TROPICAL BASES BY REGULAR PROJECTIONS

KERSTIN HEPT AND THORSTEN THEOBALD

ABSTRACT. We consider the tropical variety $\mathcal{T}(I)$ of a prime ideal I generated by the polynomials $f_1, \ldots, f_r$ and revisit the regular projection technique introduced by Bieri and Groves from a computational point of view. In particular, we show that I has a short tropical basis of cardinality at most $r + \operatorname{codim} I + 1$ at the price of increased degrees, and we provide a computational description of these bases.

1. Introduction

Given a field K endowed with a non-trivial real valuation $\operatorname{ord} : K \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, the *tropical variety* $\mathcal{T}(I)$ of an ideal $I \lhd K[x_1, \ldots, x_n]$ is defined as the topological closure of the set

$$\operatorname{ord} \mathcal{V}(I) \ = \ \{(\operatorname{ord}(z_1), \ldots, \operatorname{ord}(z_n)) : z \in \mathcal{V}(I)\} \ \subseteq \ \mathbb{R}^n \,, \tag{1}$$

where $\mathcal{V}(I)$ denotes the zero set of I in $(K^*)^n$. Tropical varieties have been the subject of intensive recent studies ([2, 4, 7, 8, 10]; see [9] for a general introduction.)

A basis $\mathcal{F} = \{f_1, \ldots, f_r\}$ of I is called a *tropical basis* of I if $\bigcap_{i=1}^r \mathcal{T}(f_i) = \mathcal{T}(I)$. Bogart, Jensen, Speyer, Sturmfels, and Thomas initiated the systematic computational investigation of tropical bases [2, 7], by providing both Gröbner-related techniques for computing tropical bases as well as by providing lower bounds on the size. They consider the field of Puiseux series $K = \mathbb{C}\{\{t\}\}$ with the natural valuation and concentrate on the "constant coefficient case", i.e., $I \lhd \mathbb{C}[x_1, \ldots, x_n]$. As a lower bound, they show that for $1 \le d \le n$ there is a d-dimensional linear ideal I in $\mathbb{C}[x_1, \ldots, x_n]$ such that any tropical basis of linear forms in I has size at least $\frac{1}{n-d+1}\binom{n}{d}$.

In this note we explain that by dropping the assumption on the degree of the polynomials there always exists a small tropical basis for a prime ideal I, thus contrasting that lower bound.

Theorem 1.1. *Let $I \lhd K[x_1, \ldots, x_n]$ be a prime ideal generated by the polynomials $f_1, \ldots, f_r$. Then there exist $g_0, \ldots, g_{n-\dim I} \in I$ with*

$$\mathcal{T}(I) \ = \ \bigcap_{i=0}^{n-\dim I} \mathcal{T}(g_i) \tag{2}$$

and thus $\mathcal{G} := \{f_1, \ldots, f_r, g_0, \ldots, g_{n-\dim I}\}$ is a tropical basis for I of cardinality $r + \operatorname{codim} I + 1$.

In particular, this also implies the universal (i.e., independent of $\dim I$) bound of $n+1$ polynomials in the representation (2).

The statement comes as a consequence of the regular projection technique introduced by Bieri and Groves [1]. The purpose of this note is to revisit this approach from the computational point of view, with the goal to provide an explicit and constructive description of the resulting tropical bases. Specifically, we apply tropical elimination on a particular class of ideals; for a general treatment of tropical elimination see the recent papers of Sturmfels, Tevelev, and Yu [11, 12].

Based on this construction, we characterize the Newton polytopes of the polynomials g_i in the tropical bases for the special case of ideals generated by two linear polynomials. The tradeoff between the cardinality and the degree of a tropical bases in the general case is subject to further study.

This paper is structured as follows. In Section 2 we introduce the relevant notation from tropical geometry and their relation to valuations. In Section 3 we provide the computational treatment of regular projections and prove Theorem 1.1. Section 4 provides some results on the characterization of the resulting Newton polytopes of the basis polynomials.

