# COMMON TANGENTS TO FOUR UNIT BALLS IN $\mathbb{R}^3$

IAN G. MACDONALD, JÁNOS PACH, AND THORSTEN THEOBALD

Abstract. We answer a question of David Larman, by proving the following result. Any four unit balls in 3-dimensional space, whose centers are not collinear, have at most twelve common tangent lines. This bound is tight.

## 1. Introduction

The screen of a computer monitor consists of small pixels. Suppose that we are given a 3-dimensional scene consisting of several objects and a viewpoint. Generating an image of this scene ("rendering" the scene) is a basic task in computer graphics and in computational geometry, which amounts to determining the visible object(s) at each pixel. The methods developed for the solution of such problems have an extensive literature under the labels "ray tracing" and "hidden surface removal" (see, e.g., [7, 20]). This field has served as a rich source of problems on geometric, combinatorial, and algebraic properties of systems of lines in their interaction with geometric objects.

For instance, we can assume that all of our objects are unit balls in a region $A$, and we want to determine which balls are not visible from *any* viewpoint outside of $A$ (see [29]). This leads to the following problem, first formulated by David Larman [19], and later discussed by Durand [9], Karger [17], and Verschelde [30]. Here, $B(c, r)$ denotes the (closed) ball with center $c$ and radius $r$.

**Given:** Four balls $B(c_i, r)$ with centers $c_i \in \mathbb{R}^3$ and radius $r$, $1 \leq i \leq 4$.

**Question:** Under what conditions can we guarantee that the balls permit only a finite number of common tangent lines? If these conditions are satisfied, what is the maximum number of common tangents?

Equivalently, we can ask for the circular cylinders with radius $r$ circumscribing the tetrahedron with vertices $c_1, \ldots, c_4$ [25, 17, 1]. Note that in the original formulation of the problem, the balls are not necessarily disjoint. In the second formulation, this means that $r$ may be larger than or equal to $\text{diam}\{c_1, c_2, c_3, c_4\}/2$.

The above problem belongs to *enumerative geometry* [27, 14]. For some rigorous modern treatises using the framework of algebraic geometry, see ([18, 11]). One of the most famous results in this field is the enumeration by Cayley and Salmon of the 27 lines on a smooth cubic surface (see [14, 13]). According to another famous result, misstated by Steiner [28] and correctly proved first by Chasles (cf. [27]), there can be 3264 conics tangent to *five* given conics. It turns out that all of them can be real (see [22] and [11], §7.2).

There are two other results in enumerative geometry, somewhat related to the above problem:

*Key words and phrases.* tangents, balls, enumerative geometry, computational geometry
*2000 Mathematics Subject Classification* 14N10, 14Q15, 51N20, 52A15, 68U05 .

(1) The number of balls touching *four* given balls in 3-space is at most *sixteen* in the generic case [26, 16]. (This can be regarded as the 3-dimensional version of the Apollonius problem).
(2) The number of lines intersecting *four* given lines in 3-space is at most *two* in the generic case [27, 15, 12].

The following result provides an answer to the above mentioned question of Larman.

**Theorem 1.** *Any* four *unit balls in 3-dimensional space, whose centers are not collinear, have at most* twelve *common tangent lines. This bound is tight.*

The second statement of this theorem answers a question of Karger [17], who asked whether there exist configurations with more than eight common tangents.

In the case when the centers are *affinely independent*, we will represent the common tangents by common solutions of a cubic and a quartic equation. We use the method from [3, 24] to characterize all lines equidistant from the four centers, by a cubic curve in projective plane. We also present an alternative approach to deduce a cubic equation, using a classical construct in projective geometry: the *pedal surface of a tetrahedron*. The condition on the radius, i.e., the actual distance of the lines from the centers, leads to a quartic equation.

If the cubic equation is *irreducible*, a detailed geometric inspection ensures that the cubic and the quartic cannot have a common component; hence, the desired result is implied by Bézout's Theorem. In case of a *reducible* cubic, we can find suitable parametrizations of the quadratic or linear factors (cf. [24]). Substituting the parametrization into the radius condition gives a univariate polynomial equation whose leading coefficient can be explicitly analyzed.

In the case when the centers of the balls are *affinely dependent*, we give a direct argument using the ellipses passing through the *four* centers, whose shorter half-axis is fixed.

The paper is structured as follows. Section 2 deals with the case where the centers of the balls are affinely independent. In Section 3 we show that 12 tangents can indeed be established in real space, and we exhibit a whole class of these configurations based on $c_1, \ldots, c_4$ constituting an equifacial tetrahedron. Finally, Section 4 contains the proof for the affinely dependent case.

After reading an earlier draft of this paper, William Fulton found an alternative proof of Theorem 1 in the generic case, using techniques from intersection theory [10].

## 2. Affinely Independent Centers

2.1. **A cubic and a quartic equation.** Let $c_1, \ldots, c_4 \in \mathbb{R}^3$ be affinely independent, and let $T$ be the tetrahedron with vertices $c_1, \ldots, c_4$. Further, let $A_i$ be the area of the face of $T$ which is opposite to $c_i$, $1 \leq i \leq 4$. First we describe the set of lines which are tangent to the balls $B(c_i, r)$ for *some* radius $r > 0$.

*First method (based on elementary geometry* [24]*)*: A line $l$ in $\mathbb{R}^3$ can be characterized by its closest point $p$ to the origin, and by its direction $s$. More precisely, it can be described by $l = \{p + \mu s \ : \ \mu \in \mathbb{R}\}$, where $p$ and $s \neq 0$ are perpendicular vectors (in notation, $p \perp s$). The direction vector $s = (s_1, s_2, s_3)^T$ can be regarded as homogeneous coordinate, i.e., multiplying $s$ by any nonzero constant still gives the same direction of the line. Since $p \perp s$, the distance of $l$ from the origin is given by $||p||$, where $||\cdot||$ refers to the Euclidean norm.

