SYMMETRIC SAGE AND SONC FORMS, EXACTNESS AND QUANTITATIVE GAPS

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Dedicated to the memory of Ágnes Szántó

Abstract. The classes of sums of arithmetic-geometric exponentials (SAGE) and of sums of nonnegative circuit polynomials (SONC) provide nonnegativity certificates which are based on the inequality of the arithmetic and geometric means. We study the cones of symmetric SAGE and SONC forms and their relations to the underlying symmetric nonnegative cone.

As main results, we provide several symmetric cases where the SAGE or SONC property coincides with nonnegativity and we present quantitative results on the differences in various situations. The results rely on characterizations of the zeroes and the minimizers for symmetric SAGE and SONC forms, which we develop. Finally, we also study symmetric monomial mean inequalities and apply SONC certificates to establish a generalized version of Muirhead's inequality.

1. Introduction

The inequality of the arithmetic and geometric means (AM/GM inequality) is one of the classical topics in calculus which also can be applied in various contexts. Building on work of Reznick [33] and further developed by Pantea, Koeppl and Craciun [31], Iliman and de Wolff [20] as well as Chandrasekaran and Shah [7], there has recently been renewed interest in polynomials and more generally signomials (i.e., exponential sums), whose nonnegativity results from applying the weighted AM/GM inequality. For example, given $\alpha_0, \ldots, \alpha_m \in \mathbb{R}^n$ and $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^n_+$ with $\sum_{i=1}^m \lambda_i = 1$ and $\sum_{i=1}^m \lambda_i \alpha_i = \alpha_0$, the signomial

$$\sum_{i=1}^m \lambda_i \exp(\langle \alpha_i, x \rangle) - \exp(\langle \alpha_0, x \rangle)$$

is nonnegative on $\mathbb{R}^n$ and a similar results holds for polynomials. To simplify notation, we abbreviate polynomials and signomials shortly as *forms*.

Since sums of nonnegative forms are nonnegative as well, this basic idea defines certain convex cones of nonnegative forms. For signomials with support set $\mathcal{T}$, that cone is denoted as the SAGE cone $C_{\mathrm{SAGE}}(\mathcal{T})$ (*Sums of Arithmetic-Geometric Exponentials* [7]) and for polynomials, it is denoted as the SONC cone $C_{\mathrm{SONC}}(\mathcal{T})$ (*Sums of Nonnegative Circuits* [20]). These nonnegativity certificates enrich and can be combined with other nonnegativity certificates such as sums of

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squares in the polynomial setting. In optimization, the cones based on the AM/GM inequality can be used to determine lower bounds of signomials (and polynomials) through

$$f^{\mathrm{SAGE}} \;=\; \sup\{\lambda \in \mathbb{R} \;:\; f - \lambda \in C_{\mathrm{SAGE}}(\mathcal{T})\},$$

which can be numerically computed using relative entropy programming. These techniques rely on the fact that every SONC form p (and similarly, SAGE forms) can be written as a sum of nonnegative circuit polynomials supported on the support of p ([38], see also [27, 32]). The AM/GM techniques can also be extended to constrained settings ([13, 28, 29, 37]). For the second-order representability of the SAGE cone and the SONC cone see [4, 24, 30] and for combining the SONC cone with the cone of sums of squares see [12].

So far, only few theoretical results are known concerning when the bounds are exact and how good are the bounds when they are not exact. Concerning exactness, Wang [38] presented a class of polynomials with several negative terms, for which the SONC bound coincides with the true minimum. A main case of this class is a Newton simplex whose supports of the negative terms are contained in the interior of the simplex, see also [20, 27] for the characterization of this class. Moreover, in [38, Theorem 4.1] a generalization of that main class is given.

