

ON THE FRONTIERS OF POLYNOMIAL COMPUTATIONS IN TROPICAL GEOMETRY

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ABSTRACT. We study some basic algorithmic problems concerning the intersection of tropical hypersurfaces in general dimension: deciding whether this intersection is non-empty, whether it is a tropical variety, and whether it is connected, as well as counting the number of connected components. We characterize the borderline between tractable and hard computations by proving \mathcal{NP} -hardness and $\#\mathcal{P}$ -hardness results under various strong restrictions of the input data, as well as providing polynomial time algorithms for various other restrictions.

1. INTRODUCTION

Geometry over the tropical semiring $(R, \oplus, \odot) := (\mathbb{R}, \min, +)$ has received much attention in the last years (see the surveys [15, 20, 25] and the references therein) with applications in counting curves [14], studying phylogenetic trees [16], and the analysis of amoebas of algebraic varieties [15]. From the viewpoint of polynomial equations, the modern birth of tropical geometry originates in the book [26] which pinpoints the central role of tropical geometry as a link between algebraic geometry, symbolic computation, and discrete geometry, thus providing computationally-accessible methods for studying algebraic-geometric problems. Indeed, one of the early roots of the developments in tropical geometry can be seen in the polyhedral homotopy methods by Huber and Sturmfels [9], providing a state-of-the-art technique for numerically solving systems of polynomial equations based on a deformation to a (discrete) tropical problem.

Some major algorithmic results in tropical geometry are based on Gröbner basis computations and thus may become intractable already for small dimensions [3]. On the positive side, there also exist some algorithmic problems (such as computing the tropical determinant) which can be efficiently solved using techniques from linear programming, polyhedral computation and combinatorial optimization (see, e.g., [10, 20]). For many tropical problems, their computational complexity has not been clarified yet.

In this paper, we make a first step towards systematically studying the frontiers of polynomial time computations in tropical geometry. For this, we consider three natural algorithmic problems concerning the intersection of tropical hypersurfaces, so-called tropical prevarieties. The algorithmic problems are to decide whether this intersection

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P is nonempty (TROPICAL INTERSECTION), whether P is a tropical variety (TROPICAL CONSISTENCY), and whether P is connected (TROPICAL CONNECTIVITY).

Our results refer to the standard Turing machine model, and we mainly aim at characterizing the borderline between tractable (in the sense of polynomial time solvable) and hard (in the sense of \mathcal{NP} -hard) computations. Our main results can roughly be stated as follows. If the number of hypersurfaces is part of the input then the three problems become \mathcal{NP} -hard or co- \mathcal{NP} -hard, and this hardness persists even under various restrictions to the input data. As a particular example, already for quadratic input polynomials it is co- \mathcal{NP} -hard to decide whether a tropical prevariety is a tropical variety. Hence, efficient algorithms cannot be expected in this setting. We contrast these hardness results by polynomial time algorithms for a fixed number of tropical hypersurfaces. For a precise statement of the results see Theorems 3.1–3.3.

The paper is structured as follows. In Section 2 we introduce the relevant notation from tropical geometry and computational complexity. In Section 3 we formally state our main results, and Section 4 contains the proofs of these theorems. We close the paper with a short discussion of related computational aspects on amoebas.

2. PRELIMINARIES

2.1. Tropical geometry. One of the original motivations for tropical varieties was a combinatorial approach to certain problems from enumerative geometry suggested by Kontsevich, and that program has been realized by Mikhalkin [14]. Tropical varieties are also related to the observation that algebraic varieties have very simple behavior at infinity when plotted on “log paper” [2, 28]. While these roots come from algebraic geometry and valuation theory, tropical varieties are profitably approached via polyhedral combinatorics.

Tropical hypersurfaces can be defined in a combinatorial and in an algebraic way (for general background we refer to [15], [20], [26, Chapter 9]). For the combinatorial definition, let $(\mathbb{R}, \oplus, \odot)$ denote the *tropical semiring*, where

$$x \oplus y = \min\{x, y\} \quad \text{and} \quad x \odot y = x + y.$$

Sometimes the underlying set \mathbb{R} of real numbers is augmented by ∞ .

A *tropical monomial* is an expression of the form $c \odot x^\alpha = c \odot x_1^{\alpha_1} \odot \cdots \odot x_n^{\alpha_n}$ where the powers of the variables are computed tropically as well (e.g., $x_1^3 = x_1 \odot x_1 \odot x_1$). This tropical monomial represents the classical linear function

$$\mathbb{R}^n \rightarrow \mathbb{R}, \quad (x_1, \dots, x_n) \mapsto \alpha_1 x_1 + \cdots + \alpha_n x_n + c.$$

A *tropical polynomial* is a finite tropical sum of tropical monomials and thus represents the (pointwise) minimum function of linear functions. At each given point $x \in \mathbb{R}^n$ the minimum is either attained at a single linear function or at more than one of the linear functions (“at least twice”). The *tropical hypersurface* $\mathcal{T}(f)$ of a tropical polynomial f is defined by

$$\mathcal{T}(f) = \{x \in \mathbb{R}^n \ : \ \text{the minimum of the tropical monomials of } f \\ \text{is attained at least twice at } x\}.$$

Rather than simply intersecting tropical hypersurfaces, the definition of tropical varieties of arbitrary codimension involves a valuation theoretic construction (Section 2.2 explains this subtlety.) Let $K = \overline{\mathbb{C}(t)}$ denote the algebraically closed field of Puiseux series, i.e., series of the form

$$p(t) = c_1 t^{q_1} + c_2 t^{q_2} + c_3 t^{q_3} + \dots$$

with $c_i \in \mathbb{C} \setminus \{0\}$ and rational $q_1 < q_2 < \dots$ with common denominator (see, e.g., [1]). The *order* $\text{ord } p(t)$ is the exponent of the lowest-order term of $p(t)$. The order of an n -tuple of Puiseux series is the n -tuple of their orders. This gives a map

$$(2.1) \quad \text{ord} : (K^*)^n \rightarrow \mathbb{Q}^n \subset \mathbb{R}^n,$$

where $K^* = K \setminus \{0\}$.

We are extending \mathcal{T} to allow also ideals in the polynomial ring $K[x_1, \dots, x_n]$ as argument. Let I be an ideal in $K[x_1, \dots, x_n]$, and consider its affine variety $V(I) \subset (K^*)^n$ over K . The image of $V(I)$ under the map (2.1) is a subset of \mathbb{Q}^n . The *tropical variety* $\mathcal{T}(I)$ is defined as the topological closure of this image, $\mathcal{T}(I) = \overline{\text{ord } V(I)}$. It is well-known that for principal ideals $I = \langle g \rangle$ the two definitions of tropical varieties coincide (see [11] or, e.g., [20, Lemma 3.2]):

Proposition 2.1. *If f is a tropical polynomial in x_1, \dots, x_n then there exists a polynomial $g \in K[x_1, \dots, x_n]$ such that $\mathcal{T}(f) = \mathcal{T}(\langle g \rangle)$, and vice versa.*

For a polynomial $f = \sum_{\alpha \in \mathcal{A}} c_\alpha(t) x^\alpha \in K[x_1, \dots, x_n]$ with a finite support set $\mathcal{A} \subset \mathbb{N}_0^n$ and $c_\alpha(t) \neq 0$ for all $\alpha \in \mathcal{A}$, the *tropicalization* of f is defined by

$$\text{trop } f = \bigoplus_{\alpha \in \mathcal{A}} \text{ord}(c_\alpha(t)) \odot x^\alpha,$$

where \bigoplus denotes a tropical summation. Whenever there is no possibility of confusion we also write \cdot instead of \odot .