The line $l$ has distance $r$ from some point $c_i$ if and only if the line $l-p$ (which passes through the origin) has distance $r$ from $c_i - p$. Therefore, we have

$$((c_i - p) \times s)^2 = r^2 s^2$$

(see Figure 1). Introducing the moment vector $m := p \times s$ yields

$$(c_i \times s)^2 - 2\langle c_i \times s, p \times s\rangle + m^2 - r^2 s^2 = 0,$$

whence

$$(c_i \times s)^2 - 2\langle c_i, p\rangle s^2 + m^2 - r^2 s^2 = 0. \tag{1}$$

Choosing $c_4$ to be at the origin, Equation (1) implies $m^2 - r^2 s^2 = 0$. Moreover, for $c_1, c_2, c_3$, we obtain linear equations in $p$:

$$\langle c_i, p\rangle = \frac{1}{2s^2}(c_i \times s)^2, \qquad 1 \le i \le 3. \tag{2}$$

Setting $M := (c_1, c_2, c_3)^T$, we obtain the vector equation

$$p = \frac{1}{2s^2} M^{-1} \begin{pmatrix} (c_1 \times s)^2 \\ (c_2 \times s)^2 \\ (c_3 \times s)^2 \end{pmatrix} \neq 0. \tag{3}$$

By Cramer's rule,

$$M^{-1} = \frac{1}{6V}(c_2 \times c_3, c_3 \times c_1, c_1 \times c_2), \tag{4}$$

where $V := \det(c_1, c_2, c_3)/6$ denotes the oriented volume of $T$. In particular, any direction vector $s$ of a line $l$ satisfying the four distance conditions determines the corresponding vector $p$ and the radius $r = ||p||$ uniquely. By introducing the normal vectors

$$n_1 := (c_2 \times c_3)/2, \quad n_2 := (c_3 \times c_1)/2, \quad n_3 := (c_1 \times c_2)/2, \tag{5}$$

and substituting (3) into $\langle p, s\rangle = 0$, we can eliminate $p$ and obtain a homogeneous cubic condition for the direction vector $s$:

$$\sum_{i=1}^{3} (c_i \times s)^2 \langle n_i, s\rangle = 0. \tag{6}$$

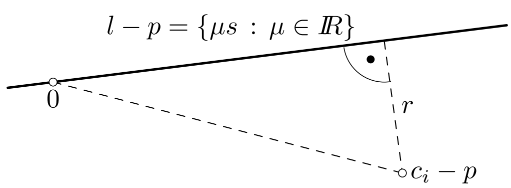



FIGURE 1. Distance of the line $l - p$ from $c_i - p$

In order to simplify this equation, we express $s$ in terms of the three centers $c_1, c_2, c_3$, i.e.,

$$s = \sum_{j=1}^{3} t_j c_j \tag{7}$$

with homogeneous coordinates $t_1, t_2, t_3$. This yields

$$\langle n_i, s \rangle = \langle n_i, \sum_{j=1}^{3} t_j c_j \rangle = t_i \langle n_i, c_i \rangle.$$

As the scalar triple product $\langle n_i, c_i \rangle$ is invariant for $1 \le i \le 3$, Equation (6) simplifies to

$$\sum_{i=1}^{3} t_i (c_i \times s)^2 = 0. \tag{8}$$

By using $A_1 = ||n_1||$, $A_2 = ||n_2||$, $A_3 = ||n_3||$, $A_4 = ||(c_1 - c_2) \times (c_3 - c_2)||/2$, and setting $F := (A_1^2 + A_2^2 + A_3^2 - A_4^2)/2 = -(\langle n_1, n_2 \rangle + \langle n_2, n_3 \rangle + \langle n_3, n_1 \rangle)$, the expansion of this sum yields

$$A_1^2 t_2 t_3 (t_2 + t_3) + A_2^2 t_3 t_1 (t_3 + t_1) + A_3^2 t_1 t_2 (t_1 + t_2) + 2F t_1 t_2 t_3 = 0. \tag{9}$$

*Second method (based on classical projective geometry)*: Note that the numbers in (7) can be interpreted as barycentric coordinates of $s$ in the projective space relative to $c_1$, $c_2$, $c_3$ (cf. [6]). If we allow $c_4$ to be an arbitrary vector again, the representation in barycentric coordinates is

$$s = \sum_{j=1}^{4} t_j c_j. \tag{10}$$

Then the equation of $\Pi_\infty$, the plane at infinity in three-dimensional projective space $\mathbb{P}^3$, is

$$t_1 + t_2 + t_3 + t_4 = 0 \tag{11}$$

(cf. [6]). The locus of all points $x$ with the property that the feet of the perpendiculars from $x$ on the planes supporting the faces of $T$ lie in a plane, is a cubic surface $\Sigma$ ([23], p. 118, Exercise 17). In the appendix we provide a proof of this statement. Namely, $\Sigma$ is given by

$$A_1^2 t_2 t_3 t_4 + A_2^2 t_1 t_3 t_4 + A_3^2 t_1 t_2 t_4 + A_4^2 t_1 t_2 t_3 = 0, \tag{12}$$

or, in a nicer (but slightly imprecise) form

$$\frac{A_1^2}{t_1} + \frac{A_2^2}{t_2} + \frac{A_3^2}{t_3} + \frac{A_4^2}{t_4} = 0. \tag{13}$$

Obviously, all six lines defined by the edges $c_i c_j$, $1 \le i \ne j \le 4$, belong to $\Sigma$. Consider now any circular cylinder $C$ circumscribing $T$ and let $x(C)$ denote the point at infinity of the axis of $C$. We claim that $x(C) \in \Sigma$, i.e., its barycentric coordinates satisfy (12). By the Wallace-Simson Theorem, the feet of the perpendiculars from $c_4$ on the planes $c_1 c_2 x(C)$, $c_1 c_3 x(C)$, $c_2 c_3 x(C)$ are collinear ([6], p. 16, Exercise 11; [8]). Consequently, the feet of the perpendiculars from $c_4$ on the four planes supporting the faces of the tetrahedron $c_1c_2c_3x(C)$ lie in a plane. But then $x(C)$ is in the same relation to the tetrahedron $c_1c_2c_3c_4$, i.e., $x(C) \in \Sigma$ (see [2], p. 25).

By solving (11) for $t_4$ and substituting this expression into (12), we obtain a cubic equation in $t_1, t_2, t_3$. It can be easily checked that for $c_4 = 0$ this equation is equivalent to (9).

Consequently, the set of lines tangent to the balls $B(c_i, r)$ for *some* radius $r$ can be characterized by the homogeneous cubic equation (9) in $s$. In addition, for a *fixed* radius $r$, Equation (3) in conjunction with $p^2 = r^2$ leads to a homogeneous equation of degree 4 in $s$. Hence, unless the cubic curve $\mathcal{C}$ and the quartic curve $\mathcal{Q}$ in projective plane $\mathbb{P}^2$ have a common component, Bézout's Theorem implies there are 12 (possibly complex) solutions including multiplicities (see, e.g., [5]).