In many related areas, the use of symmetries is a key technique to extend the scope of applicability of methods (see, for example, [5, 15, 23, 26, 34, 35]). In the current paper, we study symmetric SAGE and SONC forms. For the cone of sums of squares, symmetry has been studied in [6]. In [25], it was initiated to exploit symmetries in the computation of the SAGE and SONC lower bounds for linear group actions of a group G on $\mathbb{R}^n$. At the core is a symmetric decomposition of the SAGE and SONC forms in the symmetric cones $C^G_{\mathrm{SAGE}}(\mathcal{T})$ and $C^G_{\mathrm{SONC}}(\mathcal{T})$, see Proposition 2.1 in Section 2 below. Depending on the symmetry, this can lead to large gains in computation time. For the special case of the symmetric group, Heuer, de Wolff and Tran [18] gave an alternative derivation of some of the results using a generalized Muirhead inequality.

The goal of the current paper is to provide theoretical results on the structure and on the quantitative aspects concerning the cones of symmetric SAGE and SONC forms and on the SAGE relaxations. On the one hand, this is motivated by the question to understand further the symmetry reduction for AM/GM-based optimization. On the other hand, symmetry provides a natural framework to tackle the exactness question and the quantitative questions mentioned above, thus enabling to provide new non-trivial classes of signomials and polynomials for which exactness results or exact quantitative gaps can be shown.

Our contributions.

1. As a starting point, we characterize the structure of symmetry-induced circuit decompositions and the structure of the zeroes of symmetric SAGE and SONC forms with respect to the symmetric group. These results on the zeroes provide symmetric analogs of the characterizations of the zeroes in [11] and [14]. Our treatment departs from the known result that the zero set of a SAGE exponential constitutes a subspace and is therefore convex and that every SONC polynomial with a finite number of zeroes has at most one zero in the positive orthant.

In sharp contrast to this, for the rather structured class of SONC polynomials and SAGE exponentials, the minimal solutions of symmetric optimization problems are in general not symmetric. We say that these functions have the minimum *outside of the diagonal*, see Example 3.5.

2. The symmetric decomposition in [25] raised the natural question whether a symmetric version of Wang's result applies for certain classes of symmetric polynomials. We show in Theorem 4.1 that such a symmetric generalization holds for a class with one orbit of exterior and several orbits of interior points. For this class, we have SAGE exactness and we can explicitly

characterize the minimizer of such a polynomial or signomial in terms of the unique positive zero of a univariate signomial.

3. We provide several quantitative results concerning the question how far is the notion of being SAGE or SONC from being nonnegative.

(a) We classify the difference of SAGE polynomials to nonnegative polynomials for symmetric quadratic forms.

(b) We prove that already in a very restricted setting of quartic polynomials with two interior support points in the Newton polytope, the cone of symmetric SONC polynomials differs from the cone of all symmetric polynomials with that support. See Theorem 4.8. Moreover, for the underlying parametric class of quartic polynomials, we give a full characterization of the SONC/SAGE bounds and the true minima.

(c) We give a detailed study of SONC certificates in the context of monomial symmetric inequalities. On the one hand, we show that the normalized setup of such inequalities can be well certified with SONC certificates. We study this phenomenon especially in the case of the classical Muirhead inequality, which as we show can be seen as a SONC certificate. Based on this observation we also give a slight generalization of this classical inequality. On the other hand, we demonstrate a significant disparity between the capability of SONC and the potential of sums of squares in certifying the nonnegativity of symmetric inequalities which are not normalized.

The paper is structured as follows. Section 2 collects background on the SAGE and the SONC cone and symmetry techniques. Section 3. In Section 4, we compare the symmetric SAGE cone and the symmetric SONC cone with the symmetric nonnegative cone.

2. Preliminaries

Throughout the article, we use the notation $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. For a finite subset $\mathcal{T} \subset \mathbb{R}^n$, let $\mathbb{R}^{\mathcal{T}}$ be the set of $|\mathcal{T}|$-tuples whose components are indexed by the set $\mathcal{T}$. We denote by $\langle \cdot, \cdot \rangle$ the standard Euclidean inner product in $\mathbb{R}^n$.

