^3y + y^3z + z^3x = 0 \tag{14}$$

equation	p	q	r	$< p,q,r > /N$	Ord $< p,q,r > /N$	remarks
7	2	7	14	C_{14}	14	h
	7	7	7	C_7	7	
8	2	4	12	$S_3 \times C_4$	24	h
	2	12	12	C_{12}	12	
	4	4	6	$C_3 \rtimes C_4$	12	
9	2	4	8	$(C_8 \times C_2) \rtimes C_2$	32	h
	2	8	8	$C_8 \times C_2$	16	
	4	8	8	C_8	8	
	4	4	4	$C_4 \rtimes C_4$	16	
10	2	4	6	$S_4 \times C_2$	48	h , ncm
	2	6	6	$A_4 \times C_2$	24	
	3	4	4	S_4	24	
11	3	9	9	C_9	9	na
12	2	3	12	$(SL_2\mathbb{F}_3 \times C_4)/ < (-I,t^2) >$	48	
	3	3	6	$SL_2\mathbb{F}_3$	24	
	3	4	12	C_{12}	12	
13	2	3	8	$(C_4 \times C_4) \rtimes S_3$	96	
	3	3	4	$(C_4 \times C_4) \rtimes C_3$	48	
	2	4	8	$(C_4 \times C_4) \rtimes C_2$	32	
	4	4	4	$C_4 \times C_4$	16	
	2	8	8	$C_8 \rtimes C_2$	16	
	4	8	8	C_8	8	
14	2	3	7	$PSL_2\mathbb{F}_7$	168	
	3	3	7	$C_7 \rtimes C_3$	21	
	7	7	7	C_7	7	

Their universal covering groups N are normal subgroups of the triangle groups $< p,q,r >$ listed with their quotients and their indices in the table of the last page. The last column indicates if the curve is hyperelliptic, if the covering group is non–arithmetic and if the Jacobian is not of CM type by the abbreviations h, na, ncm. By D_m we denote the dihedral group of order $2m$. The other semidirect products are explained in the comments below.

Comments. For the curve (8), $< 4,4,6 > /N \cong C_3 \rtimes C_4$ is isomorphic to the group $< 3,4,4 > /N$ in case (5), but with different generators.
If in the case of the curve (9) the generators of the factors of $G =< 2,4,8 > /N$ of respective orders $8,2,2$ are denoted by α, β, γ, where γ acts on the normal subgroup by

$$\gamma^{-1}\alpha\gamma \;=\; \alpha^{-1}\beta \quad , \quad \gamma^{-1}\beta\gamma \;=\; \beta \, ,$$

we may choose $g_0 = \gamma$, $g_1 = \alpha\gamma$, $g_\infty = \beta\alpha$. Then, $< 4,4,4 > /N$ is generated by α^2 and $\alpha\gamma$.

The curve (10) plays an exceptional role in several respects. The polynomial on the right hand side of (10) has to be constructed in such a way that the zeros form the vertices of a regular cube or the centers of the faces of a regular octahedron inside the Riemann sphere. Using Lemma 5 and Theorem 2, X can be seen to be a 2–fold cover of several isomorphic elliptic curves, e.g. $H\backslash X$ where H is the subgroup

$$\{\,((1),1)\,,\,((12)(34),1)\,\} \quad \text{of} \quad S_4 \times C_2\,.$$

Its equation is

$$y^2 \;=\; x^4 \;-\; 14x^2 \;+\; 1$$

whose right hand side vanishes at $\sqrt{2+\sqrt{3}}$ and its algebraic conjugates. A fractional linear transformation maps these zeros to $0, 1, \infty$ and $\lambda = 4/3$, so the j–invariant of the elliptic curve becomes

$$j \;=\; 4^4\,\frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2} \;=\; \frac{16\cdot 13^3}{9}\;.$$