2.2. **The irreducible case.** Assume first that $\mathcal{C}$ is irreducible. Then $\mathcal{C}$ and $\mathcal{Q}$ have a common component if and only if $\mathcal{C} \subset \mathcal{Q}$. Now observe that any solution of (9) uniquely defines a radius $r$ via (3). Hence, if $\mathcal{C} \subset \mathcal{Q}$ then the radius is constant for all elements in $\mathcal{C}$. Since we know six points on $\mathcal{C}$, namely the six edge directions, it suffices to prove the following lemma.

**Lemma 2.** *If all six edge directions give the same radius, then $\mathcal{C}$ is reducible.*

*Proof.* Consider two directions, parallel to two skew edges of $T$, say $s := c_1 - c_4$ and $s' := c_3 - c_2$. Using (3) and (4), we can compute the corresponding radii $r_s$ and $r_{s'}$. We obtain

$$\begin{aligned} r_s &= \frac{2A_2A_3||n_1 + n_2||}{3Vc_1^2}, \\ r_{s'} &= \frac{||(c_1 \times (c_3 - c_2))^2(c_2 \times c_3) + 4A_1^2(c_3 \times c_1) + 4A_1^2(c_1 \times c_2)||}{12V(c_3 - c_2)^2}. \end{aligned}$$

Applying the relation $A_4 = ||(c_1 - c_2) \times (c_3 - c_2)||/2$, the latter expression can be compactly written as

$$r_{s'} = \frac{2A_1A_4||n_1 + n_2||}{3V(c_3 - c_2)^2}.$$

Now $r_s = r_{s'}$ implies

$$c_1^2 A_1 A_4 = (c_3 - c_2)^2 A_2 A_3. \tag{14}$$

Let $a_{ij} = ||c_i - c_j||$, $i \neq j$. Further, let $R_i$ denote the circumradius of the face opposite to $c_i$, $1 \leq i \leq 4$. In view of the well-known triangle formula "$R = (abc)/4A$", we have $R_1 = a_{23}a_{24}a_{34}/4A_1$ and three analogous equations for $R_2$, $R_3$, and $R_4$. Hence, (14) becomes

$$R_1R_4 = R_2R_3. \tag{15}$$

By our assumptions, the radii corresponding to the directions $c_2 - c_4$ and $c_3 - c_1$ as well as the radii corresponding to the directions $c_3 - c_4$ and $c_2 - c_1$ coincide. Thus, we obtain

$$R_2R_4 = R_1R_3, \qquad R_3R_4 = R_1R_2, \tag{16}$$

and hence $R_1 = R_2 = R_3 = R_4$. Therefore, the four faces of the tetrahedron are equidistant from the center of the sphere through $c_1, \ldots, c_4$. In other words, the *in-center* of $T$ coincides with its *circumcenter.* Hence, the circumcenter of a face is the point at which the inscribed sphere of $T$ touches that face. In particular, it lies inside the face, which implies that every face of $T$ has only *acute* angles.

Let $\alpha_{ij}$ denote the angle at $c_i$ in the face opposite to $c_j$. By the Law of Sines ([6], p. 13), $a_{23} = 2R_1 \sin \alpha_{41} = 2R_4 \sin \alpha_{14}$, so that

$$\sin \alpha_{ij} = \sin \alpha_{ji}, \qquad 1 \le i \ne j \le 4.$$

Altogether, any pair of faces have a common edge, identical acute angles opposite to this edge, and the same circumradius. Consequently, the two faces are congruent and have the same area, i.e., $A_1 = A_2 = A_3 = A_4$. However, if all four faces have the same area, the cubic $\mathcal{C}$ is *reducible*; this will be discussed in detail in Section 2.3. □

2.3. **The reducible cases.** Now let $\mathcal{C}$ be reducible. We distinguish between the case $A_1 = A_2 = A_3 = A_4$ and the case that not all of $A_1, A_2, A_3, A_4$ are equal.

2.3.1. *The case of an equifacial tetrahedron.* If $A_1 = A_2 = A_3 = A_4$ then the tetrahedron with vertices $c_1, \ldots, c_4$ defines a (not necessarily regular) equifacial tetrahedron. The cubic equation (9) decomposes into the union of three lines,

$$(t_1 + t_2)(t_2 + t_3)(t_3 + t_1) = 0. \tag{17}$$

We consider the line $t_1 + t_2 = 0$, the other two cases are symmetric. The line $t_1 + t_2 = 0$ can be parametrized by

$$t_1 = 1, \quad t_2 = -1, \quad t_3 = \lambda, \qquad -\infty < \lambda \le \infty. \tag{18}$$

Substituting these expressions into the square of (3) yields a polynomial equation $P_4(\lambda) = 0$ of degree at most 4 in $\lambda$. We show that the polynomial $P_4$ cannot degenerate to zero; hence, the equation has at most 4 solutions. For a polynomial $q$ in the variable $\lambda$, let $\mathrm{Coeff}_{\lambda,k}(q)$, denote the coefficient of $\lambda^k$ in the polynomial $q$. In the following computations no higher power in $\lambda$ than the inspected one can occur. Since in (18) the degree of $t_3$ is larger than the degree of $t_2$, we obtain

$$\mathrm{Coeff}_{\lambda,2}\left((c_1 \times s)^2\right) = 4A_2^2, \quad \mathrm{Coeff}_{\lambda,2}\left((c_2 \times s)^2\right) = 4A_1^2, \quad \mathrm{Coeff}_{\lambda,2}\left((c_3 \times s)^2\right) = 0.$$