2.1. **The SAGE and the SONC cone.**

For a given non-empty finite set $\mathcal{T} \subset \mathbb{R}^n$, the SAGE cone refers to signomials supported on $\mathcal{T}$. Formally, the SAGE cone $C_{\mathrm{SAGE}}(\mathcal{T})$ is defined as

$$C_{\mathrm{SAGE}}(\mathcal{T}) := \sum_{\beta \in \mathcal{T}} C_{\mathrm{AGE}}(\mathcal{T} \setminus \{\beta\}, \beta),$$

where for $\mathcal{A} := \mathcal{T} \setminus \{\beta\}$

$$C_{\mathrm{AGE}}(\mathcal{A}, \beta) := \Big\{ f = \sum_{\alpha \in \mathcal{A}} c_\alpha e^{\langle \alpha, x \rangle} + c_\beta e^{\langle \beta, x \rangle} \ : \ c_\alpha \geqslant 0 \text{ for } \alpha \in \mathcal{A},\, c_\beta \in \mathbb{R},\, f \geqslant 0 \text{ on } \mathbb{R}^n \Big\}$$

denotes the nonnegative signomial which may only have a negative coefficient in the term indexed by β (see [7]). The elements in these cones are called *SAGE signomials* and *AGE signomials*, respectively. If $f \in C_{\mathrm{AGE}}(\mathcal{A}, \beta)$ and $\mathcal{A}$ and β are clear from the context, we write in brief just *f is AGE*. Similarly, but only for $\mathcal{T} \subset \mathbb{N}^n$, define $C_{\mathrm{SONC}}(\mathcal{T})$ as

$$C_{\mathrm{SONC}}(\mathcal{T}) := \sum_{\beta \in \mathcal{T}} C_{\mathrm{AG}}(\mathcal{T} \setminus \{\beta\}, \beta),$$

where for $\mathcal{A} := \mathcal{T} \setminus \{\beta\}$

$$C_{\mathrm{AG}}(\mathcal{A},\beta) \;:=\; \Big\{f = \sum_{\alpha\in\mathcal{A}} c_\alpha x^\alpha + c_\beta x^\beta \;:\; c_\alpha \geqslant 0 \text{ for } \alpha\in\mathcal{A},\, c_\beta\in\mathbb{R},$$
$$c_\gamma = 0 \text{ for all } \gamma\in\mathcal{A} \text{ with } \gamma\notin(2\mathbb{N})^n,\; f\geqslant 0 \text{ on } \mathbb{R}^n\Big\}$$

denotes the nonnegative polynomials which may only have a negative coefficient in the term indexed by β. The elements in these cones are called *SONC polynomials* and *AG polynomials*, where the acronym SONC comes from the circuit decompositions discussed further below [20] and the equivalence of the definition given here was shown in [27, 38]. Note that $C_{AG}(\mathcal{A},\beta)$ refers to polynomials, whereas $C_{AGE}(\mathcal{A},\beta)$ refers to signomials. The cones $C_{\mathrm{SAGE}}(\mathcal{T})$ and $C_{\mathrm{SONC}}(\mathcal{T})$ are closed convex cones in $\mathbb{R}^{\mathcal{T}}$ (see [22, Proposition 2.10]). Membership in the convex cones can be decided in terms of relative entropy programming [27], see also [22] or [25].

2.2. **Circuit decompositions.** A simplicial circuit is a non-zero vector $\nu\in\mathbb{R}^{\mathcal{T}}$, whose positive support (denoted by ν^+) is affinely independent, whose components sum to zero and whose unique negative support element β satisfies $(\sum_{\alpha\in\nu^+}\nu_\alpha)\beta = \sum_{\alpha\in\nu^+}\nu_\alpha\alpha$. A simplicial circuit is normalized when the nonnegative components sum to 1, and hence the negative component is -1. Let $\Lambda(\mathcal{T})$ denote the set of normalized simplicial circuits of $\mathcal{T}$. In geometric terms, a normalized circuit $\lambda\in\Lambda(\mathcal{T})$ can be interpreted as the barycentric coordinates of $\lambda^- = \beta$ in terms of the vectors in λ^+.