2. Tropical geometry

For a field K, a real valuation is a map $\mathrm{ord} : K \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ with $K \setminus \{0\} \to \mathbb{R}$ and $0 \mapsto \infty$ such that $\mathrm{ord}(ab) = \mathrm{ord}(a) + \mathrm{ord}(b)$ and $\mathrm{ord}(a+b) \geq \min\{\mathrm{ord}(a), \mathrm{ord}(b)\}$. Thus $\mathrm{ord} = -\log ||\cdot||$ for a non-archimedean norm $||\cdot||$ on K. Examples include $K = \mathbb{Q}$ with the p-adic valuation or the field $K = \mathbb{C}\{\{t\}\}$ of Puiseux series with the natural valuation. We can extend the valuation map to K^n via

$$\mathrm{ord} : K^n \; \to \; \bar{\mathbb{R}}^n, \quad (a_1, \ldots, a_n) \; \mapsto \; (\mathrm{ord}(a_1), \ldots, \mathrm{ord}(a_n)) \, .$$

We always assume that ord is non-trivial, i.e., $\mathrm{ord}(K \setminus \{0\}) \neq \{0\}$.

Let $f = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in $K[x_1, \ldots, x_n]$. The *tropicalization of* f is defined as

$$\mathrm{trop}(f) \; = \; \min_\alpha \{\mathrm{ord}(c_\alpha) + \alpha_1 x_1 + \cdots + \alpha_n x_n\} \, ,$$

and the *tropical hypersurface of* f is

$$\mathcal{T}(f) \; = \; \{w \in \mathbb{R}^n \, : \text{ the minimum in } \mathrm{trop}(f) \text{ is attained at least twice in } w\} \, .$$

For an ideal $I \lhd K[x_1, \ldots, x_n]$, the *tropical variety of* I can be defined either by

$$\mathcal{T}(I) \; = \; \bigcap_{f \in I} \mathcal{T}(f)$$

or equivalently by (1); see [4].

We shortly review the link between tropical geometry and classical valuation theory. If I is a prime ideal then we have the field extension

$$K(y_1, \ldots, y_n) \;=\; \mathrm{Quot}(K[x_1, \ldots, x_n]/I) \;=:\; L$$

of K. It is well known (see, e.g., [5]) that each valuation on K can be extended to a valuation of L. Let Δ_L^{ord} be defined by

$$\Delta_L^{\mathrm{ord}} \;=\; \{(w(y_1), \ldots, w(y_n)) \in \mathbb{R}^n \mid w : L \to \bar{\mathbb{R}} \text{ a valuation with } w|_K = \mathrm{ord}\}\,.$$

This subset of $\mathbb{R}^n$ coincides with the tropical variety of I,

$$\Delta_L^{\mathrm{ord}} \;=\; \mathcal{T}(I)$$

(see [4]). Bieri and Groves [1] showed that Δ_L^{ord} (and thus $\mathcal{T}(I)$ as well) is a pure polyhedral complex of dimension equal to the transcendence degree of L over K, and rationally defined over the value group $\mathrm{ord}(K^*)$ of ord.

3. Projections and the main theorem

Let $I \lhd K[x_1, \ldots, x_n]$ be an m-dimensional prime ideal. The main geometric idea is to consider $n - m + 1$ different (rational) projections $\pi_0, \ldots, \pi_{n-m} : \mathbb{R}^n \to \mathbb{R}^{m+1}$. If these projections are sufficiently generic (as specified below) then we obtain

$$\bigcap_{i=0}^{n-m} \pi_i^{-1}(\pi_i(\mathcal{T}(I))) \;=\; \mathcal{T}(I)\,,$$

and each of the sets $\pi_i^{-1}(\pi_i(\mathcal{T}(I)))$ is a tropical hypersurface.

First we consider the image of the tropical variety $\mathcal{T}(I)$ under a single (rational) projection

$$\begin{aligned} \pi \,:\, \mathbb{R}^n &\;\to\; \mathbb{R}^{m+1}\,, \\ x &\;\mapsto\; Ax \end{aligned}$$

with a regular rational matrix A whose rows are denoted by $a^{(1)}, \ldots, a^{(m+1)}$. Let $u^{(1)}, \ldots, u^{(l)} \in \mathbb{Q}^n$ with $l := n - (m+1)$ be a basis of the orthogonal complement of $\mathrm{span}\{a^{(1)}, \ldots, a^{(m+1)}\}$.