For every tropical variety $\mathcal{T}(I)$ there exists a finite subset $\mathcal{B} \subset I$ such that $\mathcal{T}(I) = \bigcap_{f \in \mathcal{B}} \mathcal{T}(f)$. (However, we remark that Corollary 2.3 in [24], which claims that any universal Gröbner basis of I satisfies this condition, is not correct. See [20].)

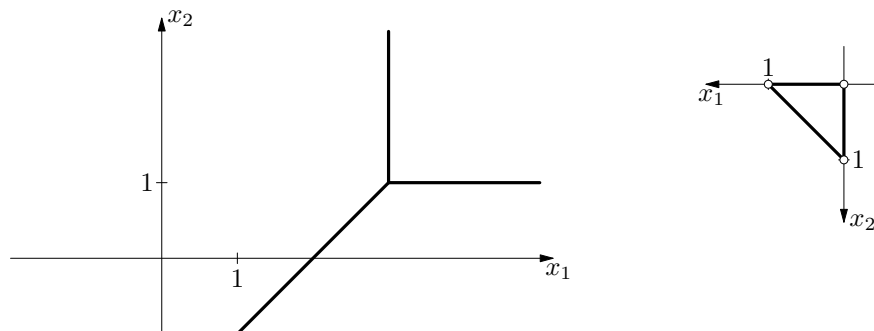


FIGURE 1. The tropical variety of a linear polynomial f in two variables and the Newton polygon of f .

2.2. The geometry of tropical hypersurfaces. Let $\mathcal{A} \subset \mathbb{N}_0^n$ be finite and $f(x_1, \dots, x_n) = \bigoplus_{\alpha \in \mathcal{A}} c_\alpha \cdot x^\alpha$ be a tropical polynomial with $c_\alpha \in \mathbb{R}$ for all $\alpha \in \mathcal{A}$. Then $\mathcal{T}(f)$ is a polyhedral complex in \mathbb{R}^n which is geometrically dual to the following regular subdivision of the Newton polytope $\text{New}(f)$ of f . Let \hat{P} be the convex hull $\text{conv}\{(\alpha, c_\alpha) \in \mathbb{R}^{n+1} : \alpha \in \mathcal{A}\}$. Then the lower faces of \hat{P} project bijectively onto $\text{conv } \mathcal{A}$ under deletion of the last coordinate, thus defining a subdivision of \mathcal{A} . Such subdivisions are called *regular* or *coherent* (see, e.g., [13]). We say that a tropical polynomial is *of degree at most d* if every term has (total) degree at most d . See Figure 1 for an example of a tropical line (i.e., the tropical variety of a linear polynomial in two variables) and Figure 2 for an example of a tropical cubic curve, as well as their dual subdivisions (whose coordinate axes are directed to the left and to the bottom to visualize the duality).

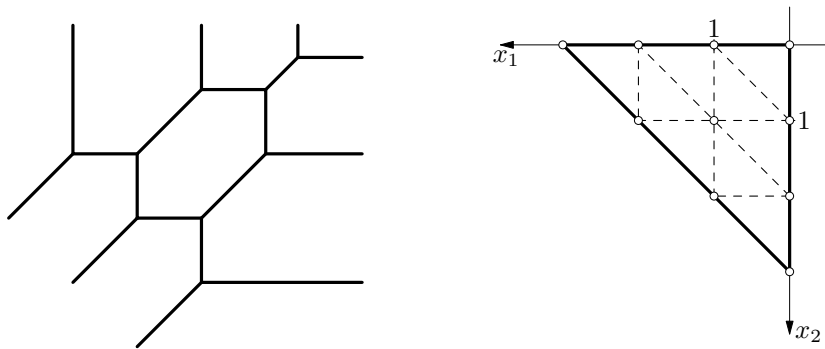


FIGURE 2. An example of a tropical cubic curve $\mathcal{T}(f)$ and the dual subdivision of the Newton polygon of f .

Following the notation in [20], a *tropical prevariety* is the intersection of tropical hypersurfaces. If f_1, \dots, f_m are linear polynomials then the tropical prevariety $P = \bigcap_{i=1}^m \mathcal{T}(f_i)$ is called *linear*. If additionally P is a tropical variety, then it is called a *linear tropical variety*. In dimension 2, a linear tropical variety is either a translate of the left-hand set in Figure 1, a classical line (in the x_1 -, x_2 -, or the main diagonal direction), a single point, or the empty set. A tropical prevariety in \mathbb{R}^2 can also be a one-sided infinite ray. Understanding the geometry and combinatorics of tropical prevarieties or varieties in general dimension is still a widely open problem. Even for the case of linear tropical varieties, the maximum number of bounded i -dimensional faces of such polyhedral complexes is unknown. The recent *f -vector conjecture* in [23] conjectures that (in our affine setting) the number of bounded i -dimensional faces of a k -dimensional linear tropical variety in \mathbb{R}^n is at most $\binom{n-2i+1}{k-i+1} \binom{n-i}{i-1}$ and that this bound is tight.

With respect to our investigations on the consistency problem, we remark that there are linear tropical spaces of dimension $n - 2$ which are not complete intersections, i.e., which are not the intersection of two tropical hypersurfaces (see [24, Proposition 6.3]).

2.3. Model of computation. Our model of computation is the binary Turing machine: all relevant data are presented by certain rational numbers, and the size of the input is

defined as the length of the binary encoding of the input data. A rational number is specified as the concatenation of the numerator a and the denominator b , and we may assume without loss of generality that a and b are relatively prime. Polynomials of degree d are specified by the binary encoding of all $\binom{n+d}{n}$ coefficients (even if a coefficient is zero); this encoding is sometimes referred as the *dense* encoding. For general background on algorithms and complexity theory we refer to [6, 17], and in particular for complexity aspects of geometric problems to [8].

In the realm of the complexity classes \mathcal{P} and \mathcal{NP} , complexity theory usually deals with decision problems: those whose answers are YES or NO. The class \mathcal{P} denotes the set of all decision problems which can be solved in polynomial time in the input size. The class \mathcal{NP} (nondeterministic polynomial time) denotes the class of all problems such that every YES-instance has a short (i.e. polynomial-size) certificate that can be verified in polynomial time. Recall that a problem is called *co- \mathcal{NP} -hard* if its complement is \mathcal{NP} -hard, where the complement of a problem is defined by switching the answers YES and NO for all inputs.