Since it is not an integer, the elliptic curve has no complex multiplication and by Lemma 3, the Jacobian of (10) is not of CM type. Apparently, (10) is an example for the hyperelliptic exception mentioned in Theorems 5 and 6. With the index 2 factor $G' = S_4$ of G there is a complex 1–dimensional family of curves X_τ of genus 3 with automorphism group $\mathrm{Aut}\,X_\tau \supseteq G'$ and $\mathrm{Aut}\,\mathrm{Jac}\,X_\tau \supseteq G'_J \cong G$. The existence of this family can be deduced from the fact that the Fuchsian (genus 0) quadrangle groups of signature $< 2,2,2,3 >$ form a complex 1–dimensional family and N is contained in such a quadrangle group. By Singerman's theorem (Lemma 5) every member of this quadrangle group family contains a normal torsion–free genus 3 subgroup with quotient $\cong S_4$, so we have a 1–dimensional Shimura family of Jacobians parametrized by a complex submanifold $\mathbb{H}(L) \cong \mathcal{H}$ of the Siegel upper half space $\mathcal{H}_3$. All these Jacobians are isogenous to a third power of an elliptic curve, and $\mathbb{H}(L)$ intersects the subset of $\mathcal{H}_3$ of period quotients for hyperelliptic curves just in the (transcendental) period quotient for the curve (10). With the curve (13) we will meet another (but algebraic) point of this family $\mathbb{H}(L)$.

In the notation for the automorphism group $G \cong < 2,3,12 > /N$ of the curve (12), I denotes the identity matrix of $SL_2\mathbb{F}_3$ and t a generator of C_4. The center Z of G is of order 4, generated by g_∞^3 and with quotient $G/Z \cong A_4$. In G we have the index 2 subgroup

$$< 3,3,6 > /N \;\cong\; SL_2\mathbb{F}_3 \quad \text{via} \quad g_0 \;=\; \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\,,\; g_1 \;=\; \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\,,\; g_\infty \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

(recall that the center is $\cong C_2$ with quotient $PSL_2\mathbb{F}_3 \cong A_4$).

For the automorphism group of the Fermat curve (13) imagine that C_4 is written additively, $C_4 \times C_4$ given by

$$\{\,(\xi,\eta,\zeta) \in C_4^3 \mid \xi+\eta+\zeta \;=\; 0\,\}$$

on which S_3 acts by permutation of the coordinates. The symmetric group S_4 is contained in G in form of $(C_2 \times C_2) \rtimes S_3$, and the corresponding subgroup of $< 2,3,8 >$ is again a quadrangle group of signature $< 2,2,2,3 >$ as announced in the discussion of equation (10). For the quotients of the other triangle groups we mention only the presentation

$$< 2,8,8 > /N \;\cong\; C_8 \rtimes C_2 \;=\; < g_0, g_1 \,|\, g_0^2 = g_1^8 = 1\,,\; g_0 g_1 g_0 = g_1^5 > \;\;.$$

Finally, (14) is Klein's quartic. From the last quotient of the table $< 7,7,7 > /N \cong C_7 = < \alpha >$ we can derive

$$y^7 \;=\; x\,(x-1)^2$$

as another useful model and the (known) conclusion that it is a quotient of the Fermat curve of exponent 7. By [KR], the Jacobian is isogenous to the third power of an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-7})$. The generators

$$g_0 \;=\; \alpha\;, \qquad g_1 \;=\; \alpha^2\;, \qquad g_\infty \;=\; \alpha^4$$

for C_7 used for this model of (14) cannot be transformed by an automorphism of C_7 into those

$$g_0 \;=\; \alpha\;, \qquad g_1 \;=\; \alpha\;, \qquad g_\infty \;=\; \alpha^5$$

for the curve (7), hence both epimorphisms $< 7,7,7 > \to C_7$ have different (and even not $PSL_2\mathbb{R}$–conjugate) kernels.

6.4 Genus 4

Here we have eleven isomorphism classes of curves with many automorphisms given by the models

$$
\begin{aligned}
y^2 &= x^9 - 1 && (15)\\
y^2 &= x\,(3x^4+1)\,(3x^4+6x^2-1) && (16)\\
y^2 &= x^9 - x && (17)\\
y^2 &= x^{10} - 1 && (18)\\
y^{10} &= x^2\,(x-1) && (19)\\
y^{12} &= x^3\,(x-1)^2 && (20)\\
y^{15} &= x^5\,(x-1)^3 && (21)\\
y^3 &= 1 - x^6 && (22)\\
y^{12} &= x^4 - x^5 && (23)
\end{aligned}
$$