Set $R = K[x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_l]$, and define the ideal $J \lhd R$ by

$$J \;=\; \left\langle g \in R \,:\, g = f(x_1 \prod_{j=1}^{l} \lambda_j^{u_1^{(j)}}, \ldots, x_n \prod_{j=1}^{l} \lambda_j^{u_n^{(j)}}) \text{ for some } f \in I \right\rangle.$$

We show the following characterization of $\pi^{-1}(\pi(\mathcal{T}(I)))$ in terms of elimination.

Theorem 3.1. *Let $I \lhd K[x_1, \ldots, x_n]$ be an m-dimensional prime ideal and $\pi : \mathbb{R}^n \to \mathbb{R}^{m+1}$ be a rational projection. Then $\pi^{-1}(\pi(\mathcal{T}(I)))$ is a tropical variety with*

$$\pi^{-1}(\pi(\mathcal{T}(I))) \;=\; \mathcal{T}(J \cap K[x_1, \ldots, x_n])\,. \tag{3}$$

In order to prove Theorem 3.1, we first consider *algebraically regular* projections (as defined below). At the end of this section we also cover the remaining special cases.

We start with an auxiliary statement which holds for an arbitrary rational projection π.

Lemma 3.2. *For any* $w \in \mathcal{T}(J \cap K[x_1, \ldots, x_n])$ *and* $u \in \operatorname{span}\{u^{(1)}, \ldots, u^{(l)}\}$ *we have* $w + u \in \mathcal{T}(J \cap K[x_1, \ldots, x_n])$.

Proof. Let $u = \sum_{i=1}^{l} \mu_j u^{(j)}$ with $\mu_1, \ldots, \mu_l \in \mathbb{Q}$. The case of real μ_i then follows as well.

Let $w \in \mathcal{T}(J \cap K[x_1, \ldots, x_n])$. Since $\mathcal{T}(J \cap K[x_1, \ldots, x_n])$ is closed, we can assume without loss of generality that there exists $z \in \mathcal{V}(J \cap K[x_1, \ldots, x_n])$ with $\operatorname{ord} z = w$. Define $y = (y', y'') \in K^{n+l}$ by

$$y \;=\; (y', y'') \;=\; \left(z_1 t^{\sum_{j=1}^{l} \mu_j u_1^{(j)}}, \ldots, z_n t^{\sum_{j=1}^{l} \mu_j u_n^{(j)}}, t^{-\mu_1}, \ldots, t^{-\mu_l} \right) .$$

For any $f \in I$, the point y is a zero of the polynomial

$$f(x_1 \prod_{j=1}^{l} \lambda_j^{u_1^{(j)}}, \ldots, x_n \prod_{j=1}^{l} \lambda_j^{u_n^{(j)}}) \;\in\; R \,,$$

and thus $y \in \mathcal{V}(J)$. Hence, $y' \in \mathcal{V}(J \cap K[x_1, \ldots, x_n])$. Moreover,

$$\operatorname{ord} y' \;=\; (w_1 + \sum_{j=1}^{l} \mu_j u_1^{(j)}, \ldots, w_n + \sum_{j=1}^{l} \mu_j u_n^{(j)}) \;=\; w + \sum_{j=1}^{l} \mu_j u^{(j)} \;=\; w + u \,,$$

which proves our claim. □

Lemma 3.3. *Let* $I \lhd K[x_1, \ldots, x_n]$ *be an ideal. Then* $J \cap K[x_1, \ldots, x_n] \subseteq I$.