Murray, Chandrasekaran and Wierman [27] have shown the following circuit decomposition theorem for the SAGE cone (see also Wang [38] for the variant regarding the SONC variant).

Proposition 2.1. *The cone $C_{\mathrm{SAGE}}(\mathcal{T})$ decomposes as the finite Minkowski sum*

$$C_{\mathrm{SAGE}}(\mathcal{T}) = \sum_{\lambda\in\Lambda(\mathcal{T})} C_{\mathrm{SAGE}}(\mathcal{T},\lambda) + \sum_{\alpha\in\mathcal{A}} \mathbb{R}_+\cdot\exp(\langle\alpha,x\rangle). \tag{2.1}$$

where $C_{\mathrm{SAGE}}(\mathcal{T},\lambda)$ denotes the λ-witnessed cone, that is, with $\beta := \lambda^-$,

$$C_{\mathrm{SAGE}}(\mathcal{T},\lambda) = \left\{\sum_{\alpha\in\mathcal{T}} c_\alpha \exp(\langle\alpha,x\rangle) \;:\; \prod_{\alpha\in\lambda^+}\left(\frac{c_\alpha}{\lambda_\alpha}\right)^{\lambda_\alpha} \geqslant -c_\beta,\; c_\alpha\geqslant 0 \textit{ for } \alpha\in\mathcal{T}\setminus\{\beta\}\right\}.$$

Since the SONC setting refers to nonnegativity of polynomials on the whole space $\mathbb{R}^n$, the circuit concept has to be slightly adapted. Namely, in the definition of a circuit we have to add the requirements that support vectors in λ^+ and λ^- have nonnegative integer coordinates and that the vectors in λ^+ can only have even coordinates. See [22] for an exact characterization of the extreme rays of $C_{\mathrm{SAGE}}(\mathcal{T})$ and $C_{\mathrm{SONC}}(\mathcal{T})$ in terms of the circuits.

For disjoint sets $\emptyset\neq\mathcal{A}\subset\mathbb{R}^n$ and $\mathcal{B}\subset\mathbb{R}^n$, it is convenient to denote by

$$C_{\mathrm{SAGE}}(\mathcal{A},\mathcal{B}) := \sum_{\beta\in\mathcal{B}} C_{\mathrm{AGE}}(\mathcal{A}\cup\mathcal{B}\setminus\{\beta\},\beta) \tag{2.2}$$

the *signed SAGE cone*, which allows negative coefficients only in a certain subset $\mathcal{B}$ of the support $\mathcal{A}\cup\mathcal{B}$ (see, e.g., [21, 27]).

Finally, in the case where all the exponent vectors have nonnegative coordinates, the decomposition in Proposition 2.1 can be refined with some further information on the possible positive coefficients used in the decomposition for a given β. For $\alpha\in\mathbb{R}^n$, we introduce its support

$$\mathrm{supp}(\alpha) = \{i\in\{1,\ldots,n\} \;:\; \alpha_i\neq 0\}.$$

Then we have:

Proposition 2.2. *Let* $f = \sum_{\alpha\in\mathcal{A}} c_\alpha e^{\alpha x} - de^{\beta x}$ *where* $c_\alpha \geqslant 0$ *and* $d > 0$*. Assume that every* $\alpha \in \mathcal{A}$ *is a nonnegative vector and set* $\mathcal{A}' = \{\alpha \in \mathcal{A} \ : \ \operatorname{supp}(\alpha) \subset \operatorname{supp}(\beta)\}$*. Then*

$$f \text{ is AGE } \Leftrightarrow \ \tilde{f} = \sum_{\alpha\in\mathcal{A}'} c_\alpha e^{\alpha x} - de^{\beta x} \text{ is AGE.}$$