$$
\begin{aligned}
\frac{4}{27}\,\frac{(x^2-x+1)^3}{x^2(x-1)^2} + \frac{4}{27}\,\frac{(y^2-y+1)^3}{y^2(y-1)^2} &= 1 && (24)\\
x_1^n + \ldots + x_5^n &= 0 \qquad \text{for} \quad n \;=\; 1\,,\,2\,,\,3\,. && (25)
\end{aligned}
$$

equation	p	q	r	$< p,q,r > /N$	Ord $< p,q,r > /N$	remarks
15	2	9	18	C_{18}	18	h
	9	9	9	C_9	9	
16	3	4	6	$SL_2\mathbb{F}_3$	24	h
17	2	4	16	$C_{16} \rtimes C_2$	32	h , na
	2	16	16	C_{16}	16	
	4	4	8	$C_8 \rtimes_{C_2} C_4$	16	
18	2	4	10	$(C_{10} \times C_2) \rtimes C_2$	40	h
	2	10	10	$C_{10} \times C_2$	20	
	5	10	10	C_{10}	10	
	4	4	5	$C_5 \rtimes C_4$	20	
19	5	10	10	C_{10}	10	
20	4	6	12	C_{12}	12	na
21	3	5	15	C_{15}	15	na
22	2	6	6	$S_3 \times C_6$	36	
	3	6	6	$C_3 \times C_6$	18	
	3	6	6	$S_3 \times C_3$	18	
23	2	3	12	$C_3 \times ((C_2 \times C_2) \rtimes S_3)$	72	
	3	3	6	$C_3 \times ((C_2 \times C_2) \rtimes C_3)$	36	
	2	6	12	$C_3 \times D_4$	24	
	3	12	12	C_{12}	12	
	6	6	6	$C_3 \times C_2 \times C_2$	12	
24	2	4	6	$(S_3 \times S_3) \rtimes C_2$	72	ncm
	2	6	6	$S_3 \times S_3$	36	
	3	6	6	$C_3 \times S_3$	18	
	3	4	4	$\ker \chi\psi\phi$	36	
25	2	4	5	S_5	120	ncm
	2	5	5	A_5	60	
	4	4	5	$C_5 \rtimes C_4$	20	

With the same notations as in genus 3 we give the corresponding table of triangle groups containing their universal covering groups N as normal subgroups of genus 4.

Comment. For (16), the point ∞ and the zeros of the right hand side form the vertices and the midpoints of the edges of a regular tetrahedron. Therefore, the automorphism group must have a quotient $\cong A_4$.

$$< 3,4,6 > /N \;\cong\; SL_2\mathbb{F}_3 \quad \text{via} \quad g_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \,,\; g_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \,,\; g_\infty = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

— recall again that $PSL_2\mathbb{F}_3 \cong A_4$. Schindler [Sr1] calculated an algebraic period quotient for (16) (his equation is easily transformed into ours), so its Jacobian is of CM type by Theorem 3.

For more information see Remark 9 at the end of the next section.
For (17), the generator β of C_2 acts on the generator α of C_{16} by $\beta^{-1}\alpha\beta = \alpha^7$. The group in the third line denotes an amalgamated product.
For (18), denote the generators of the factors of $\mathrm{Aut}\, X \cong (C_{10} \times C_2) \rtimes C_2$ by α, β, γ of respective orders 10, 2, 2. Then γ acts on the normal subgroup by

$$\gamma^{-1}\alpha\gamma = \alpha^{-1}\beta \quad , \qquad \gamma^{-1}\beta\gamma = \beta \, .$$

Observe that $\delta = \alpha\gamma$ is of order 4 and acts on $\lambda = \alpha^2 = (\alpha\beta)^2$ by $\delta^{-1}\lambda\delta = \lambda^{-1}$. This gives the definition of $< 4,4,5 > /N = C_5 \rtimes C_4$ (not isomorphic to that one for the curve (25)!). The group $< 5,10,10 > /N \cong C_{10} =< \alpha >$ is isomorphic to the automorphism group of (19), but here with generators $g_0 = \alpha^8$, $g_1 = g_\infty = \alpha$ and for (19) with generators $g_0 = \alpha^6$, $g_1 = \alpha$, $g_\infty = \alpha^3$.
The curve (22) is a 3–fold cover of the curve described in equation (5). Between N and $< 2,6,6 >$ there are two triangle groups of type $< 3,6,6 >$, conjugate in $< 2,4,6 >$ but not in $< 2,6,6 >$.
In the automorphism group of (23), S_3 acts on $C_2 \times C_2$ by permutation of the elements $\neq 1$.