Proof. Let $p = \sum_i h_i g_i$ be a polynomial in $J \cap K[x_1, \ldots, x_n]$ with

$$g_i \;=\; f_i(x_1 \prod_{j=1}^{l} \lambda_j^{u_1^{(j)}}, \ldots, x_n \prod_{j=1}^{l} \lambda_j^{u_n^{(j)}}) \;\in\; R \;\text{ and } f_i \in I \,.$$

Since p is independent of $\lambda_1, \ldots, \lambda_l$ we have

$$p \;=\; p|_{\lambda_1=1,\ldots,\lambda_l=1} \;=\; \sum_i h_i|_{\lambda_1=1,\ldots,\lambda_l=1} f_i \;\in\; I.$$

□

We call a projection *algebraically regular* for I if for each $i \in \{1, \ldots, l\}$ the elimination ideal $J \cap K[x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_i]$ has a finite basis $\mathcal{F}_i$ such that in every polynomial $f \in \mathcal{F}_i$ the coefficients of the powers of λ_i (when considering f as a polynomial in λ_i) are monomials in $x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_{i-1}$.

The following statement shows that the set of algebraically regular projections is dense in the set of all real projections $\pi : \mathbb{R}^n \to \mathbb{R}^{m+1}$.

Lemma 3.4. *The set of projections which are not algebraically regular is contained in a finite union of hyperplanes within the space all projections $\pi : \mathbb{R}^n \to \mathbb{R}^{m+1}$*

Proof. It suffices to show that for the choice of $u^{(l)}$, we just have to avoid a lower-dimensional subset of $\mathbb{R}^n \setminus \{0\}$. For $u^{(1)}, \ldots, u^{(l-1)}$ we can then argue inductively (however, an explicit description then becomes more involved). Assume that J is generated by the polynomials

$$f_j(x_1 \prod_{i=1}^{l} \lambda_i^{u_1^{(i)}}, \ldots, x_n \prod_{i=1}^{l} \lambda_i^{u_n^{(i)}}), \quad 1 \le j \le s,$$

where $f_1, \ldots, f_s \in I$. Let f_j be any of these polynomials. f_j is of the form

$$f_j \;=\; \sum_{\alpha \in \mathcal{A}_j} c_\alpha x^\alpha \lambda_1^{\sum \alpha_i u_i^{(1)}} \cdots \lambda_l^{\sum \alpha_i u_i^{(l)}}$$

with $\mathcal{A}_j \subset \mathbb{Z}^n$ finite. Thus all λ_l^k have monomial coefficients if

$$\sum \alpha_i u_i^{(l)} \;\neq\; \sum \beta_i u_i^{(l)}$$

for all $\alpha, \beta \in \mathcal{A}_j$ with $\alpha \neq \beta$. So we have to choose $u^{(l)}$ from the subset

$$\bigcap_j \{u \in \mathbb{R}^n \,:\, \sum \alpha_i u_i^{(l)} \neq \sum \beta_i u_i^{(l)} \text{ for all } \alpha, \beta \in \mathcal{A}_j \text{ with } \alpha \neq \beta\}.$$

Hence, the algebraically non-regular projections are contained in a finite number of hyperplanes. ☐

Theorem 3.5. *Let $I \lhd K[x_1, \ldots, x_n]$ be a prime ideal and $\pi : \mathbb{R}^n \to \mathbb{R}^{m+1}$ be an algebraically regular projection. Then $\pi^{-1}\pi(\mathcal{T}(I))$ is a tropical variety with*

$$\pi^{-1}\pi(\mathcal{T}(I)) \;=\; \mathcal{T}(J \cap K[x_1, \ldots, x_n]). \tag{4}$$

Proof. Let $w \in \pi^{-1}\pi(\mathcal{T}(I))$. Since the right hand set of (4) is closed, we can assume without loss of generality that there exists $z' \in \mathcal{V}(I)$ and $u \in \operatorname{span}\{u^{(1)}, \ldots, u^{(l)}\}$ with $\operatorname{ord} z' = w + u$. For any $f \in I$, the point

$$z \;:=\; (z', 1)$$

is a zero of the polynomial

$$f(x_1 \prod_{j=1}^{l} \lambda_j^{u_1^{(j)}}, \ldots, x_n \prod_{j=1}^{l} \lambda_j^{u_n^{(j)}}) \;\in\; R,$$

and thus $z \in \mathcal{V}(J)$. Hence, $z' \in \mathcal{V}(J \cap K[x_1, \ldots, x_n])$. By Lemma 3.2, $w \in \mathcal{T}(J \cap K[x_1, \ldots, x_n])$ as well.