Proof. One direction is obvious. Suppose now that f is AGE, and let λ be a normalized simplicial circuit appearing in the decomposition. Since

$$\sum_{\alpha\in\lambda^+} \lambda_\alpha \alpha = \beta,$$

for $i \notin \operatorname{supp}(\beta)$, we must have $\sum_{\alpha\in\lambda^+} \lambda_\alpha \alpha_i = 0$, which forces $\alpha_i = 0$ because by assumption $\alpha_i \geqslant 0$ for every $\alpha \in \lambda^+$. □

2.3. **Optimizing over the SAGE and SONC cones.** Since the SAGE cone is contained in the cone of nonnegative signomials, relaxing to the SAGE cone gives an approximation of the global infimum f^* of a signomial f supported on $\mathcal{T}$:

$$f^{\mathrm{SAGE}} = \sup\{\lambda \in \mathbb{R} \ : \ f - \lambda \in C_{\mathrm{SAGE}}(\mathcal{T})\}$$

satisfying $f^{\mathrm{SAGE}} \leqslant f^*$. The following statement is closely related to the strong duality statement for the SAGE bound in [27, Proposition 2]:

Proposition 2.3. *Let*

$$f = \sum_{\alpha\in\mathcal{A}} c_\alpha \exp(\langle \alpha, x\rangle) + \sum_{\beta\in\mathcal{B}} c_\beta \exp(\langle \beta, x\rangle)$$

with $c_\alpha > 0$ *for* $\alpha \in \mathcal{A}$*. Assume* $\mathcal{B} \subset \operatorname{relint}(\operatorname{conv}(\mathcal{A} \cup \{(0,\ldots,0)^T\}))$*. Then* $f^{\mathrm{SAGE}} > -\infty$*.*

Proof. The finiteness of $\mathcal{B}$ and [27, Theorem 3] allow to assume $|\mathcal{B}| = 1$. In the special case $\beta = (0,\ldots,0)^T$, consider f without its constant term f_0 is a SAGE signomial and thus $f^{\mathrm{SAGE}} \geqslant f_0 > -\infty$.

Hence, we may assume that $\beta \neq (0,\ldots,0)^T$. Then, the ray with initial point $(0,\ldots,0)^T$ and passing through β meets a facet of the convex hull of $\mathcal{A}$ at a point γ. We can express γ as

$$\gamma = \sum_{\alpha\in\mathcal{A}'} \nu'_\alpha \alpha,$$

where $\nu'_\alpha \geqslant 0$ for $\alpha \in \mathcal{A}'$, $\sum_{\alpha\in\mathcal{A}'} \nu'_\alpha = 1$, and $\mathcal{A}' \subset \mathcal{A}\backslash\{(0,\ldots,0)^T\}$. In turn, since $\beta \neq \gamma$, we can write

$$\beta = \sum_{\alpha\in\mathcal{A}'\cup\{(0,\ldots,0)\}} \nu_\alpha \alpha,$$

where $\nu_\alpha \geqslant 0$, for $\alpha \in \mathcal{A}'$, $\nu_{(0,\ldots,0)} > 0$ and $\sum_{\alpha\in\mathcal{A}'\cup\{(0,\ldots,0)\}} \nu_\alpha = 1$. To conclude, it is enough to verify that for λ small enough, the relative entropy conditions for the SAGE containment [7] in the version of [25, Prop. 2.1] are satisfied by $f - \lambda$: the first condition follows from the previous decomposition of β, while the third one is trivially satisfied when $|\mathcal{B}| = 1$. For the second one, we observe that the function

$$l(\lambda) = \sum_{\alpha\in\mathcal{A}'} \nu_\alpha \ln \frac{\nu_\alpha}{e \cdot c_\alpha} + \nu_{(0,\ldots,0)} \ln \frac{\nu_{(0,\ldots,0)}}{e \cdot (c_{(0,\ldots,0)} - \lambda)}$$

tends to $-\infty$ when $\lambda \to -\infty$. Hence, there exists λ such that $l(\lambda) < c_\beta$. □

The finiteness of f^{SAGE} in Proposition 2.3 can be seen as an advantage with respect to the sum of squares analogue f^{SOS}. Indeed, the Motzkin polynomial $f = x^4 + y^4 + x^2 + y^2 - 3x^2y^2 + 1$ satisfies $f^{\mathrm{SOS}} = -\infty$ while $f^{\mathrm{SAGE}} = f^* = 0$.