The order of the automorphism group of (24) is also 72 but $\mathrm{Aut}\, X$ is a wreath product, i.e. C_2 acts on the normal subgroup by exchanging the two factors S_3. If the non–trivial quadratic characters of the three factors are denoted by χ, ψ, ϕ, we get the definition of the subgroup $< 3,4,4 > /N$ given in the table. Manfred Streit determined as a model the affine equation (24) using the facts that the quotient of X by both factors S_3 has genus 0 and that one can easily determine the Belyi functions on both quotients induced by the triangle group $< 2,6,6 >$. This triangle group has a genus 1 subgroup of signature $< 1; 2,2 >$. Via its dessin Streit determined the equation

$$y^2 = (z-1)\,(27z^3 - 27z - 4)$$

of the corresponding elliptic curve E. It has non–integral j–invariant, hence no complex multiplication. At the end of the next section we will see that $\mathrm{Jac}\, X$ is in fact isogenous to E^4.

Finally, there is Bring's curve (25), given by three projective equations in $\mathbb{P}^4$. Here, the subgroup $< 4,4,5 > /N$ of the automorphism group is generated e.g. by the permutations $\alpha = (12345)$, $\beta = (2354)$ such that

$$\beta^{-1}\alpha\beta = \alpha^{-2} \, ,$$

defining therefore a semidirect product $C_5 \rtimes C_4$ different from that for the curve (18). There are several possible proofs ([RR], Section 8.3.2 of [Se]) that the Jacobian of Bring's curve is isogenous to a fourth power E^4 of an elliptic curve of invariant

$$j(E) = -\frac{25}{2} \, .$$

Since it is not integral, E has no complex multiplication whence the Jacobian is again not of CM type: the first case of Theorem 7 occurs or, in the terminology of Theorem 5, $\dim \mathbb{H}(L) = 1$.

It is remarkable that all these low–genera–curves X with many automorphisms are uniquely determined up to isomorphism by Δ and $\mathrm{Aut}\, X$. According to [Br], this observation remains true for $g = 5$ and 6 and fails first for $g = 7$.

6.5 Macbeath's curve

Macbeath's curve X of genus $g = 7$ is still uniquely determined by its automorphism group of order 504, according to Hurwitz the maximal possible order $84(g-1)$ for compact Riemann surfaces of genus $g > 1$. The automorphism group $\mathrm{Aut}\, X$ is the simple group

$$PSL_2\mathbb{F}_8 \;=\; G \;\cong\; \Delta/N \quad \text{for} \quad \Delta \;= < 2, 3, 7 > ,$$

and by work of Macbeath, his student Jennifer Whitworth and Berry and Tretkoff [BT] it is known that $X = N\backslash\mathcal{H}$ has a Jacobian isogenous to a product of elliptic curves E described by a model

$$y^2 \;=\; (x-1)(\zeta x - 1)(\zeta^2 x - 1)(\zeta^4 x - 1) \quad \text{with} \quad \zeta \;=\; e^{2\pi i/7} \,.$$

As already indicated in [BT], the question of whether $\mathrm{Jac}\, X$ has CM type and if its period quotient matrix gives therefore an algebraic point of the Siegel upper half space (see the Sections 3 and 5) reduces to the question 'does E have complex multiplication?'. The usual transformation into a cubic equation (replace $1/(x-1)$ by x and $y/(\sqrt[4]{-7}(x-1)^2)$ by y) leads to

$$y^2 \;=\; (x-\beta_1)(x-\beta_2)(x-\beta_4) \quad \text{with} \quad \beta_k := \frac{\zeta^k}{1-\zeta^k} \,.$$

The calculation of the coefficients of the cubic can be done by taking traces and norms because the zeros β_k are Galois–conjugated. A lengthy calculation gives the Weierstrass model and the invariant of E

$$j \;=\; 2^8 \cdot 7 \;=\; 1792 \,.$$

This invariant does not belong to the list of 13 rational invariants of elliptic curves with complex multiplication (see e.g. [Cr]), whence E has no complex multiplication and the Jacobian of Macbeath's curve X is not of CM type.