Let now $w \in \mathcal{T}(J \cap K[x_1, \ldots, x_n])$. Again we can assume that there is a $z \in \mathcal{V}(J \cap K[x_1, \ldots, x_n] \subseteq (K^*)^n$ with $w = \operatorname{ord}(z)$. The projection is algebraically regular which means that the generators of the elimination ideals

$J \cap K[x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_i]$ have only monomials as coefficients with respect to λ_i. By the Extension Theorem (see, e.g., [3]), we can extend the root z inductively to a root $\tilde{z} \in \mathcal{V}(J)$ with the same first n entries. The definition of J says that

$$z' := (z_1 \tilde{z}_{n+1}^{u_1^{(1)}} \cdots \tilde{z}_{n+l}^{u_1^{(l)}}, \ldots, z_n \tilde{z}_{n+1}^{u_n^{(1)}} \cdots \tilde{z}_{n+l}^{u_n^{(l)}})$$

is a root of I. Then

$$\operatorname{ord}(z') = \operatorname{ord}(z) + \sum_{i=1}^{l} \operatorname{ord}(\tilde{z}_{n+i}) u^{(i)}$$

which means that $\operatorname{ord}(z) = w \in \pi^{-1}\pi(\mathcal{T}(I))$. □

This completes the proof of Theorem 3.1 for the case of algebraically regular projections.

In the following, we consider the notion of *geometric regularity.*

Definition 3.6. Let $\mathcal{C}$ be a polyhedral complex in $\mathbb{R}^n$. A projection $\pi : \mathbb{R}^n \to \mathbb{R}^{m+1}$ is called *geometrically regular* if the following two conditions hold.

(1) For any k-face σ of $\mathcal{C}$ we have $\dim(\pi(\sigma)) = k$, $0 \le k \le \dim \mathcal{C}$.
(2) If $\pi(\sigma) \subseteq \pi(\tau)$ then $\sigma \subseteq \tau$ for all $\sigma, \tau \in \mathcal{C}$.

These conditions ensure that we can recover the whole complex $\mathcal{C}$ from the projections.

Corollary 3.7. *In the situation of Theorem 3.1, if* $\dim \pi(\mathcal{T}(I)) = m$ *then* $\pi^{-1}\pi(\mathcal{T}(I))$ *is a tropical hypersurface.*

In particular, this holds when the projection is geometrically regular.

Proof. $\dim \pi^{-1}\pi(\mathcal{T}(I)) = \dim \pi(\mathcal{T}(I)) + \dim \ker \pi = m + (n-(m+1)) = n-1$. □

Let $I \lhd K[x_1, \ldots, x_n]$ be a prime ideal and $m = \dim I$. Then $\mathcal{T}(I)$ is a pure m-dimensional polyhedral complex. Bieri and Groves [1] used the following geometric technique (which actually was also used to prove that $\mathcal{T}(I)$ has this polyhedral property).

There exists a finite family $\mathcal{X} = \{\mathcal{X}_1, \ldots, \mathcal{X}_s\}$ of m-dimensional affine subspaces with $\mathcal{T}(I) \subseteq \bigcup_{i=1}^{s} \mathcal{X}_s$. By the finiteness of $\mathcal{X}$, for a sufficiently generic choice of $n-m+1$ geometrically regular projections $\pi_0, \ldots, \pi_{n-m}$ the set-theoretic intersection of the inverse projections exactly yields the original polyhedral complex:

Proposition 3.8 (Bieri, Groves [1])**.** *Let* $I \lhd K[x_1, \ldots, x_n]$ *be a prime ideal. Then there exist* $\operatorname{codim} I + 1$ *projections* $\pi_0, \ldots, \pi_{\operatorname{codim} I}$ *such that*

$$\mathcal{T}(I) = \bigcap_{i=0}^{\operatorname{codim} I} \pi_i^{-1}\pi_i(\mathcal{T}(I)).$$

By considering algebraically regular projections, and combining this proposition with Theorems 3.1 (so far only proved for algebraically regular projections) and 3.5 yields Theorem 1.1. Note that by Lemma 3.3 the generators g_i are actually contained in I.