Remark 2.4. When $\beta \notin \mathrm{conv}(\mathcal{A} \cup \{(0,\ldots,0)^T\})$, the hyperplane separation theorem implies $\inf f = -\infty$, forcing $f^{\mathrm{SAGE}} = -\infty$. If β is on the boundary of $\mathrm{conv}(\mathcal{A} \cup \{(0,\ldots,0)^T\})$, then we cannot conclude in general. For example, consider the function

$$f(x,y) = \mu + e^{2x} + e^{2y} - \delta e^{x+y}.$$

Then $f^{\mathrm{SAGE}} = \mu$ when $\delta \leqslant 2$, while $f^{\mathrm{SAGE}} = -\infty$ when $\delta > 2$.

In the same spirit, in the corresponding setting for polynomials, we can define f^{SONC} as

$$f^{\mathrm{SONC}} \;=\; \sup\{\lambda \in \mathbb{R} \,:\, f - \lambda \in C_{\mathrm{SONC}}(\mathcal{A})\}.$$

2.4. **Relations between SAGE and SONC.** Since the two notions come from the arithmetic mean/geometric mean inequality, SONC and SAGE forms are closely related, and most of the statements for SAGE forms can be transferred to the SONC setting, following [27]: For a polynomial $f = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha$, with $\mathcal{A} \subset \mathbb{N}^n$, let

$$\mathrm{sig}(f) := f(\exp(y_1),\ldots,\exp(y_n)) = \sum_{\alpha \in \mathcal{A}} c_\alpha \exp(\langle \alpha, y\rangle)$$

be the the signomial associated with f. Studying $\mathrm{sig}(f)$ on $\mathbb{R}^n$ is equivalent to studying f on the positive open orthant $\{x \in \mathbb{R}^n : x_i > 0,\, 1 \leqslant i \leqslant n\}$. In general, for $\omega \in \{\pm 1\}^n$, studying the restriction of f to the open orthant $\{x \in \mathbb{R}^n : \omega_i x_i > 0,\, 1 \leqslant i \leqslant n\}$ boils down to studying the signomial $\mathrm{sig}(f^\omega)$, where

$$f^\omega(x) = f(\omega_1 x_1, \ldots, \omega_n x_n).$$

Finally, we define

$$\tilde{f} = \sum_{\alpha \in \mathcal{A} \cap (2\mathbb{N})^n} c_\alpha x^\alpha - \sum_{\gamma \in \mathcal{A} \setminus (2\mathbb{N})^n} |c_\gamma| x^\gamma,$$

and we call f *orthant-dominated* if there is some $\omega \in \{\pm 1\}^n$ such that $f^\omega = \tilde{f}$. In this case, f is nonnegative on $\mathbb{R}^n$ if and only if $\tilde{f}$ is nonnegative on the positive orthant, namely if and only if $\mathrm{sig}(\tilde{f})$ is nonnegative on $\mathbb{R}^n$. In general, we only have $\mathrm{sig}(\tilde{f}) \leqslant \mathrm{sig}(f^\omega)$ for every $\omega \in \{\pm 1\}^n$.

According to [27], the polynomial f admits a SONC certificate if and only if the signomial $\mathrm{sig}(\tilde{f})$ admits a SAGE certificate. From an optimization point of view, this discussion can be summed up in the following proposition:

Proposition 2.5. *Let $f(x) = \sum_\alpha c_\alpha x^\alpha$ be a polynomial. Then*

$$f^{\mathrm{SONC}} = \mathrm{sig}(\tilde{f})^{\mathrm{SAGE}} \leqslant \min_{\omega \in \{-1,1\}^n} \mathrm{sig}(f^\omega)^{\mathrm{SAGE}} \leqslant f^*,$$

where the first inequality is an equality when f is orthant-dominated.