7 G–invariant and endomorphism–invariant subspaces

The previous section shows that general results will be more complicated than for $g = 2$ and 3. In the present section we will further develop the ideas of Section 4. Recall that Φ is the canonical representation of the group $\Delta/\Gamma \subseteq \mathrm{Aut}\, X$ on the $\bar{\mathbb{Q}}$–vector space of differential of the first kind on X (or on its Jacobian, both defined over $\bar{\mathbb{Q}}$: as always, we identify X with the complex points of a nonsingular projective algebraic curve defined over $\bar{\mathbb{Q}}$).

Lemma 14 *Let* $\operatorname{Jac} X$ *be defined over* $\bar{\mathbb{Q}}$ *and isogenous to the direct product*

$$A_1^{k_1} \times \ldots \times A_m^{k_m}$$

of simple, pairwise non–isogenous abelian varieties A_ν , $\nu = 1, \ldots, m$ *and let* U_ν , $\nu = 1, \ldots, N$, *denote the pullback of* $H^0(A_\nu^{k_\nu}, \Omega_{\bar{\mathbb{Q}}})$ *in the space of differentials of the first kind on* $\operatorname{Jac} X$, *defined over* $\bar{\mathbb{Q}}$. *Then every* U_ν *is an invariant subspace for the representation* Φ .

For the *proof* of the Lemma recall that the action of G on X induces an action on $\operatorname{Jac} X$ such that all $\alpha \in G$ map simple subvarieties of $\operatorname{Jac} X$ onto isomorphic simple subvarieties. Consider the *isotypic components* $I_\nu \subseteq \operatorname{Jac} X$ isogenous to the factors $A_\nu^{k_\nu}$ which may be constructed as identity component of the intersection of the kernels of all projections $\operatorname{Jac} X \to A_\mu$ onto other factors, $\mu \neq \nu$. Clearly, I_ν is G–invariant with $U_\nu \cong H^0(I_\nu, \Omega_{\bar{\mathbb{Q}}})$ and all the factors, the isogenies and the projections can be defined over $\bar{\mathbb{Q}}$.

As a consequence, we may suppose without loss of generality that in the decomposition of the representation Φ into irreducible factors, any factor is contained in some U_ν whence we can use the decomposition of $\operatorname{Jac} X$ into isotypic components for the decomposition of Φ as follows.

Lemma 15 *Identify the subspace* $U_\nu \subseteq H^0(\operatorname{Jac} X, \Omega_{\bar{\mathbb{Q}}})$ *with the space of differentials* $H^0(A_\nu^{k_\nu}, \Omega_{\bar{\mathbb{Q}}})$ *on* $A_\nu^{k_\nu}$. *With the notation of Lemma 14, denote* $D_\nu := \operatorname{End}_0 A_\nu$ *and use further the isomorphisms*

$$H^0(A_\nu, \Omega_{\bar{\mathbb{Q}}})^{k_\nu} \;\cong\; H^0(A_\nu^{k_\nu}, \Omega_{\bar{\mathbb{Q}}}) \;\cong\; U_\nu$$

as identifications. Decompose $H^0(A_\nu, \Omega_{\bar{\mathbb{Q}}})$ *into a direct sum of* D_ν*–invariant and* D_ν*–irreducible subspaces* $U_{\nu\mu}$, $\mu = 1, \ldots, m_\nu$. *Then we may identify* U_ν *with a direct sum*

$$\sum_{\mu=1}^{m_\nu} (U_{\nu\mu})^{k_\nu} \ .$$

The components $(U_{\nu\mu})^{k_\nu}$ *are* $\Phi(G)$*–invariant, i.e. invariant subspaces for the representation* Φ . *In particular, there is a decomposition of* $H^0(\operatorname{Jac} X, \Omega_{\bar{\mathbb{Q}}})$ *into* Φ*–irreducible subspaces* U *such that any* U *is contained in some* $(U_{\nu\mu})^{k_\nu}$.