In this paper, we also deal with counting problems, which refer to problems whose answer is a bit string encoding an integer. A counting problem Π is in the class $\#\mathcal{P}$ if there is a decision problem $\Pi' \in \mathcal{NP}$ such that, for all inputs I , the output of Π is exactly the number of accepting solutions to Π' on input I . A counting problem Π is *$\#\mathcal{P}$ -hard* if every problem in $\#\mathcal{P}$ can be reduced in polynomial time to Π , i.e., if for every problem $\Pi' \in \#\mathcal{P}$ there is a polynomial-time computable function f such that for any input I to Π'

- (1) $f(I)$ is a valid input to Π ,
- (2) the output of Π' on input I is exactly the output of Π on input $f(I)$.

3. STATEMENT OF PROBLEMS AND MAIN RESULTS

We consider three basic problems on the intersection of tropical hypersurfaces. Let $\mathbb{Q}[x_1, \dots, x_n]^\oplus$ denote the set of tropical polynomials with rational coefficients in n variables. Given n, m, d_1, \dots, d_m and a set of tropical polynomials $f_1, \dots, f_m \in \mathbb{Q}[x_1, \dots, x_n]^\oplus$ of degrees at most d_1, \dots, d_m , respectively, the first problem asks whether the tropical prevariety $\bigcap_{i=1}^m \mathcal{T}(f_i)$ is nonempty. For the complexity results it is quite crucial which information is part of the input of the problem. In particular, note that in the formal definitions of the three problems the dimension and the number of hypersurfaces is part of the input.

Problem TROPICAL INTERSECTION:

Instance: n, m, d_1, \dots, d_m , polynomials $f_1, \dots, f_m \in \mathbb{Q}[x_1, \dots, x_n]^\oplus$ of degrees at most d_1, \dots, d_m .

Question: Decide whether there exists a point in $\bigcap_{i=1}^m \mathcal{T}(f_i)$.

The next problem asks whether an intersection of tropical hypersurfaces (i.e., a prevariety) is a tropical variety.

Problem TROPICAL CONSISTENCY:

Instance: n, m, d_1, \dots, d_m , polynomials $f_1, \dots, f_m \in \mathbb{Q}[x_1, \dots, x_n]^\oplus$ of degrees at most d_1, \dots, d_m .

Question: Decide whether $\bigcap_{i=1}^m \mathcal{T}(f_i)$ is a tropical variety.

We also consider the variant **TROPICAL m -CONSISTENCY** which asks whether $\bigcap_{i=1}^m \mathcal{T}(f_i)$ is a tropical variety of codimension m . The third problem asks for topological connectivity of the set $\bigcap_{i=1}^m \mathcal{T}(f_i)$.

Problem TROPICAL CONNECTIVITY:

Instance: n, m, d_1, \dots, d_m , polynomials $f_1, \dots, f_m \in \mathbb{Q}[x_1, \dots, x_n]^\oplus$ of degrees at most d_1, \dots, d_m with $\bigcap_{i=1}^m \mathcal{T}(f_i) \neq \emptyset$.

Question: Decide whether $\bigcap_{i=1}^m \mathcal{T}(f_i)$ is connected.

Besides these decision problems, we consider the counting problem **#CONNECTED COMPONENTS** whose input is the same one as for **TROPICAL INTERSECTION** and whose task is to determine the number of connected components of $\bigcap_{i=1}^m \mathcal{T}(f_i)$.

Our main results can be stated as follows.

Theorem 3.1. *The problem TROPICAL INTERSECTION is \mathcal{NP} -complete, and the problems TROPICAL CONSISTENCY and TROPICAL CONNECTIVITY are co- \mathcal{NP} -hard. For TROPICAL INTERSECTION and TROPICAL CONNECTIVITY these hardness results persist if the instances are restricted to those where $\bigcap_{i=1}^m \mathcal{T}(f_i)$ is a tropical variety.*

Moreover, for TROPICAL INTERSECTION and TROPICAL CONSISTENCY the hardness persists if all polynomials are restricted to be of degree at most 2. For TROPICAL CONNECTIVITY, the hardness persists if all polynomials are restricted to be of degree at most 3.

These hardness results are contrasted by the following positive algorithmic results for restricted input classes.

Theorem 3.2. *(i) If the number m of tropical hypersurfaces is a fixed constant, then TROPICAL INTERSECTION can be solved in polynomial time.*

(ii) For fixed m and if all input polynomials are restricted to be linear polynomials then the problem TROPICAL m -CONSISTENCY can be solved in polynomial time.

(iii) If the number m of tropical hypersurfaces is a fixed constant, then TROPICAL CONNECTIVITY can be solved in polynomial time. Moreover, any linear tropical prevariety is connected; hence, if all polynomials are restricted to be linear polynomials, the output of TROPICAL CONNECTIVITY is always YES.

Finally, we show $\#\mathcal{P}$ -hardness of counting the number of solutions.

Theorem 3.3. *#CONNECTED COMPONENTS is $\#\mathcal{P}$ -hard. This statement persists if all polynomials are restricted to be of degree at most 2.*

Remark 3.4. Obviously, TROPICAL INTERSECTION (and similarly, TROPICAL CONNECTIVITY and CONNECTED COMPONENTS) can be solved (not necessarily efficiently) by explicitly constructing the polyhedral complexes $\mathcal{T}(f_1), \dots, \mathcal{T}(f_m)$ in \mathbb{R}^n and intersecting them.

Solving TROPICAL CONSISTENCY in a similar way can be done based on a synthetic definition of the tropical varieties under investigation. For tropical hypersurfaces such a definition can be found in [14, Prop. 3.15] and for tropical lines in \mathbb{R}^n in [20, Example 3.8].

Several questions remain open. In particular, the question of polynomial time solvability remains open for the following restrictions.

Open problem 3.5. Can TROPICAL INTERSECTION and TROPICAL CONSISTENCY be solved in polynomial time if the input polynomials are restricted to be linear? Can TROPICAL CONNECTIVITY for quadratic polynomials be solved in polynomial time?

4. PROOFS OF THE RESULTS

4.1. Linear tropical prevarieties. We begin with a statement on TROPICAL CONSISTENCY AND TROPICAL m -CONSISTENCY for linear varieties.

Lemma 4.1. *Let all input polynomials $f_1, \dots, f_m \in \mathbb{Q}[x_1, \dots, x_n]^\oplus$ be restricted to be linear polynomials.*

- (a) *If $m \leq n$ then the output of TROPICAL CONSISTENCY is always YES.*
- (b) *For a fixed constant m , the problems TROPICAL m -CONSISTENCY can be solved in polynomial time.*

Before providing the proof, we recall and collect some statements about linear tropical varieties. Let f_1, \dots, f_m be linear tropical polynomials in x_1, \dots, x_n . If $m \leq n$ and the tropical hyperplanes $\mathcal{T}(f_i)$ are in general position then P is a linear tropical variety of dimension $n - m$. Moreover, P always contains a well-defined *stable intersection* which is a linear tropical variety of dimension $n - m$ (see [20, 23]). In particular, this implies that for $m \leq n$ the answer to TROPICAL CONSISTENCY is always YES.