It follows immediately from the definition that $C_{\mathrm{SAGE}}(\mathcal{T})$ is a full-dimensional cone in the vector space of signomials supported by $\mathcal{T}$. In the SONC case, we need some additional condition on the support. Using the SONC characterization in terms of the circuit number [20] in connection with Carathéodory's Theorem implies:

Proposition 2.6. *Let $\mathcal{T} \subset \mathbb{N}^n$ and $\mathcal{T}^+ = \mathcal{T} \cap (2\mathbb{N})^n$. Assume that for every $\beta \in \mathcal{T} \setminus \mathcal{T}^+$, $\beta \in \mathrm{conv}(\mathcal{T}^+ \cup \{0\})$. Then $\mathrm{C}_{\mathrm{SONC}}(\mathcal{T})$ is a full-dimensional cone in the space of polynomials supported by $\mathcal{T}$.*

2.5. **The symmetric cones and circuit decompositions.** Finally, we provide the symmetric setup for the SAGE and the SONC cones. We summarize and revisit the results from [25], in particular the symmetric circuit decomposition, see Theorem 2.8.

A linear action of a finite group G on $\mathbb{R}^n$ naturally induces an action on the dual space of exponent vectors and an action on the signomials. With a small abuse of notation, for $\sigma \in G$, we will denote respectively by $\sigma(x)$ for a variable vector $x \in \mathbb{R}^n$, $\sigma(\alpha)$ for an exponent vector $\alpha \in \mathbb{R}^n$, and by σf for a signomial f the action of σ on these elements, even if these actions are slightly different. For a detailed discussion of these actions, we refer to [25].

Define, for a G-invariant support $\mathcal{T}$, the cone $C^G_{\mathrm{SAGE}}(\mathcal{T})$ of G-invariant signomials in $C_{\mathrm{SAGE}}(\mathcal{T})$. Here, we write a G-invariant signomial f supported on $\mathcal{T} = \mathcal{A} \cup \mathcal{B}$ in the form

$$f = \sum_{\alpha \in \mathcal{A}} c_\alpha \exp(\langle \alpha, x \rangle) + \sum_{\beta \in \mathcal{B}} c_\beta \exp(\langle \beta, x \rangle) \tag{2.3}$$

with $c_\alpha > 0$ for $\alpha \in \mathcal{A}$ and $c_\beta < 0$ for $\beta \in \mathcal{B}$.

For a set $\mathcal{S} \subset \mathbb{R}^n$ of exponent vectors, the *orbit of* $\mathcal{S}$ under G is

$$G \cdot \mathcal{S} = \{\sigma(s) : s \in \mathcal{S}, \sigma \in G\},$$

simply denoted $G \cdot \alpha$ when $\mathcal{S} = \{\alpha\}$. Then, a subset $\hat{\mathcal{S}} \subset \mathcal{S}$ is *a set of orbit representatives for* $\mathcal{S}$ if $\hat{\mathcal{S}}$ is an inclusion-minimal set with $(G \cdot \hat{\mathcal{S}}) = \mathcal{S}$. Moreover, let $\operatorname{Stab} \beta := \{\sigma \in G \ : \ \sigma(\beta) = \beta\}$ denote the *stabilizer* of an exponent vector β. For a given G-invariant signomial, the following symmetric decomposition was shown in [25].

Proposition 2.7. [25] *Let f be a G-invariant signomial of the form* (2.3) *and $\hat{\mathcal{B}}$ be a set of orbit representatives for $\mathcal{B}$. Then $f \in C^G_{\mathrm{SAGE}}(\mathcal{A}, \mathcal{B})$ if and only if for every $\hat{\beta} \in \hat{\mathcal{B}}$, there exists an AGE signomial $h_{\hat{\beta}} \in C_{\mathrm{SAGE}}(\mathcal{A}, \hat{\beta})$ such that*

$$f = \sum_{\hat{\beta} \in \hat{\mathcal{B}}} \sum_{\rho \in G / \operatorname{Stab}(\hat{\beta})} \rho h_{\hat{\beta}}. \tag{2.4}$$

The functions $h_{\hat{\beta}}$ can be chosen to be invariant under the action of $\operatorname{Stab}(\hat{\beta})$.