The *proof* of the isomorphisms is obvious, so it remains to prove the G–invariance of the spaces $(U_{\nu\mu})^{k_\nu}$. This can be done by using the obvious morphism of the group algebra $\mathbb{Z}[G]$ into $\operatorname{End}_0 \operatorname{Jac} X$. The representation Φ of G restricted to U_ν is then embedded into the complex representation of $\operatorname{End}_0 A_\nu^{k_\nu} \cong M_{k_\nu}(D_\nu)$ and has at least the same decomposition into invariant subspaces. A second possibility is given by the transcendence tools explained in Section 3 and the substitution rule

$$\int_{\alpha(\gamma)} \omega \;=\; \int_\gamma \omega \circ \alpha$$

for all $\omega \in (U_{\nu\mu})^{k_\nu}$, all $\alpha \in G$, and all $\gamma \in H_1(\mathrm{Jac}\, X, \mathbb{Z})$. They indicate that the periods of all differentials $\omega \circ \alpha \in \Phi(G)(U_{\nu\mu})^{k_\nu}$ already lie in the $\bar{\mathbb{Q}}$–vector space generated by

$$\{ \int_\gamma \omega \;\mid\; \gamma \in H_1(A_\nu^{k_\nu}, \mathbb{Z}) \,,\, \omega \in U_{\nu\mu}^{k_\nu} \} \,,$$

hence in the $\bar{\mathbb{Q}}$–vector space $V_{U_{\nu\mu}}$ generated by

$$\{ \int_\gamma \omega \;\mid\; \gamma \in H_1(A_\nu, \mathbb{Z}) \,,\, \omega \in U_{\nu\mu} \}$$

and of dimension

$$\frac{2 \dim_{\bar{\mathbb{Q}}} U_{\nu\mu} \dim_{\mathbb{C}} A_\nu}{\dim_{\mathbb{Q}} D_\nu} \;=:\; d_{\nu\mu} \,,$$

see Lemma 6. Since

$$H^0(A_\nu, \Omega_{\bar{\mathbb{Q}}}) \;=\; \sum_\mu U_{\nu\mu} \quad \text{with} \quad \sum_\mu \dim_{\bar{\mathbb{Q}}} U_{\nu\mu} \;=\; \dim_{\mathbb{C}} A_\nu$$

is a direct sum and, again by Lemma 6,

$$\dim_{\bar{\mathbb{Q}}} V_{A_\nu} \;=\; \frac{2\,(\dim_{\mathbb{C}} A_\nu)^2}{\dim_{\mathbb{Q}} D_\nu} \;=\; \sum_\mu d_{\nu\mu} \,,$$

the corresponding decomposition of the space of periods

$$V_{A_\nu} \;=\; \sum_\mu V_{U_{\nu\mu}}$$

is also a direct sum. Therefore, a bijection exists between the set of subspaces $U_{\nu\mu}$ and the set of period spaces $V_{U_{\nu\mu}}$. In particular, since we know the corresponding property for their periods, no

$$\omega \circ \alpha \,, \quad \omega \in (U_{\nu\mu})^{k_\nu} \,,\ \alpha \in G \,,$$

can have a component outside $(U_{\nu\mu})^{k_\nu}$.

Remark 4. It should be emphasized that the decomposition of $H^0(A_\nu, \Omega_{\bar{\mathbb{Q}}})$ is in general not unique since D_ν–irreducible subspaces may occur with some multiplicity, say $v_{\nu\lambda}$ where λ runs over the isomorphism classes of complex representations of D_ν . The first proof given above can be easily extended to see that any irreducible representation of G corresponding to a G–invariant subspace $U \subseteq (U_{\nu\mu})^{k_\nu}$ occurs also at least with multiplicity $v_{\nu\lambda}$ in Φ .