For a matrix $A = (a_{ij}) \in (\mathbb{R} \cup \{\infty\})^{k \times k}$, the tropical determinant is defined by

$$(4.1) \quad \det_{\text{trop}}(A) = \bigoplus_{\sigma \in S_k} (a_{1,\sigma_1} \odot \dots \odot a_{k,\sigma_k}) = \min_{\sigma \in S_k} (a_{1,\sigma_1} + \dots + a_{k,\sigma_k}),$$

where S_k denotes the symmetric group on $\{1, \dots, k\}$. It was observed in [20] that the computation of the tropical determinant can be phrased as an assignment problem from combinatorial optimization. Hence, using well-known algorithms (see [22, Corollary 17.4b]), a tropical determinant can be computed in polynomial time.

A tropical $k \times k$ -matrix is *singular* if the minimum in (4.1) is attained at least twice. In order to decide in polynomial time whether a $k \times k$ -matrix is singular, first compute the tropical determinant of A . Let $\sigma \in S_k$ be a permutation of $\{1, \dots, k\}$ for which the minimum in (4.1) is attained. For every $j \in \{1, \dots, k\}$ let A_j be the matrix which is obtained from A by replacing the entry (j, σ_j) by an arbitrary larger value. Then A is

tropically singular if and only if $\det_{\text{trop}} A_j = \det_{\text{trop}} A$ for some $j \in \{1, \dots, k\}$. Hence, this can be decided in polynomial time.

Proof of Lemma 4.1. It just remains to prove b). Let $f_i = \bigoplus_{j=1}^n a_{ij} \cdot x_j \oplus a_{i,n+1}$, $1 \leq i \leq m$, and $A = (a_{ij}) \in \mathbb{R}^{m \times (n+1)}$ be the coefficient matrix of f_1, \dots, f_m . Since for $m > n + 1$ the answer of TROPICAL m -CONSISTENCY is always NO, we can assume $m \leq n + 1$. For $m = n + 1$, the problem is equivalent to ask whether the tropical prevariety is empty, which will be treated in Lemma 4.8. For $m \leq n$, by Theorem 5.3 in [20] the tropical prevariety $\bigcap_{i=1}^m \mathcal{T}(f_i)$ is a linear tropical variety of codimension m if and only if none of the $m \times m$ -submatrices of A is tropically singular. For each of the $m \times m$ -submatrices of A it can be checked in polynomial time (in the binary length of the input data) whether it is singular. Since for fixed m , the number $\binom{n}{m}$ of those submatrices is polynomial in n , the claim follows. \square

4.2. Tropical intersection and tropical consistency.

Lemma 4.2. TROPICAL INTERSECTION is \mathcal{NP} -hard. This statement persists if the instances are restricted to those where $\bigcap_{i=1}^m \mathcal{T}(f_i)$ is a tropical variety. Moreover, this statement persists if all polynomials are restricted to be of degree at most 2.

In order to prove \mathcal{NP} -hardness of TROPICAL INTERSECTION, we provide a polynomial time reduction from the well-known \mathcal{NP} -complete 3-satisfiability (3-SAT) problem [6]. Let \wedge and \vee denote the Boolean conjunction and disjunction, respectively, and let $\mathcal{C} = \mathcal{C}_1 \wedge \dots \wedge \mathcal{C}_k$ denote an instance of 3-SAT with clauses $\mathcal{C}_1, \dots, \mathcal{C}_k$ in the variables y_1, \dots, y_n . Furthermore, let $\overline{y_i}$ denote the complement of a variable y_i , and let the literals y_i^1 and y_i^0 be defined by $y_i^1 = y_i$, $y_i^0 = \overline{y_i}$. Let the clause \mathcal{C}_i be of the form

$$(4.2) \quad \mathcal{C}_i = y_{i_1}^{\tau_{i_1}} \vee y_{i_2}^{\tau_{i_2}} \vee y_{i_3}^{\tau_{i_3}},$$

where $\tau_{i_1}, \tau_{i_2}, \tau_{i_3} \in \{0, 1\}$ and $i_1, i_2, i_3 \in \{1, \dots, n\}$ are pairwise different indices.

The reduction consists of two ingredients. First we construct an intersection of suitable tropical hypersurfaces $\bigcap_{i=1}^n \mathcal{T}(h_i)$ in \mathbb{R}^n with $\bigcap_{i=1}^n \mathcal{T}(h_i) = \{0, 1\}^n$ (see Figure 3). We call these hypersurfaces “structural” tropical hypersurfaces.

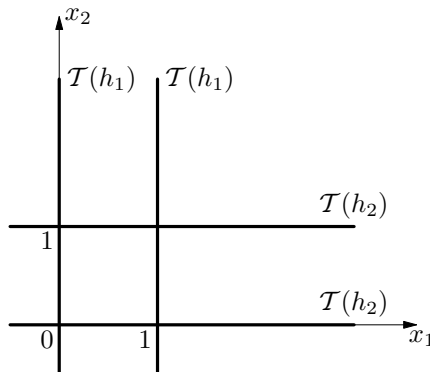


FIGURE 3. Structural hypersurfaces $\mathcal{T}(h_i)$, $1 \leq i \leq n$, for $n = 2$.

In the second step, we relate satisfying assignments of a given clause (4.2) to solutions of some “clause hypersurface”. Let $s : \{\text{TRUE}, \text{FALSE}\} \rightarrow \{0, 1\}$ be defined by $s(\text{TRUE}) = 1$ and $s(\text{FALSE}) = 0$. We utilize the correspondence between a truth assignment $a = (a_1, \dots, a_n)^T \in \{\text{TRUE}, \text{FALSE}\}^n$ to the variables y_1, \dots, y_n and the point $(s(a_1), \dots, s(a_n))^T \in \{0, 1\}^n$ of the tropical prevariety $\bigcap_{i=1}^n \mathcal{T}(h_i)$. To achieve this, we construct one or, in some cases, several tropical hypersurfaces representing the clause.

In order to construct the structural tropical hypersurfaces, let $h'_i \in K[x_1, \dots, x_n]$ be the polynomial

$$h'_i(x) = (t^0 \cdot x_i + t^1) \cdot (t^0 \cdot x_i + t^0) = t^0 \cdot x_i^2 + (t^0 + t^1) \cdot x_i + t^1$$

over K , $1 \leq i \leq n$. Since the tropical hypersurface of a product of polynomials is the union of the tropical hypersurfaces of the factors, we have $\mathcal{T}(h'_i) = \{x \in \mathbb{R}^n : x_i \in \{0, 1\}\}$, and h'_i tropicalizes to

$$(4.3) \quad h_i := \text{trop}(h'_i) = 0 \cdot x_i^2 \oplus 0 \cdot x_i \oplus 1.$$

Hence, $\bigcap_{i=1}^n \mathcal{T}(h_i) = \{0, 1\}^n$.