Using this knowledge about the existence of some tropical basis, we can also provide the proof of Theorem 3.1 for arbitrary rational projections.

Theorem 3.9 (Tropical Extension Theorem). *Let $I \lhd K[x_0, \ldots, x_n]$ be an ideal and $I_1 = I \cap K[x_1, \ldots, x_n]$ be its first elimination ideal. For any $w \in \mathcal{T}(I_1)$ there exists a point $\tilde{w} = (w_0, \ldots, w_n) \in \mathbb{R}^{n+1}$ with $w_i = \tilde{w}_i$ for $1 \leq i \leq n$ and $\tilde{w} \in \mathcal{T}(I)$.*

Proof. First let $w \in \operatorname{ord}(\mathcal{V}(I_1))$, so that there exists $z \in \mathcal{V}(I_1)$ with $\operatorname{ord}(z) = w$. Let $\mathcal{G} = \{g_1, \ldots, g_s\}$ be a reduced Gröbner basis of I with respect to a lexicographical term order with $x_0 > x_i$, $1 \leq i \leq n$. I.e.,

$$g_i \;=\; h_i(x_1, \ldots, x_n) x_0^{\deg_{x_0} g_i} + \text{ terms of lower degree in } x_0 \,.$$

There are two cases to consider:

(1) $z \notin \mathcal{V}(h_1, \ldots, h_s)$. Then by the classical Extension Theorem there is a root $\tilde{z}$ of I which extends z, so $\operatorname{ord}(\tilde{z}) =: \tilde{w}$ extends w.

(2) $z \in \mathcal{V}(h_1, \ldots, h_s)$. Then $w = \operatorname{ord}(z) \in \mathcal{T}(h_1, \ldots, h_s)$. Let $\mathcal{P} = \{p_1, \ldots, p_t\}$ be a tropical basis of I.

Let p_j be any of these polynomials. p_j has the form

$$p_j \;=\; q_j(x_1, \ldots, x_n) x_0^{\deg_{x_0} p_j} + \text{ terms of lower degree in } x_0 \,.$$

Since $\mathcal{G}$ is a lexicographic Gröbner basis, we have $q_j(x_1, \ldots, x_n) =: \sum k_\alpha x^\alpha \in \langle h_1, \ldots, h_s \rangle$. Hence, the minimum

$$\min_\alpha \{\operatorname{ord}(k_\alpha) + \alpha_1 x_1 + \cdots + \alpha_n x_n\}$$

is attained twice at w. We can pick a sufficiently small value $w_0^{(j)} \in \mathbb{R}$ so that all terms $x_1^{m_1} \cdots x_n^{m_n} x_0^{m_0}$ of p_j with $m_0 < \deg_{x_0} p_j$ have a larger value $m_1 w_1 + \cdots + m_n w_n + m_0 w_0^{(j)}$. But then the minimum of all values of all terms of p_j is attained at least twice; it is

$$\min_\alpha \{\operatorname{ord}(k_\alpha) + \alpha_1 x_1 + \cdots + \alpha_n x_n\} + \deg_{x_0} p_j \cdot w_0^{(j)} \,.$$

So $(w_0^{(j)}, w_1, \ldots, w_n) \in \mathcal{T}(h_j)$.

By setting $w_0 = \min_j \{w_0^{(j)}\}$ and $\tilde{w} := (w_0, \ldots, w_n) \in \mathcal{T}(I)$, we obtain the desired extension of w.