We will see that the dimensions $d_{\nu\mu}$ are of particular interest. First, Lemma 10 gives lower bounds for the degrees of the irreducible subrepresentations of Φ :

$$\dim U \;\geq\; \dim V(U) \;\geq\; \dim V_U$$

Second, this last dimension equals the dimension $d_{\nu\mu}$ of the period vector space $V_{U_{\nu\mu}}$ by the following reason: again by the substitution rule $\int_{\alpha(\gamma)} \omega_0 = \int_\gamma \omega_0 \circ \alpha$, but this time with $\omega_0 \in U_{\nu\mu}$, $\gamma \in H_1(A_\nu, \mathbb{Z})$ and $\alpha \in \mathrm{End}\, A_\nu$, any $\bar{\mathbb{Q}}$–vector space generated by all periods of some differential ω_0 coincides with the $\bar{\mathbb{Q}}$–vector space generated by all periods of all D_ν–images of ω_0 , i.e. with $V_{U_{\nu\mu}}$ if $\omega_0 \neq 0$. Since for the evaluation of the periods we can replace any nonzero $\omega \in U$ by a suitable nonzero $\omega_0 \in U_{\nu\mu}$ we get finally

Lemma 16 *Let U be a Φ–irreducible subspace of $(U_{\nu\mu})^{k_\nu}$. The period vector spaces V_U and $V_{U_{\nu\mu}}$ coincide and satisfy*

$$d_{\nu\mu} \leq \dim U \, .$$

The dimensions $d_{\nu\mu}$ can be made more explicit using classical facts about the classification of simple abelian varieties and their endomorphism algebras ([A], [Sh]). Omitting the index we denote the simple abelian variety by A and its endomorphism algebra by D with center $\mathbb{K}$. The polarization defines an involution ρ of D . The numbers in $\mathbb{K}$ fixed by ρ form a totally real field $\mathbb{F}$. Then, D falls under the following four types:

1. $D = \mathbb{F}$
2. D is a totally indefinite quaternion algebra over $\mathbb{F}$
3. D is a totally definite quaternion algebra over $\mathbb{F}$
4. $\mathbb{K}$ is a totally imaginary quadratic extension of $\mathbb{F}$ and D is a central simple algebra over $\mathbb{K}$ with

$$[D : \mathbb{K}] = q^2 \quad , \quad q \in \mathbb{N} \, .$$

If we define $q = 1$ for type 1 and $q = 2$ for the types 2 and 3, this number q is the degree of D over a maximal commutative subfield L of D , and the action of D on $H^0(A, \Omega_{\bar{\mathbb{Q}}})$ splits this space into irreducible subspaces of dimension q (generate these subspaces by L–eigendifferentials and their D–images or use results of [Sh]). For types 1 to 3, all irreducible representations of D occur with the same multiplicity $v = (\dim_{\mathbb{C}} A)/(q[\mathbb{F} : \mathbb{Q}])$ whence the number $v_{\nu\lambda}$ in Remark 4 does in fact not depend on λ with the possible exception of type 4 factors.— Collecting the essentials of the last lemmas we can summarize our results as

Theorem 8 *Let $X = \Gamma\backslash\mathcal{H}$ be a Riemann surface with many automorphisms of genus $g > 1$, Γ be a normal torsion free subgroup of a Fuchsian triangle group Δ , let $G = \Delta/\Gamma \subseteq \mathrm{Aut}\, X$ be the monodromy group of its Belyi function $X \to \Delta\backslash\mathcal{H}$, let Φ be the canonical representation of G on the space of differentials of the first kind on X .*
For any irreducible subspace U of Φ there is a simple factor A_ν of $\mathrm{Jac}\, X$ occurring with multiplicity k_ν and with endomorphism algebra D_ν of dimension q_ν^2 over its center such that

$$\frac{2\, q_\nu \dim_{\mathbb{C}} A_\nu}{\dim_{\mathbb{Q}} D_\nu} \leq \dim_{\bar{\mathbb{Q}}} U \leq k_\nu\, q_\nu \, .$$

In particular, the endomorphism algebra satisfies

$$2 \dim_{\mathbb{C}} A_\nu \ \leq \ k_\nu \dim_{\mathbb{Q}} D_\nu \ .$$

Remark 5. $\dim_{\mathbb{Q}} D_\nu$ is always a divisor of $2\dim_{\mathbb{C}} A_\nu$, and the left hand side of the first inequality equals 1 if and only if A_ν has complex multiplication whence Theorem 8 contains Theorem 4 as a special case.