Now we construct the quadratic polynomials which represent the 3-clauses. In order to illustrate the construction, and since this will be needed explicitly later on, we begin with a 2-clause. Let \mathcal{C}_i denote the 2-clause $\mathcal{C}_i = y_{i_1}^{\tau_{i_1}} \vee y_{i_2}^{\tau_{i_2}}$. Let $f'_i = (x_{i_1} + t^{\tau_{i_1}})(x_{i_2} + t^{\tau_{i_2}})$, which tropicalizes to

$$\begin{aligned} f_i &= (0 \cdot x_{i_1} \oplus \tau_{i_1}) \cdot (0 \cdot x_{i_2} \oplus \tau_{i_2}) \\ &= 0 \cdot x_{i_1} \cdot x_{i_2} \oplus \tau_{i_1} \cdot x_{i_1} \oplus \tau_{i_2} \cdot x_{i_2} \oplus \tau_{i_1} \cdot \tau_{i_2}. \end{aligned}$$

Hence, $\mathcal{T}(f_i) = \{x \in \mathbb{R}^2 : x_{i_1} = \tau_{i_1} \text{ or } x_{i_2} = \tau_{i_2}\}$. In particular, for any $x \in \{0, 1\}^n$ we have $x \in \mathcal{T}(f_i)$ if and only if $x_{i_1} = \tau_{i_1}$ or $x_{i_2} = \tau_{i_2}$. Figure 4 (a) shows the clause curve for the case $n = 2$, $\tau_1 = \tau_2 = 1$.

Now consider a 3-clause \mathcal{C}_i of the form (4.2). Here, the straightforward approach to consider $\mathcal{T}(f_i) = \prod_{j=1}^3 (0 \cdot x_{i_j} \oplus \tau_{i_j})$ leads to cubic polynomials. In order to show hardness even for quadratic polynomials we distinguish several cases corresponding to the number p of positive literals in \mathcal{C}_i .

Case $p \in \{0, 1\}$: Here we can use the following more general lemma.

Lemma 4.3. *Let $C(y_1, \dots, y_k, z_1, \dots, z_l)$ be the clause in the variables $y_1, \dots, y_k, z_1, \dots, z_l$ defined by*

$$C(y_1, \dots, y_k, z_1, \dots, z_l) = y_1 \vee \dots \vee y_k \vee \overline{z_1} \vee \dots \vee \overline{z_l}.$$

Then for $(a_1, \dots, a_k, b_1, \dots, b_l) \in \{\text{TRUE}, \text{FALSE}\}^{k+l}$ we have $C(a_1, \dots, a_k, b_1, \dots, b_l) = \text{TRUE}$ if and only if

$$(4.4) \quad (s(a_1), \dots, s(a_k), s(b_1), \dots, s(b_l)) \in \mathcal{T} \left(\left\langle \prod_{i=1}^k (t^0 \cdot y_i + t^1) \cdot \left(\sum_{j=1}^l t^0 \cdot z_j + t^0 \right) \right\rangle \right).$$

Proof. Let $C_i(y_i) = y_i$, $1 \leq i \leq k$. Then for $a_i \in \{\text{TRUE}, \text{FALSE}\}$, we have $C_i(a_i) = \text{TRUE}$ if and only if $s(a_i) \in \mathcal{T}((t^0 \cdot y_i + t^1))$. Let $C_{k+1}(z_1, \dots, z_l) = \overline{z_1} \vee \dots \vee \overline{z_l}$. Then for $(b_1, \dots, b_l) \in \{\text{TRUE}, \text{FALSE}\}^l$, we have $C_{k+1}(z_1, \dots, z_l) = \text{TRUE}$ if and only if $s(b_1, \dots, b_l) \in \mathcal{T}((\sum_{j=1}^l t^0 \cdot z_j + t^0))$. Considering the disjunction $C_1 \vee \dots \vee C_{k+1}$ proves the claim. \square

For every clause \mathcal{C}_i which contains 0 or 1 positive literals, we associate a tropical hypersurface $\mathcal{T}(f_i)$ as defined in (4.4). Since $p \in \{0, 1\}$ the degree of f_i is at most 2.

In particular, for the case $p = 0$ and $i_1 = 1, i_2 = 2, i_3 = 3$, we have $\mathcal{C}_i = \overline{y_1} \vee \overline{y_2} \vee \overline{y_3}$, and the hypersurface in (4.4) is $\mathcal{T}(\langle t^0 \cdot y_1 + t^0 \cdot y_2 + t^0 \cdot y_3 + t^0 \rangle)$, which is the hypersurface given by the linear tropical polynomial $0 \cdot y_1 \oplus 0 \cdot y_2 \oplus 0 \cdot y_3$. Figure 4(b) visualizes this situation for the smaller-dimensional case of a 2-clause.

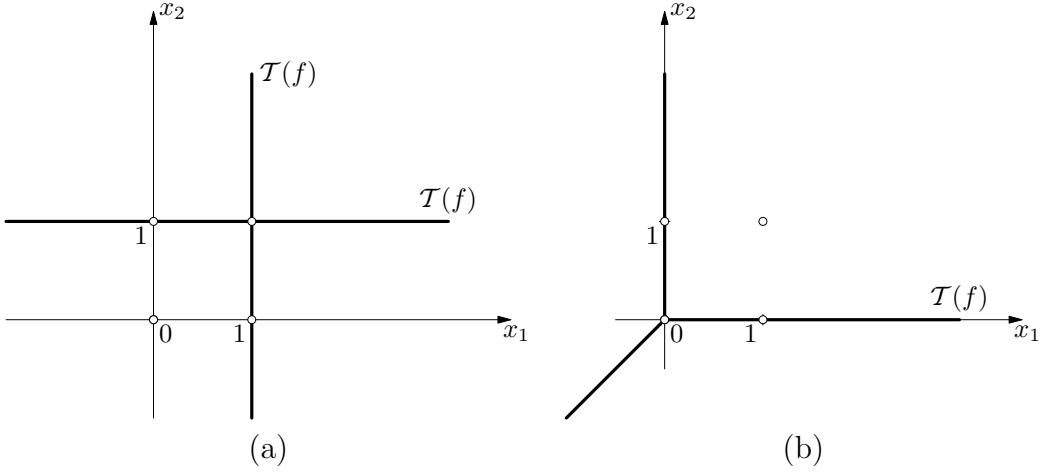


FIGURE 4. Figure (a) shows a quadratic clause curve for the 2-clause $y_1 \vee y_2$ in \mathbb{R}^2 . Figure (b) shows a linear clause curve for the 2-clause $\overline{y_1} \vee \overline{y_2}$ in \mathbb{R}^2 .