Let now $w = \lim_{i \to \infty} w^{(i)}$ be in the closure of $\operatorname{ord}(\mathcal{V}(I_1))$. Then there exist $\tilde{w}^{(i)} \in \mathcal{T}(I)$ with $\tilde{w}_j^{(i)} = w_j^{(i)}$ for $1 \leq j \leq n$. Let $\mathcal{P} = \{p_1, \ldots, p_t\}$ be again a tropical basis of I. Then we can assume w.l.og. that the minimum of $\operatorname{trop}(p_k)$, $1 \leq k \leq t$ for $\tilde{w}^{(i)}$ is attained at the same terms. This gives us conditions for the $\tilde{w}_0^{(i)}$:

$$k^{(i)} \leq \tilde{w}_0^{(i)} \leq l^{(i)} \quad (\text{one of them can be } \pm\infty) \,.$$

These bounds vary continuously with $w^{(i)}$. So we can choose $\tilde{w}_0$ arbitrarily in $[\lim k^{(i)}, \lim l^{(i)}]$ (only one of the limites can be $\pm\infty$). □

4. The Newton polytopes for the linear case

As mentioned earlier, an ideal generated by linear forms may not have a small tropical basis if we restrict the basis to consist of linear forms. Using our results from Section 3, we can provide a short basis at the price of increased degrees. A natural question is to provide a good characterization for the Newton polytopes of the resulting basis polynomials. Here, we briefly discuss the special case of a prime ideal I generated by two linear polynomials $F = \sum_{i=1}^{n} a_i x_i + a_{n+1}$, $G = \sum_{i=1}^{n} b_i x_i + b_{n+1} \in K[x_1, \ldots, x_n]$.

In order to characterize the Newton polytope of the additional polynomials in the tropical basis, we consider the resultant of the polynomials f, g

$$\begin{aligned} f &= a_1 x_1 \lambda^{v_1} + \cdots + a_n x_n \lambda^{v_n} + a_{n+1}\,, \\ g &= b_1 x_1 \lambda^{v_1} + \cdots + b_n x_n \lambda^{v_n} + b_{n+1} \end{aligned}$$

in $K[x_1, \ldots, x_n, \lambda]$. Assume that the components v_i are distinct. Then w.l.o.g. we can assume $v_1 > v_2 > \cdots > v_n > v_{n+1} := 0$.

In order to apply the results of Gelfand, Kapranov and Zelevinsky [6] regarding the Newton polytope of the resultant, we consider the representation

$$\mathrm{Res}_\lambda(f,g) = \sum_{p,q} c_{p,q} a^p b^q x^{p+q}$$

with $p = (p_1, \ldots, p_{n+1}), q = (q_1, \ldots, q_{n+1}) \in \mathbb{Z}_+^{n+1}$. The Newton polytope is contained in the set $\mathcal{Q}_n \subset \mathbb{Z}^{2n+2}$ of nonnegative integer points (p,q) with

(1) $\sum_{i=1}^{n+1} p_i = \sum_{j=1}^{n+1} q_j = v_1\,,$

(2) $\sum_{i=1}^{n+1} v_i p_i + \sum_{j=1}^{n+1} v_j q_j = v_1^2\,,$

(3) $\sum_{\substack{1 \le k \le n \\ 0 \le v_1 - v_k \le i}} (i - v_1 + v_k) p_k + \sum_{\substack{1 \le l \le n \\ 0 \le v_1 - v_l \le j}} (j - v_1 + v_l) q_l \ \ge\ ij \quad (0 \le i, j \le v_1)\,.$

Hence, we can conclude:

Corollary 4.1. *The set of integer points in the Newton polytope* $\mathrm{New}(\mathrm{Res}_\lambda(f,g)) \subset \mathbb{Z}^n$ *is contained in the image of* $\mathcal{Q}_n$ *under the mapping*

$$(p_1, \ldots, p_{n+1}, q_1, \ldots, q_{n+1}) \ \mapsto\ (p_1 + q_1, \ldots, p_n + q_n)\,.$$