Hence, (4) implies

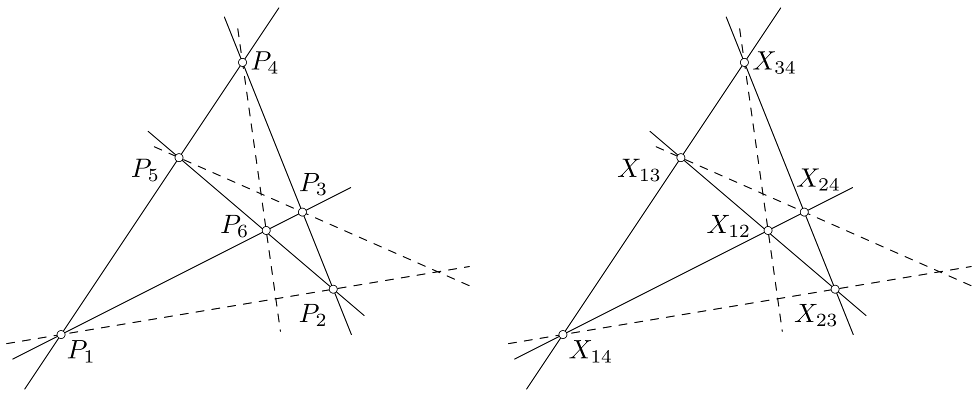
$$\mathrm{Coeff}_{\lambda,4}\left(\left(M^{-1}((c_1 \times s)^2, (c_2 \times s)^2, (c_3 \times s)^2)^T\right)^2\right) = \left(\frac{4A_1A_2||n_1 + n_2||}{3V}\right)^2.$$

Since $\mathrm{Coeff}_{\lambda,2}(s^2) = c_3^2$, the coefficient of degree 4 in $P_4$ vanishes if and only if

$$\frac{2A_1A_2||n_1 + n_2||}{3V} = rc_3^2. \tag{19}$$

Let $r_0 > 0$ be the radius defined by this equation. For $0 < r \ne r_0$, the leading coefficient of $P_4$ does not vanish, and $P_4$ has exactly 4 zeroes in $\mathbb{C}$ counted with multiplicity.

For $r = r_0$, the polynomial $P_4$ is of degree at most 3. However, it cannot degenerate to the zero polynomial, since the polynomials for $r \ne r_0$ have (possibly complex) zeroes. In particular, at any of these zeroes $\lambda$ the polynomial $P_4$ for $r = r_0$ does not evaluate to 0. Hence, for $r = r_0$ there are at most 3 solutions in $\mathbb{C}$. Additionally, in this




(a) Complete quadrilateral (b) Configuration of the points $X_{ij}$

FIGURE 2. A complete quadrilateral consists of 4 lines and 6 vertices $P_1, \ldots, P_6$; the three diagonals are drawn by dashed lines. Figure (b) shows a complete quadrilateral stemming from the reducible case.

case we have to consider the solution $\lambda = \infty$. More precisely, $r_0$ can be interpreted as follows. For $\lambda = \infty$ within the parametrization, the resulting radius $r_\infty$ is computed – in the same way as $r_0$ – by using the leading coefficients. This implies $r_\infty = r_0$.

Altogether, for any given radius $r > 0$, there are at most $3 \cdot 4 = 12$ common tangents to the four balls $B(c_i, r)$.

2.3.2. *The remaining reducible cases.* Now consider the case that not all of the faces have the same area. We can interpret the homogeneous cubic equation (6) as a cubic curve in projective plane $\mathbb{P}^2$ (for the theory of plane algebraic curves the reader is referred to [4, 31], see also [5]).

We discuss and parametrize the plane algebraic curve $\mathcal{C}$ defined by (9). As already mentioned, the directions of the six tetrahedron edges give points on $\mathcal{C}$. In particular, let $X_{ij} := c_i - c_j$, $1 \le i < j \le 4$.

Following [24], we characterize the relationships between those six points on $\mathcal{C}$. Due to (7) the $t$-coordinates of $X_{14}$, $X_{24}$, $X_{34}$, $X_{12}$, $X_{13}$, $X_{23}$ are $(1,0,0)^T$, $(0,1,0)^T$, $(0,0,1)^T$, $(1,-1,0)^T$, $(1,0,-1)^T$, and $(0,1,-1)^T$, respectively.

For any of the four tetrahedron faces, the set of directions parallel to that face defines a hyperplane through the origin (excluding the origin itself); hence, this set of directions defines a line in $\mathbb{P}^2$. Of course, this remains true even after applying the linear variable transformations.

In order to characterize this configuration of four lines, the following notation will be useful. A complete quadrilateral in projective plane consists of four lines in general position and the six points in which the lines intersect [6], see Figure 2(a); here, general position means that no three lines have a common point of intersection.

Since there does not exist a vector which is parallel to more than two faces, the four lines define a complete quadrilateral. One line contains the set of points $\{X_{12}, X_{23}, X_{34}\}$, another one contains $\{X_{12}, X_{24}, X_{14}\}$, the third one contains $\{X_{13}, X_{34}, X_{24}\}$, and the fourth one contains $\{X_{23}, X_{34}, X_{24}\}$. In particular, the points $X_{ij}$ are the 6 vertices of the complete quadrilateral. Figure 2(b) illustrates this configuration.

Since the cubic $\mathcal{C}$ is reducible, it can be decomposed into a line and a (not necessarily irreducible) conic section. An irreducible conic section intersects with any given line in at most two points; this implies that an irreducible conic section does not contain three collinear points. Hence, one of the factors of $\mathcal{C}$ is a line $l$ that contains at least two of the six points $X_{ij}$.

Whenever some direction vector $s$ of a common tangent is parallel to a face of the tetrahedron, $s$ can only take the direction of an edge; otherwise, the tangent cannot have the same distance from all three vertices of that face. For this reason, $l$ cannot contain two points from the same line of the complete quadrilateral. Hence, $l$ must be one of the three diagonals of the complete quadrilateral. Any of these diagonals contains two points $X_{ij}$, $X_{kl}$ which do not have any common index.

Without loss of generality we can assume that $l$ contains $X_{13}$ and $X_{24}$. First we show that this implies $A_1 = A_3$ and $A_2 = A_4$. Since the $t$-coordinates of $X_{13}$ and $X_{24}$ are $(1, 0, -1)^T$ and $(0, 1, 0)^T$, $l$ is given by $t_1 + t_3 = 0$. The coefficient $\tau$ of $t_2^2$ in the remaining conic section must be nonzero, because the coefficient of $t_1 t_2^2$ in (9) is nonzero. Comparing the coefficients of $t_1 t_2^2$ and $t_3 t_2^2$ in (9) with the corresponding coefficients in the decomposed representation yields $\tau = A_1^2 = A_3^2$; hence $A_1 = A_3$. Furthermore, let $\tau_1$ and $\tau_2$ denote the coefficients of $t_1 t_2$ and $t_2 t_3$ in the remaining conic section, respectively. Comparing the coefficients of $t_1^2 t_2$ yields $\tau_1 = A_3^2 = A_1^2$. In the same way, with regard to $t_2 t_3^2$ and $t_1 t_2 t_3$ we obtain $\tau_1 = A_1^2$, and $2F = 2A_1^2$, whence (by definition of $F$): $A_2 = A_4$. Hence, the remaining conic section results to

$$A_1^2(t_1 t_2 + t_2^2 + t_2 t_3) + A_2^2 t_1 t_3 = 0. \tag{20}$$

Since, by assumption, not all of the faces have the same area, we have $A_1 \neq A_2$. Furthermore, it can be verified that for $A_1 \neq A_2$ the conic section (20) is irreducible.