For the case $p = 1$ and the clause $y_1 \vee \overline{y_2} \vee \overline{y_3}$, the hypersurface in (4.4) is

$$\mathcal{T}(\langle (t^0 \cdot y_1 + t^1) \cdot (t^0 \cdot y_2 + t^0 \cdot y_3 + t^0) \rangle),$$

which is the hypersurface of the tropical polynomial $0 \cdot y_1 \cdot y_2 \oplus 0 \cdot y_1 \cdot y_3 \oplus 0 \cdot y_1 \oplus 1 \cdot y_2 \oplus 1 \cdot y_3 \oplus 1$.

Case $p = 2$: By renumbering the variables, we can assume $\mathcal{C}_i = y_{i_1} \vee y_{i_2} \vee \overline{y_{i_3}}$. Let f_i be the quadratic tropical polynomial defined by

$$f_i = 0 \cdot x_{i_1} \cdot x_{i_2} \oplus 1 \cdot x_{i_1} \oplus 1 \cdot x_{i_2} \oplus 0 \cdot x_{i_3} \oplus 0 \cdot x_{i_3}^2 \oplus 1.$$

Then for $(x_{i_1}, x_{i_2}, x_{i_3}) \in \{0, 1\}^3$ we have $f_i(x_{i_1}, x_{i_2}, x_{i_3}) = 1$ if and only if $x_{i_1} = 1$ or $x_{i_2} = 1$ or $x_{i_3} = 0$.

Case $p = 3$: Let $\mathcal{C}_i = y_{i_1} \vee y_{i_2} \vee y_{i_3}$. The following lemma (in the spirit of the nine associated points theorem for complex cubic curves) states that it is *not* possible to find a single polynomial f_i for the clause \mathcal{C}_i .

Lemma 4.4. *If $f = f(x_1, x_2, x_3)$ is a tropical quadratic polynomial with*

$$(4.5) \quad \{x \in \{0, 1\}^3 : x_1 = 1 \text{ or } x_2 = 1 \text{ or } x_3 = 1\} \subset \mathcal{T}(f),$$

then $\mathcal{T}(f)$ also contains $(0, 0, 0)$.

Proof. Assume that there exists a tropical quadratic polynomial f satisfying (4.5) such that the minimum of the linear forms at $(0, 0, 0)$ is attained only once. Let l be the linear form where the minimum is attained. Since f is quadratic, l depends on at most two variables, and the exponents of these variables are 1 or 2. Let x_k be a variable which does not occur in l , and let x' be obtained from x by switching x_k from 0 to 1. Then the value of each linear form at x' is larger than or equal to the value of that linear form of x . Since the value of the linear form l at x' is equal to the value of l at x , the minimum of all linear forms at x' is the same one as at x , and it is attained only once. \square

In order to encode a clause with three positive literals into tropical quadrics, we embed it into higher-dimensional space by introducing an additional variable z . Let \mathcal{C}'_i be the Boolean formula

$$\mathcal{C}'_i = (y_{i_1} \vee y_{i_2} \vee \bar{z}) \wedge (y_{i_3} \vee z) \wedge (\overline{y_{i_3}} \vee \bar{z})$$

in the variables $y_{i_1}, y_{i_2}, y_{i_3}, z$. The last two clauses of this formula imply that any satisfying assignment of \mathcal{C}'_i has the property $y_{i_3} = \bar{z}$. Hence, there exists a satisfying assignment for the original clause \mathcal{C}_i if and only if the formula \mathcal{C}'_i can be satisfied. \mathcal{C}'_i consists of one 3-clause that belongs to the case $p = 2$ and of two 2-clauses, which can be encoded into tropical geometry as described above. Hence, there exist three tropical quadratic polynomials g_1, g_2, g_3 in $y_{i_1}, y_{i_2}, y_{i_3}, z$ such that \mathcal{C}_i can be satisfied if and only if $\mathcal{T}(g_1), \mathcal{T}(g_2)$, and $\mathcal{T}(g_3)$ have a common point in $\{0, 1\}^4$.

For $p \in \{0, \dots, 3\}$ let $\#_p(\mathcal{C})$ denote the number of clauses in the 3-SAT formula \mathcal{C} with p positive terms. Then the construction for the clauses yields $k' := k + 2\#_3(\mathcal{C})$ tropical hypersurfaces, which we denote by $f_1, \dots, f_{k'}$. Moreover, due to the additional auxiliary variables the actual number of total variables is $n' := n + \#_3(\mathcal{C})$. Let P be the tropical prevariety

$$(4.6) \quad P = \mathcal{T}(h_1) \cap \dots \cap \mathcal{T}(h_{n'}) \cap \mathcal{T}(f_1) \cap \dots \cap \mathcal{T}(f_{k'}) \subset \mathbb{R}^{n'}.$$

Lemma 4.5. *P is nonempty if and only if \mathcal{C} can be satisfied.*

Proof. Let $y \in \{\text{TRUE}, \text{FALSE}\}^n$ be a satisfying assignment for \mathcal{C} and $x := s(y) \in \{0, 1\}^{n'}$. By construction, x is contained in all the structural hypersurfaces and in all the clause surfaces.

Conversely, let $x \in P$. Since $x \in \bigcap_{i=1}^{n'} \mathcal{T}(h_i)$ we have $x \in \{0, 1\}^{n'}$. Set $y = s^{-1}(x) \in \{\text{TRUE}, \text{FALSE}\}^n$. Since x is contained in all clause hypersurfaces representing the clause \mathcal{C}_i , the truth assignment y satisfies the clause \mathcal{C}_i , $1 \leq i \leq k$. Hence, \mathcal{C} can be satisfied. \square

All the polynomials in the construction of the tropical prevariety P are of degree at most 2. Moreover, P is a finite set and therefore even a tropical variety. Since the reduction from 3-SAT to TROPICAL INTERSECTION is doable in polynomial time, this finishes the proof of Lemma 4.2 and hence of the \mathcal{NP} -hardness statement for TROPICAL INTERSECTION in Theorem 3.1.

Corollary 4.6. *$\#\text{CONNECTED COMPONENTS}$ is $\#\mathcal{P}$ -hard.*

Proof. It suffices to observe that the reduction given above is parsimonious, i.e., the number of solutions of the tropical prevariety is the number of satisfying assignments of

the 3-SAT formula \mathcal{C} . Since counting the number of satisfying assignments of a 3-SAT formula is a $\#\mathcal{P}$ -hard problem [27], the statement follows. \square

Lemma 4.7. TROPICAL INTERSECTION $\in \mathcal{NP}$.

Proof. We have to show that for every YES instance of TROPICAL INTERSECTION, there exists a certificate of polynomial size, as well as a polynomial time verification procedure for these certificates.