Remark 6. The decomposition of the representation $\Phi+\bar\Phi$ depends on $\Theta : \Delta \to G$ only: recent work of M. Streit ([St1], Prop. 1 and 2, [St2]) gives effective group–theoretic algorithms for this decomposition giving in particular all dimensions $\dim_{\bar{\mathbb{Q}}} U$ in question.

Remark 7. There is a further restriction on the possible representations of G on $H^0(A_\nu^{k_\nu}, \Omega_{\bar{\mathbb{Q}}})$ coming from the fact that on $\mathbb{Q}\otimes H_1(A_\nu^{k_\nu}, \mathbb{Z})$ the representation $\Phi+\bar\Phi$ acts as a rational one.

Remark 8. Theorem 8 gives an instructive second look on the examples of the last section. With the representation matrices found in [KK] it is easy to see that the canonical representation is irreducible e.g. for the curves (10), (24) and (25) which give Jacobians not of CM type. Since the curves are covers of elliptic curves it is now clear that their Jacobians are isogenous to powers of these elliptic curves. The (finite!) list of all curves with many automorphisms and irreducible representation Φ has been established recently by Breuer [Br]. Forthcoming work of Streit will show that the Schur indicator of Φ provides a sufficient criterion for their Jacobian being of CM type [St3]. As an example how Theorem 8 works we mention the following consequence on curves with irreducible representation.

Theorem 9 *Under the hypotheses of Theorem 8 let Φ be irreducible. Then* $\mathrm{Jac}\, X$ *is isogenous to a power* A^k *of a simple abelian variety* A *, and the endomorphism algebra* $\mathrm{End}_0 A$ *acts irreducibly on* $H^0(A,\Omega)$ *. Moreover,*

1. *either* $g = k$ *, i.e.* A *is an elliptic curve,*
2. *or* $g = 2k$ *,* $\dim A = 2$ *, and* $\mathrm{End}_0 A$ *is an indefinite quaternion algebra* $\mathbb{B}$ *over* $\mathbb{Q}$ *.*

Proof. The first sentence follows directly from Theorem 8 with $U = H^0(\mathrm{Jac}\, X, \Omega)$. Now, $\dim U = k \dim A$ implies $\dim A \leq q$ if the endomorphism algebra $D = \mathrm{End}_0 A$ has degree q^2 over its center $\mathbb{F}$ or $\mathbb{K}$, compare the list of possible endomorphism algebras above. For type 4 algebras, Remark 5 gives

$$[\mathbb{K}:\mathbb{Q}]\cdot q^2 \ = \ \dim_{\mathbb{Q}} D \ \mid \ 2\dim A \ \leq \ 2q$$

and therefore $q = 1$, $\dim A = 1$ as claimed. For the other possible types of D , the same reasoning gives $\mathbb{F} = \mathbb{Q}$. A classical result of Shimura ([Sh], Prop. 15) says that in this case endomorphism algebras of type 3 cannot occur, so we are left with the cases mentioned in the Theorem.

Remark 9. For the curve (16), all intermediate groups between the covering group and its normalizer $< 3,4,6 >$ have genus 0 and give therefore no hint on possible factors of the Jacobian.

According to [KK] the canonical representation splits into two non–equivalent irreducible representations of degree 2 . So Theorem 8 says that each simple factor of the Jacobian (of CM type by [Sr1] and Theorem 3, hence with $q_\nu = 1$) occurs at least twice. Schindler's period matrix shows that the period lattice of (16) is contained in $\mathbb{K}^4 \subset \mathbb{C}^4$ where $\mathbb{K}$ denotes the CM field $\mathbb{Q}(\sqrt{-3}, \sqrt{-6})$. But there is no simple abelian variety with CM by this field: by [Sch], abelian varieties with complex multiplication by this biquadratic field are always isogenous to squares of elliptic curves with CM by either $\mathbb{Q}(\sqrt{-3})$ or $\mathbb{Q}(\sqrt{-6})$. Therefore, the Jacobian is isogenous to a product of elliptic curves with complex multiplication by one or two of these imaginary quadratic fields.

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