Parametrizing the line $l$ can be done like in the case $A_1 = A_2 = A_3 = A_4$. In particular, the line $l$ gives at most 4 common tangents.

In order to parametrize (20), we intersect the conic with a suitable pencil of lines. First observe that $X_{14}$ is a regular point on the conic with tangent $A_1^2 t_2 + A_2^2 t_3 = 0$. Then consider the pencil of lines

$$\lambda A_1^2 t_2 - (A_1^2 t_2 + A_2^2 t_3) = 0, \qquad -\infty < \lambda \leq \infty$$

with apex $X_{14}$. In particular, solving for $t_3$ gives $t_3 = A_1^2(\lambda - 1)t_2 / A_2^2$. The parameter value $\lambda = 0$ gives the tangent in $X_{14}$; the parameter value $\lambda = \infty$ yields $t_2 = 0$, which is the line through $X_{14}$ and $X_{34}$. Replacing $t_3$ in (20) via the pencil equation and eliminating the linear factor $t_2$ caused by the apex $(1, 0, 0)^T$ yields $(A_1^2(\lambda - 1) + A_2^2)t_2 + A_2^2 \lambda t_1 = 0$. This gives the parametrization

$$(t_1, t_2, t_3)^T = (-A_1^2(\lambda - 1) - A_2^2, A_2^2 \lambda, A_1^2(\lambda - 1)\lambda)^T, \qquad -\infty < \lambda \leq \infty.$$

Consequently,

$$\operatorname{Coeff}_{\lambda^4}((c_1 \times s)^2) = 4A_1^4 A_2^2, \quad \operatorname{Coeff}_{\lambda^4}((c_2 \times s)^2) = 4A_1^6, \quad \operatorname{Coeff}_{\lambda^4}((c_3 \times s)^2) = 0.$$

Here, the radius $r_0$ where the leading coefficient vanishes is the same one as in (19) and refers to the situation $\lambda = \infty$. Hence, the conic section gives at most 8 common tangents. Altogether, we obtain at most $4 + 8 = 12$ common tangents in this reducible case.

## 3. A Configuration With 12 Common Tangents

The easiest example of a construction with 12 real tangents stems from a regular tetrahedron configuration of $c_1, \ldots, c_4$. However, since in Section 4 we will relate the affinely dependent configurations to the limit case of affinely independent configurations, we exhibit a more general class of configurations with 12 real tangents.

Namely, consider an equifacial tetrahedron, as in Section 2.3.1. It is well-known that the vertices of such a tetrahedron $T$ can be regarded as four pairwise non-adjacent vertices of a rectangular box. Hence, there exists a representation $c_1 = (\lambda_1, \lambda_2, \lambda_3)^T$, $c_2 = (\lambda_1, -\lambda_2, -\lambda_3)^T$, $c_3 = (-\lambda_1, \lambda_2, -\lambda_3)^T$, $c_4 = (-\lambda_1, -\lambda_2, \lambda_3)^T$ with $\lambda_1, \lambda_2, \lambda_3 > 0$.

By assuming $s^2 = 1$, we have $p = s \times m$, and Equation (1) takes the form

$$\langle c_i, s\rangle^2 + 2\langle c_i, p\rangle = \sum_{j=1}^{3} \lambda_j^2 + p^2 - r^2. \tag{21}$$

Subtracting these equations pairwise gives

$$4(\lambda_2 p_2 + \lambda_3 p_3) = -4(\lambda_1\lambda_3 s_1 s_3 + \lambda_1\lambda_2 s_1 s_2)$$

(for indices 1, 2) and analogous equations, so that

$$\lambda_1 p_1 = -\lambda_2\lambda_3 s_2 s_3, \quad \lambda_2 p_2 = -\lambda_1\lambda_3 s_1 s_3, \quad \lambda_3 p_3 = -\lambda_1\lambda_2 s_1 s_2.$$

Since $\langle p, s\rangle = 0$, this yields $s_1 s_2 s_3 = 0$. By assuming without loss of generality $s_1 = 0$, we obtain

$$p = \left(-\frac{\lambda_2\lambda_3}{\lambda_1} s_2 s_3, 0, 0\right).$$

So (21) becomes

$$\lambda_2^2 s_2^2 + \lambda_3^2 s_3^2 = \sum_{j=1}^{3} \lambda_j^2 + \left(-\frac{\lambda_2\lambda_3}{\lambda_1} s_2 s_3\right)^2 - r^2,$$

which, by using $s_2^2 + s_3^2 = 1$, gives

$$\lambda_2^2\lambda_3^2 s_2^4 + (\lambda_1^2\lambda_2^2 - \lambda_1^2\lambda_3^2 - \lambda_2^2\lambda_3^2) s_2^2 + \lambda_1^2(r^2 - \lambda_1^2 - \lambda_2^2) = 0.$$

There are two distinct real solutions for $s_2^2$ if and only if

$$\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 > 2\lambda_1\lambda_2\lambda_3 r. \tag{22}$$

Since the volume of $T$ is $8\lambda_1\lambda_2\lambda_3/3$ and the area $A$ of a face is $2\sqrt{\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2}$, (22) becomes $A^2/4 > 3Vr/4$. In case of reality, both solutions for $s_2^2$ are positive if and only if

$$r^2 > \lambda_1^2 + \lambda_2^2 \tag{23}$$

and

$$\lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 > \lambda_1^2\lambda_2^2. \tag{24}$$