Let f_i be of the form $f_i = f_i(x_1, \dots, x_n) = \bigoplus_{\alpha \in \mathcal{A}_i} c_\alpha \cdot x^\alpha$ for some support set \mathcal{A}_i , $1 \leq i \leq m$. If there exists a point z in the intersection of tropical varieties, then there exist $(\beta_1, \gamma_1) \in \mathcal{A}_1^2, \dots, (\beta_m, \gamma_m) \in \mathcal{A}_m^2$ such that the minimum in f_i at z is attained at the terms given by (β_i, γ_i) . Hence, z is a solution of system of linear equations and inequalities

$$c_{\beta_i} + \sum_{j=1}^n \beta_{ij} x_j = c_{\gamma_i} + \sum_{j=1}^n \gamma_{ij} x_j \leq c_\alpha + \sum_{j=1}^n \alpha_j x_j \quad \text{for all } \alpha \in \mathcal{A}_i, \quad 1 \leq i \leq m.$$

The size of this linear program is linear in the size of the input. Moreover, checking whether a given point z is contained in a given tropical hypersurface can be done in polynomial time. Consequently, checking whether z is contained in the intersection of tropical hypersurfaces can be done in polynomial time. \square

Hence, by Lemmas 4.2 and 4.7, TROPICAL INTERSECTION is \mathcal{NP} -complete. In contrast to this, the following theorem provides a positive complexity result and yields a linear programming-based algorithm.

Lemma 4.8. *If the number m of tropical hypersurfaces is a fixed constant, then TROPICAL INTERSECTION can be solved in polynomial time.*

Proof. Let $\mathcal{A}_i = \{\alpha \in \mathbb{N}_0^n : \sum_{j=1}^{d_j} \alpha_j \leq d_i\}$ and $f_i = f_i(x_1, \dots, x_n) = \bigoplus_{\alpha \in \mathcal{A}_i} c_\alpha \cdot x^\alpha$, $1 \leq i \leq m$. If L denotes the binary encoding length of the TROPICAL INTERSECTION-instance then the size $|\mathcal{A}_i|$ of \mathcal{A}_i satisfies $|\mathcal{A}_i| \leq L$. Hence, for any $i \in \{1, \dots, m\}$ the polynomial f has at most L terms, and thus there are at most $\binom{L}{2}$ choices of two terms where the minimum in f_i is attained. Since there are at most $\binom{L}{2}^m$ choices of two terms in all the polynomials f_1, \dots, f_m , it suffices to show that for any fixed choice of two vectors $\beta_i, \gamma_i \in \mathcal{A}_i$ where the minimum is attained in f_i , $1 \leq i \leq m$, the resulting linear program

$$c_{\beta_i} + \sum_{j=1}^n \beta_{ij} x_j = c_{\gamma_i} + \sum_{j=1}^n \gamma_{ij} x_j \leq c_\alpha + \sum_{j=1}^n \alpha_j x_j \quad \text{for all } \alpha \in \mathcal{A}_i, \quad 1 \leq i \leq m$$

can be solved in polynomial time. However, since the size of the linear program is polynomial in the size of the input of TROPICAL INTERSECTION, this follows from the polynomial solvability of linear programming [12]. \square

Lemma 4.9. TROPICAL CONSISTENCY is $\text{co-}\mathcal{NP}$ -hard. *This hardness persists if all polynomials are restricted to be of degree at most 2.*

Proof. Since the empty set is a tropical variety, it suffices to provide a polynomial time reduction from 3-SAT with the following properties. For every NO-instance of 3-SAT the constructed tropical prevariety is the empty set. For every YES-instance of 3-SAT, the

constructed prevariety is not a tropical variety. In order to simplify notation, we assume from now on that all clauses contain at most 2 positive literals, since otherwise we can apply the same auxiliary construction as in the proof of Lemma 4.2.

We embed the construction from the proof of Lemma 4.5 into \mathbb{R}^{n+1} by considering all polynomials formally to be polynomials in $n + 1$ variables. Since the definition of the structural hypersurfaces in (4.3) and the definition of the clause hypersurfaces do not depend on x_{n+1} , the embedding of the tropical prevariety P from (4.6) into \mathbb{R}^{n+1} gives a prevariety $P' = P \times \mathbb{R} \subset \mathbb{R}^{n+1}$. Recall that the structural hypersurfaces are given by the tropical polynomials $h_i = 0 \cdot x_i^2 \oplus 0 \cdot x_i \oplus 1$. Let

$$g_i = 0 \cdot x_i^2 \oplus 0 \cdot x_i \oplus 1 \oplus 1 \cdot x_{n+1}$$

for $1 \leq i \leq n$. For $i = 1$ and $n + 1 = 2$, the tropical variety $\mathcal{T}(g_i)$ is shown in Figure 5.

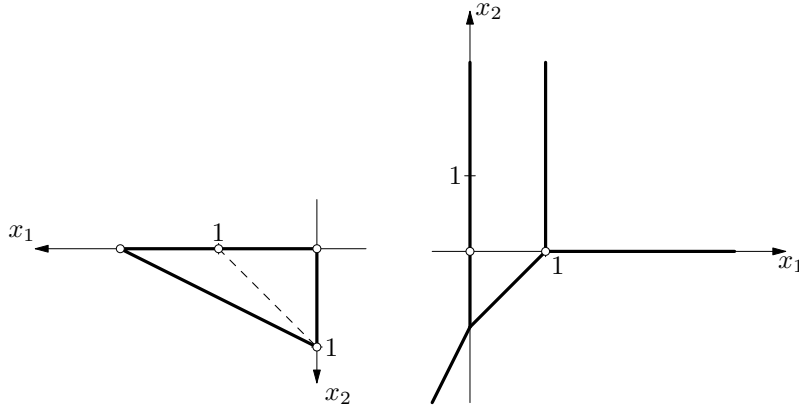


FIGURE 5. Newton polygon of the polynomial g_i for $i = 1$, $n + 1 = 2$ and $\mathcal{T}(g_i)$.

The intersection of $\mathcal{T}(g_i)$ and $\mathcal{T}(h_i)$ is

$$\mathcal{T}(g_i) \cap \mathcal{T}(h_i) = \{x \in \mathbb{R}^{n+1} : (x_i = 0 \text{ and } x_{n+1} \geq -1) \text{ or } (x_i = 1 \text{ and } x_{n+1} \geq 0)\}.$$

If we imagine the x_{n+1} -axis to be pointing upwards, the intersection of all structural hypersurfaces with the hypersurfaces defined by g_1, \dots, g_n is a union of 2^n half rays which are unbounded in the upward pointing directions,

$$\bigcap_{i=1}^n g_i \cap \bigcap_{i=1}^n h_i = \{(\{0, 1\}^n \setminus (0, \dots, 0)) \times \mathbb{R}_+ \cup (0, \dots, 0) \times \{x_{n+1} \in \mathbb{R} : x_{n+1} \geq -1\}\}.$$

Using the same clause hypersurfaces as in the proof of Lemma 4.2, embedded into \mathbb{R}^{n+1} , we obtain the empty set for every NO-instance of 3-SAT. Moreover, every YES-instance of 3-SAT gives a finite union of disjoint half rays which is not a tropical variety. All the polynomials in the construction are of degree at most 2. Since the reduction is polynomial time, the statement follows. \square

4.3. Connectivity. In order to concentrate on the aspect of connectivity (rather than a non-emptiness test in disguise), note that in the definition of TROPICAL CONNECTIVITY we have excluded inputs leading to an empty prevariety.