Example 4.2. Let $I = \langle 2x+y-4, x+2y+z-1 \rangle$ and $\mathrm{ord}(\cdot)$ be the 2-adic valuation (see Figure 4 for a figure of $\mathcal{T}(I)$). Actually, the first projection can be chosen arbitrarily (even geometrically non-regular). We choose a projection π_1 whose kernel is generated by $(0,0,1)$. Then the tropical hypersurface $\pi_1^{-1}\pi_1(\mathcal{T}(I))$ satisfies $\pi_1^{-1}\pi_1(\mathcal{T}(I)) = \mathcal{T}(2x+y-4)$, and the Newton polytope of that polynomial is a triangle (so the projection is geometrically non-regular). By choosing π_2 and π_3 with kernels generated by $(1, 2, 0)$ and $(1, 0, 1)$, respectively, we obtain the polynomials $6x^2 + 6x^2z + 49y + 14yz + yz^2$ and $3xy + 2x - yz + 4z$. Both Newton polytopes are quadrangles.

Adding these three nonlinear polynomials to the basis of I yields a tropical basis.

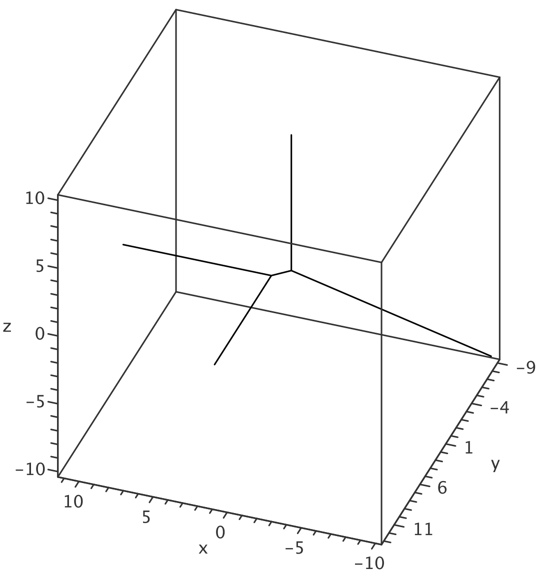

FIGURE 1. Tropical line $\mathcal{T}(I)$ in 3-space

References

1. R. Bieri and J.R.J. Groves, *The geometry of the set of characters induced by valuations*, J. Reine Angew. Math. **347** (1984), 168–195.
2. T. Bogart, A.N. Jensen, D. Speyer, B. Sturmfels, and R.R. Thomas, *Computing tropical varieties*, J. Symb. Comp. **42** (2007), no. 1-2, 54–73.
3. D. Cox, J. Little, and D. O'Shea, *Ideals, varieties, and algorithms: An introduction to computational algebraic geometry and commutative algebra*, 3rd ed., Springer-Verlag, New York, 2005.
4. M. Einsiedler, M.M. Kapranov, and D. Lind, *Non-archimedean amoebas and tropical varieties*, J. Reine Angew. Math. **601** (2006), 139–157.
5. O. Endler, *Valuation theory*, Universitext, Springer-Verlag, New York, 1972.
6. I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, *Newton polytopes of the classical resultant and discriminant*, Adv. Math. **84** (1990), 237–254.
7. A.N. Jensen, *Algorithmic aspects of Groebner fans and tropical varieties*, Ph.D. thesis, University of Aarhus, 2007.
8. A.N. Jensen, H. Markwig, and T. Markwig, *An algorithm for lifting points in a tropical variety*, Preprint, arXiv:0705.2441, 2007.

9. J. Richter-Gebert, B. Sturmfels, and T. Theobald, *First steps in tropical geometry*, Idempotent Mathematics and Mathematical Physics, Contemp. Math. **377** (2005), 289–317.
10. D. Speyer and B. Sturmfels, *The tropical Grassmannian*, Adv. Geom. **4** (2004), 389–411.
11. B. Sturmfels and E. Tevelev, *Elimination theory for tropical varieties*, Preprint, arXiv:0704.3471, 2007.
12. B. Sturmfels and J. Yu, *Tropical implicitization and mixed fiber polytopes*, Preprint, arXiv:0706.0564, 2007.

FB 12 – Institut für Mathematik, J.W. Goethe-Universität, Postfach 111932, D-60054 Frankfurt am Main, Germany

E-mail address: {hept,theobald}@math.uni-frankfurt.de