Hence, there will be 12 distinct real common tangents to $B(c_1, r)$, ..., $B(c_4, r)$ if and only if $r$ satisfies (22) and the three inequalities such as (23), and if in addition the tetrahedron $c_1, \ldots, c_4$ satisfies the three inequalities such as (24). Since $2\sqrt{\lambda_1^2 + \lambda_2^2}$ is the length of one of the edges, it follows that we require

$$\frac{e}{2} < r < \frac{A^2}{3V},$$

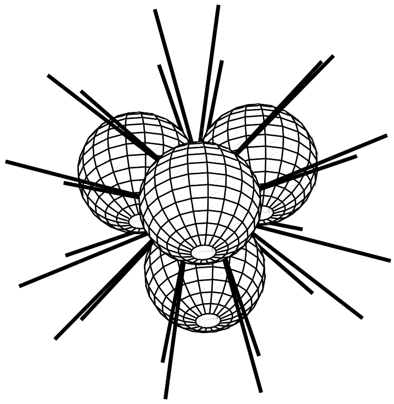


FIGURE 3. Construction with 12 tangents. Note that the four balls slightly intersect with each other.

where $e$ is the length of the longest edge; also, expressing (24) by using the area $A$ gives

$$A^2 > 8\lambda_1^2\lambda_2^2.$$

Applying the formula "$A = \frac{1}{2}ab\sin\gamma$" on the left side and the Laws of Cosines on the right side establishes a relation among the angles $\alpha$, $\beta$, and $\gamma$ of the face triangle:

$$\tan\beta\tan\gamma > 2.$$

Since $\tan\alpha\tan\beta\tan\gamma = \tan\alpha + \tan\beta + \tan\gamma$ in a triangle and since all three angles are acute, we can conclude:

**Lemma 3.** *Let $c_1, \ldots, c_4$ constitute an equifacial tetrahedron, and let $r > 0$. Then there are exactly 12 distinct real common tangents to $B(c_1, r), \ldots, B(c_4, r)$ if and only if*

a)
$$\frac{e}{2} < r < \frac{A^2}{3V},$$
*where $e$ is the length of the longest edge, $A$ is the area of a face, and $V$ is the volume of the tetrahedron; and*

b) *the angles in one (and hence in all) of the face triangles satisfy*

$$\tan\beta + \tan\gamma > \tan\alpha, \tag{25}$$

*where $\alpha$ is the largest of the three angles.*

Figure 3 depicts the configuration $c_1 = (4,4,4)^T$, $c_2 = (4,-4,-4)^T$, $c_3 = (-4,4,-4)^T$, $c_4 = (-4,-4,4)^T$ and radius $\sqrt{33}$, which gives 12 tangents by Lemma 3.

## 4. Affinely Dependent Centers

Let $c_1, \ldots, c_4$ be non-collinear points in the $xy$-plane. We now look for circular cylinders $C$ with radius $r$, whose surface contains $c_1, \ldots, c_4$. Unless the axis of $C$ is parallel to the $xy$-plane, the intersection of $C$ with the $xy$-plane is an ellipse with smaller half-axis $r$. We can assume that none of the given points is contained in the convex hull of the other points; otherwise, three points are collinear (giving at most two circular cylinders) or there is no circular cylinder.

An axis parallel to the $xy$-plane is only possible if the quadrangle formed by $c_1, \ldots, c_4$ is a trapezoid. Since such an axis can be located above or below the $xy$-plane, and since a parallelogram has two pairs of parallel edges, we obtain at most 4 circular cylinders with axis parallel to the $xy$-plane. If $c_1, \ldots, c_4$ constitute a trapezoid but not a parallelogram, this number reduces to 2.

Now any ellipse with smaller half-axis $r$ passing through $c_1, \ldots, c_4$ defines two circular cylinders with radius $r$, whose intersection with the $xy$-plane gives the ellipse; in case of a circle these two cylinders coincide.

Consider a general ellipse

$$E : ax^2 + 2hxy + by^2 + 2gx + 2fy + d = 0,$$

in other form

$$a(x - x_0)^2 + 2h(x - x_0)(y - y_0) + b(y - y_0)^2 + d' = 0. \tag{26}$$

Comparing the coefficients of the two forms yields

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -g \\ -f \end{pmatrix}.$$

With the standard invariants of conic section classification

$$\begin{aligned} I_1 &= \operatorname{tr} \begin{pmatrix} a & h \\ h & b \end{pmatrix} = a + b, \\ I_2 &= \det \begin{pmatrix} a & h \\ h & b \end{pmatrix} = ab - h^2, \\ I_3 &= \det \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & d \end{pmatrix}, \end{aligned}$$

and the notation $F := gh - af$, $G := fh - bg$, we obtain $x_0 = G/I_2$, $y_0 = F/I_2$. In particular, since $E$ is an ellipse, we have $I_3 \neq 0$, $I_2 > 0$, and $I_1 I_3 < 0$. Consequently, the absolute term $d'$ in (26) results to

$$\begin{aligned} d' &= \frac{1}{I_2^2} \begin{pmatrix} G & F & I_2 \end{pmatrix} \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & d \end{pmatrix} \begin{pmatrix} G \\ F \\ I_2 \end{pmatrix} \\ &= \frac{1}{I_2} (gG + fF + dI_2) \\ &= \frac{I_3}{I_2}. \end{aligned}$$

$E$ has smaller half-axis $r$ if and only if both eigenvalues of the matrix

$$-\frac{I_2}{I_3} \begin{pmatrix} a & h \\ h & b \end{pmatrix}$$

are positive and the larger one is $1/r^2$, i.e., if the largest solution of the quadratic equation in $\lambda$

$$I_3^2 \lambda^2 + I_1 I_2 I_3 \lambda + I_2^3 = 0$$

is $1/r^2$ and both solutions are positive.

It is well-known that the set of ellipses passing through four given points are members of the pencil of conics $S_1 + \mu S_2$, with $S_1$, $S_2$ equations of two arbitrary conics passing through the four points (see, e.g., [21]). Let $I_1(\mu)$, $I_2(\mu)$, $I_3(\mu)$ be the invariants of $S(\mu) := S_1 + \mu S_2$, so that $I_i(\mu)$ is a polynomial in $\mu$ of degree $i$. Any ellipse $S(\mu)$ with smaller half-axis $r$ passing through $c_1, \ldots, c_4$ must necessarily satisfy the condition

$$\frac{I_3(\mu)^2}{r^4} + \frac{I_1(\mu)I_2(\mu)I_3(\mu)}{r^2} + I_2(\mu)^3 = 0. \tag{27}$$

Equation (27) is of order 6 in $\mu$. The two cases where the coefficient of degree 6 vanishes stem from our affine notation of a pencil and refer to the case $\mu = \infty$.