Lemma 4.10. *TROPICAL CONNECTIVITY is co- \mathcal{NP} -hard. This statement persists if the instances are restricted to those where $\bigcap_{i=1}^m \mathcal{T}(f_i)$ is a tropical variety. Moreover, this statement persists if all polynomials are restricted to be of degree at most 3.*

Proof. We choose a point q which is always contained in the tropical variety and modify the construction from the proof of Lemma 4.2. In order to achieve that our choice of q does not interfere with the remaining construction, we embed the construction into \mathbb{R}^{n+1} , similar to the proof of Lemma 4.9.

In the modification, the structural hypersurfaces are now given by the polynomials

$$h'_i(x) = (t^0 \cdot x_i + t^1) \cdot (t^0 \cdot x_i + t^0) \cdot (t^0 \cdot x_i + t^2), \quad 1 \leq i \leq n.$$

Hence, the intersection of all hypersurfaces gives $\{0, 1, 2\}^{n+1}$. By constructing additional polynomials of the form

$$(t^0 \cdot x_i + t^1) \cdot (t^0 \cdot x_i + t^0) \cdot g'_i(x)$$

with linear forms g'_i as well as of the form

$$(t^0 \cdot x_{n+1} + t^0) \cdot (t^0 \cdot x_{n+1} + t^2),$$

we can easily achieve that the intersection of all these hypersurfaces is the set $\{0, 1\}^n \times \{0\} \cup \{(2, 2, 2)\}$.

By multiplying all polynomials of the clause surfaces by the polynomials $t^0 \cdot x_{n+1} + t^2$, we can achieve that the point $(2, 2, 2)$ remains contained in the prevariety. Note that the degree of all polynomials is increased by only 1. Altogether, the constructed tropical prevariety P is always nonempty. If the 3-SAT-formula can be satisfied, then there are at least two connected components in P . If the 3-SAT-formula cannot be satisfied then P has exactly one component.

All the constructed polynomials are of degree at most 3. The resulting tropical prevariety is a finite set and therefore a tropical variety. Moreover, the reduction is polynomial time. \square

Lemma 4.11. *If the number m of tropical prevarieties is a fixed constant, then TROPICAL CONNECTIVITY can be solved in polynomial time.*

Proof. Similar to the proof of Lemma 4.8, for fixed m we can compute in polynomial time faces F_1, \dots, F_t of the polyhedral complex $P = \bigcap_{i=1}^m \mathcal{T}(f_i)$ such that $P = \bigcup_{i=1}^t F_i$. Hence, we can construct a graph G with vertices F_1, \dots, F_t in which two faces F_i and F_j are connected by an edge if and only if they intersect. Then one computes the number of connected components of G . This can be done in polynomial time. \square

The hardness result 4.11 is also contrasted by the statement that linear tropical prevarieties are always connected.

Lemma 4.12. *Every nonempty linear tropical prevariety $P \subset \mathbb{R}^n$ is connected.*

Proof. Let $P := \bigcap_{i=1}^m \mathcal{T}(f_i)$ and $x, y \in P$. The notions \oplus, \odot , previously defined for scalars, can also be defined for vectors, by applying the operations componentwise. With this notation it suffices to show that for every $\lambda, \mu \in \mathbb{R}$, the point $z := \lambda \odot x \oplus \mu \odot y$ is contained in each $\mathcal{T}(f_i)$, $1 \leq i \leq m$. Fix an $i \in \{1, \dots, m\}$, and let $f_i = a_0 \oplus \bigoplus_{i=1}^n a_i \cdot x_i$. For convenience of notation set $x_0 = y_0 = z_0 = 0$, and let r be an index which minimizes $\{a_j + z_j : 0 \leq j \leq n\}$. By definition of z , we have $z_r = \lambda + x_r$ or $z_r = \mu + y_r$. Without loss of generality we can assume $z_r = \lambda + x_r$. Note that then $a_r + x_r \leq a_s + x_s$ for every index s .

Since $x \in \mathcal{T}(f_i)$, there exists an index $s \neq r$ with $a_r + x_r = a_s + x_s$. The definition of z implies $a_s + z_s \leq a_s + \lambda + x_s = a_r + \lambda + x_r = a_r + z_r$. Hence, by the choice of r , $a_r + z_r = a_s + z_s$. In other words, the minimum in f_i is attained at least twice at the point z , i.e., $z \in \mathcal{T}(f_i)$. \square

Remark 4.13. Using the framework of tropical convexity from [4], Lemma 4.12 also follows from the fact that tropical hyperplanes are tropically convex [4, Proposition 6] in connection with the observations that the intersection of tropically convex sets is tropically convex and that tropically convex sets are connected.

Statements 4.1–4.12 prove all claims in Theorems 3.1–3.3

5. RELATED ASPECTS ON AMOEBAS

Our work is related to (and was partially inspired by) questions on algorithmic complexity of basic problems on the amoebas that were introduced in by Gel'fand, Kapranov, Zelevinsky [7]. Let I be an ideal in the ring $\mathbb{R}[x_1, \dots, x_n]$ of Laurent polynomials. Then the *amoeba* of I is defined by the image of the complex subvariety $V(I) \subset (\mathbb{C}^*)^n$ under the mapping

$$\begin{aligned} \text{Log} : (\mathbb{C}^*)^n &\rightarrow \mathbb{R}^n, \\ z &\mapsto (\log |z_1|, \dots, \log |z_n|), \end{aligned}$$

where $|\cdot|$ denotes the absolute value of a complex number and \log is the natural logarithm. Since any hypersurface amoeba contains a tropical variety (the so-called *spine*) that is a strong deformational retract of the amoeba (see, e.g., [15, Theorem 2.6]), algorithmic questions on amoebas are closely related to algorithmic questions on tropical varieties (see [26, Chapter 9]).

A central question by Einsiedler and Lind asks for an efficient algorithm to test whether the complex amoeba of an ideal contains the origin [5]. This comes from applications in dynamical systems, where this test determines whether a dynamical system has a finiteness condition called expansiveness. Only little is known about the computational hardness of algorithmic problems on amoebas, and the computational complexity of the membership problem for amoebas (with rational input data and the dimension being part of the input) is still open. If the polynomials are given in sparse encoding (i.e., only the non-vanishing coefficients are listed in the input), then the problem becomes \mathcal{NP} -hard even for $n = 1$ ([18], see also [21]). Recently, Rojas and Stella [21] have established an algorithmic fewnomial theory providing further hardness results for amoebas in sparse encoding (e.g.,

\mathcal{NP} -hardness of deciding whether an amoeba intersects a coordinate hyperplane.) For some Nullstellensatz-type algorithmic results see [19].

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