Altogether, there are at most 12 circular cylinders with smaller half-axis $r$ passing through $c_1, \ldots, c_4$, whose axis is not parallel to the $xy$-plane. It remains to show that this number can be decreased in the case of parallelograms and trapezoids.

For the parallelogram case, suppose that the parallelogram is given by the two pairs of parallel lines $y = \gamma$, $y = -\gamma$, and $y = \alpha x + \beta$, $y = \alpha x - \beta$ for some constants $\alpha, \beta, \gamma > 0$. As generators $S_1$, $S_2$ of the pencil of conics through the four vertices, we can choose the two degenerated conics given by the two pairs of lines

$$S_1 : (y - \gamma)(y + \gamma) = 0,$$
$$S_2 : (y - \alpha x - \beta)(y - \alpha x + \beta) = 0.$$

Since both the center of $S_1$ and the center of $S_2$ is $(x_0, y_0) = (0, 0)^T$, each ellipse in the pencil $S_1 + \mu S_2$ has center $(0, 0)^T$. Hence, any ellipse $S(\mu)$ in the pencil is of the form

$$ax^2 + 2hxy + by^2 + 1 = 0.$$

Since

$$I_3(\mu) = \det \begin{pmatrix} a_1 + \mu a_2 & h_1 + \mu h_2 & 0 \\ h_1 + \mu h_2 & b_1 + \mu b_2 & 0 \\ 0 & 0 & 1 + \mu \end{pmatrix} = I_2(\mu)(1 + \mu),$$

Equation (27) becomes

$$I_2(\mu)^2 \left((1 + \mu)^2 r^4 + I_1(\mu)(1 + \mu) r^2 + I_2(\mu)\right) = 0.$$

Consequently, since $I_2(\mu) \neq 0$ for any ellipse in the pencil, we obtain a quadratic condition in $\mu$.

For the trapezoid case, suppose that two vertices are located on the line $y = 0$ and that two vertices are located on the line $y = 2\alpha$ with $\alpha > 0$. Then $S_2$ can be chosen as the degenerated conic consisting of two parallel lines

$$S_2 : y(y - 2\alpha) = 0.$$

The representation matrix of the ellipse $S_1 + \mu S_2$ is of the form

$$\begin{pmatrix} a_1 & h_1 & f_1 \\ h_1 & b_1 + \mu & g_1 - \alpha\mu \\ f_1 & g_1 - \alpha\mu & d_1 \end{pmatrix}.$$

Therefore $I_2(\mu)$ is only linear in $\mu$, and $I_3(\mu)$ is only quadratic in $\mu$. Hence, Equation (27) is only of degree 4 in $\mu$. We can conclude:

**Corollary 4.** *Let $c_1, \ldots, c_4$ be affinely dependent, and let $r > 0$. If $c_1, \ldots, c_4$ form a trapezoid, then there are at most 10 common tangents to $B(c_1, r), \ldots, B(c_4, r)$. If $c_1, \ldots, c_4$ form a parallelogram, then there are at most 8 common tangents to $B(c_1, r), \ldots, B(c_4, r)$.*

Concerning constructions with many real tangents in the affinely dependent case, our best construction gives 8 real tangents. For an easy construction with 8 real tangents, let $c_1, \ldots, c_4$ constitute a square with edge length $e$. For $e/2 < r < \sqrt{2}e/2$ two neighboring balls intersect with each other, but a ball does not intersect with its opposite partner.

Hence, the opposite pairs of the intersection circles are disjoint, and they lie on the vertical planes bisecting opposite edges of the square. The four common tangents to such a pair of intersection circles are common tangents to the four balls which altogether gives 8 common tangents.

It might be possible that the bound of 12 is not tight in the affinely dependent case. In fact, our proof replaces the condition "$1/r^2$ is the largest eigenvalue and both eigenvalues are positive" by the weaker condition "$1/r^2$ is an eigenvalue". In contrast to the affinely independent case (where our construction with 12 tangents was based on symmetry), Corollary 4 implies that symmetric constructions yield fewer than 12 tangents in the affinely dependent case.

Finally, we want to explain what happens to some of the tangents when trying to approach a rectangle configuration (with at most 8 common tangents) as a limit case of affinely independent centers. Let $c_1, \ldots, c_4$ constitute a rectangle in the $xy$-plane. By lifting two opposite of the four centers appropriately, we can establish a configuration with 12 tangents by Lemma 3. By reducing the height of the resulting box with base rectangle in the $xy$-plane, we can interpret the rectangle as limit case of this flattening process. Now Lemma 3 explains where some of the 12 tangents get lost in this limit process. Namely, flattening of the box implies that the triangular faces of the tetrahedron tend towards rectangular triangles. However, then $\tan \alpha$ in (25) tends to infinity, and (25) is violated at some stage of this process. Intuitively, this means that some of the tangents get lost even before the limit case is reached.

## Appendix: The Pedal Surface of a Tetrahedron

Let $c_1, \ldots, c_4 \in \mathbb{R}^3$ be the vertices of a tetrahedron $T$, and let $N_i$ denote the unit outer normal vector of the face opposite to $c_i$. Further, let $A_i$ denote the area of that face. An elementary computation (using (5), $n_4 := ((c_1 - c_2) \times (c_3 - c_2))/2$ and a suitable orientation) shows

$$A_1 N_1 + A_2 N_2 + A_3 N_3 + A_4 N_4 = 0. \tag{28}$$

We would like to write up the equation of the so-called *pedal surface* $\Sigma$ of the tetrahedron, i.e., the locus of the points $x$ such that the feet of the perpendiculars from $x$ to the planes supporting the faces of the tetrahedron lie in a plane.

Let $v_i \in \mathbb{R}^3$ be the vector connecting $x$ to the foot of the perpendicular from $x$ to the plane supporting the face opposite to $c_i$. The feet of these perpendiculars (i.e., the endpoints of these vectors) are co-planar if and only if the determinant of the $4 \times 4$-matrix with $i$-th row $(v_i, 1)$ vanishes. The latter condition is equivalent to

$$(v_2\, v_3\, v_4) - (v_1\, v_3\, v_4) + (v_1\, v_2\, v_4) - (v_1\, v_2\, v_3) = 0,$$

where $(a\,b\,c) = \langle a \times b, c\rangle$ is the scalar triple product. If $b_i$ is defined by $v_i = b_i N_i$, then the equation becomes

$$\frac{(N_2\,N_3\,N_4)}{b_1} - \frac{(N_1\,N_3\,N_4)}{b_2} + \frac{(N_1\,N_2\,N_4)}{b_3} - \frac{(N_1\,N_2\,N_3)}{b_4} = 0. \tag{29}$$

It follows from (28) by taking scalar products with $N_2 \times N_3$ that

$$A_1(N_1\,N_2\,N_3) + A_4(N_2\,N_3\,N_4) = 0,$$

and from the analogous relations we obtain that for some $b \in \mathbb{R}$,

$$(N_2\,N_3\,N_4) = bA_1, \quad (N_1\,N_3\,N_4) = -bA_2, \quad (N_1\,N_2\,N_4) = bA_3, \quad (N_1\,N_2\,N_3) = -bA_4.$$

Comparing this with (29) yields

$$\frac{A_1}{b_1} + \frac{A_2}{b_2} + \frac{A_3}{b_3} + \frac{A_4}{b_4} = 0. \tag{30}$$

Let $t_1, \ldots, t_4$ denote the projective barycentric coordinates of $x$ relative to $c_1, \ldots, c_4$. Notice that $t_i$ is proportional to $b_i A_i$ (cf. [6]). Therefore, $x$ satisfies the required property if and only if

$$\frac{A_1^2}{t_1} + \frac{A_2^2}{t_2} + \frac{A_3^2}{t_3} + \frac{A_4^2}{t_4} = 0, \tag{31}$$

as desired.

## References


[1] P.K. Agarwal, B. Aronov, and M Sharir. Line transversals of balls and smallest enclosing cylinders in three dimensions. *Discrete Comput. Geom.*, 21:373–388, 1999.

[2] H.F. Baker. *Principles of Geometry*, volume 4 (Higher Geometry). Cambridge University Press, 1925.

[3] O. Bottema and G.R. Veldkamp. On the lines in space with equal distances to $n$ given points. *Geom. Dedicata*, 6:121–129, 1977.

[4] E. Brieskorn and H. Knörrer. *Plane Algebraic Curves*. Birkhäuser, Basel, 1986.

[5] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms*. Springer-Verlag, New York, 2nd edition, 1996.

[6] H.S.M. Coxeter. *Introduction to Geometry*. John Wiley & Sons, 1961.

[7] M. de Berg. *Ray Shooting, Depth Orders and Hidden Surface Removal*, volume 703 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, 1993.

[8] M. de Guzmán. An extension of the Wallace-Simson Theorem: Projecting in arbitrary directions. *Amer. Math. Monthly*, 106(6):574–580, 1999.

[9] C. Durand. *Symbolic and Numerical Techniques for Constraint Solving*. PhD thesis, Purdue University, 1998.

[10] W. Fulton. Personal communication.

[11] W. Fulton. *Introduction to Intersection Theory in Algebraic Geometry*. CBMS 54. AMS, 2nd edition, 1996.

[12] J. Harris. *Algebraic Geometry. Graduate Texts in Mathematics*, volume 133. Springer-Verlag, New York, 1992.

[13] R. Hartshorne. *Algebraic Geometry. Graduate Texts in Mathematics*, volume 52. Springer-Verlag, New York, 1977.

[14] A. Henderson. *The Twenty–Seven Lines upon the Cubic Surface*. Cambridge Tracts, vol. 13, 1911.

[15] W. Hodge and D. Pedoe. *Methods of Algebraic Geometry*, volume 2. Cambridge University Press, 1952.
[16] F. Hohenberg. Das Apollonische Problem im $R^n$. *Deutsche Mathematik*, 7(1):78–81, 1942.
[17] A. Karger. Classical geometry and computers. *J. Geom. Graph.*, 2:7–15, 1998.
[18] S.L. Kleiman and D. Laksov. Schubert calculus. *Amer. Math. Monthly*, 79:1061–1082, 1972.
[19] D. Larman. Problem posed in the Problem Session of the DIMACS Workshop on Arrangements, Rutgers University, New Brunswick, NJ, 1990.
[20] M. Pellegrini. Ray shooting and lines in spaces. In J.E. Goodman and J. O'Rourke, editors, *The CRC Handbook of Discrete and Computational Geometry*, pages 599–614. CRC Press, Boca Raton, 1997.
[21] A. Robson. *An Introduction to Analytical Geometry*, volume 1. Cambridge University Press, 1940.
[22] F. Ronga, A. Tognoli, and T. Vust. The number of conics tangent to 5 given conics: the real case. *Rev. Mat. Univ. Complut. Madrid*, 10:391–421, 1997.
[23] G. Salmon. *A Treatise on the Analytic Geometry of Three Dimensions*, volume 1. Longmans and Green, London, 7th edition, 1928. Reprinted by Chelsea Publishing Company, New York, 1958.
[24] H. Schaal. Ein geometrisches Problem der metrischen Getriebesynthese. In *Sitzungsber., Abt. II, Österr. Akad. Wiss.*, volume 194, pages 39–53, 1985.
[25] E. Schömer, J. Sellen, M. Teichmann, and C.K. Yap. Efficient algorithms for the smallest enclosing cylinder problem. In *Proc. 8th Canad. Conf. Comput. Geom.*, pages 264–269, 1996.
[26] H. Schubert. Eine geometrische Eigenschaft der 16 Kugeln, welche 4 beliebig gegebene Kugeln berühren. *Zeitschrift für Mathematik und Physik*, 14:506–513, 1869.
[27] H. Schubert. *Kalkül der abzählenden Geometrie*. Springer-Verlag, Berlin, 1879. Reprinted by Springer-Verlag, Berlin (1979).
[28] J. Steiner. Elementare Lösung einer geometrischen Aufgabe und über einige damit in Beziehung stehende Eigenschaften der Kegelschnitte. *Journal für die reine und angewandte Mathematik*, 37:161–192, 1847.
[29] T. Theobald. Work in preparation.
[30] J. Verschelde. Polynomial homotopies for dense, sparse and determinantal systems. Preprint, 1999.
[31] G. Walker. *Algebraic Curves*. Princeton University Press, 1950. Reprinted by Springer-Verlag, New York (1979).

I.G. Macdonald: 56 High Street, Steventon, Oxfordshire OX13 6RS, U.K.

János Pach: Hungarian Academy of Sciences, Pf. 127, Budapest, Hungary; and Courant Institute of Mathematical Sciences, New York University, New York, NY 10012

Thorsten Theobald: Zentrum Mathematik, Technische Universität München, D–80290 München, Germany