This result implies an additional structure in the decomposition of $C^G_{\mathrm{SAGE}}(\mathcal{T})$ with respect to Proposition 2.1. For a circuit λ with $\beta = \lambda^-$, we then introduce $C^G_{\mathrm{SAGE}}(\lambda)$ the *symmetrized λ-witnessed cone*

$$C^G_{\mathrm{SAGE}}(\lambda) := \left\{ \sum_{\rho \in G} \rho g, \quad g \in C_{\mathrm{AGE}}(\lambda^+, \lambda^-) \right\}.$$

In Proposition 2.7, every $h_{\hat{\beta}}$ comes from the symmetrization under $\operatorname{Stab} \hat{\beta}$ of a sum of signomials supported on circuits λ with $\lambda^- = \hat{\beta}$. Hence, we obtain the following symmetric version of Proposition 2.1.

Theorem 2.8 (Symmetry-adapted circuit decomposition). *Let $\hat{\mathcal{T}}$ be a set of orbit representatives for the whole support set $\mathcal{T}$. Then the G-symmetric cone $C^G_{\mathrm{SAGE}}(\mathcal{T})$ decomposes as*

$$C^G_{\mathrm{SAGE}}(\mathcal{T}) \ = \sum_{\hat{\beta} \in \hat{\mathcal{T}}} \sum_{\substack{\lambda \in \Lambda(\mathcal{T}) \\ \lambda^- = \hat{\beta}}} C^G_{\mathrm{SAGE}}(\lambda) + \sum_{\hat{\alpha} \in \hat{\mathcal{T}}} \mathbb{R}_+ \sum_{\rho \in G / \operatorname{Stab}(\hat{\alpha})} \rho \exp(\langle \hat{\alpha}, x \rangle).$$

Even if these statements are valid for any linear group action, from now on, we restrict our attention to the most natural case: the action of the symmetric group by permutation of variables.

3. Zeroes and minimizers of symmetric SAGE and SONC forms

One of the goals of the paper is to provide information on the gap between the SAGE and SONC bounds and the minimum of a symmetric signomial/polynomial, and in particular to find situations in which there is no gap. In this case, $f - f^*$ is a SAGE respectively SONC form whose infimum is 0. This encourages to understand the structure of the zeroes of symmetric forms of this kind, that will lead to examples and counterexamples about the exactness of the bounds.

In general, the set of all zeros of any SAGE exponential is convex and when finite, this zero set has cardinality at most one (see [14, Theorem 4.1]). Similarly, any SONC polynomial in $f \in \mathbb{R}[x_1, \ldots, x_n]$ with a finite number of zeroes has at most 2^n real zeroes in $(\mathbb{R} \setminus \{0\})^n$ (see [11, Corollary 4.1]), because it has at most one zero in each open orthant. The invariance under the action of the symmetric group forces additional structure on the zeroes of SAGE signomials:

Lemma 3.1. *Let f be a non-constant SAGE signomial in $n \geqslant 2$ variables that is invariant under the action of $\mathcal{S}_n$. If the zero set $V_{\mathbb{R}}(f)$ of f is non-empty, then there are three possibilities:*

(1) $V_{\mathbb{R}}(f)$ is a singleton on the diagonal.
(2) $V_{\mathbb{R}}(f)$ is the diagonal.
(3) $V_{\mathbb{R}}(f)$ is an affine hyperplane of the form $\{x : \sum_{i=1}^n x_i = \tau\}$ for some constant $\tau \in \mathbb{R}$.

In particular, if a symmetric SAGE signomial has a zero, then it has at least one zero on the diagonal.

Proof. Recall that the set of zeroes of any SAGE signomial is an affine subspace, see ([14, Theorem 3.1]). Clearly, the zero set of f is invariant under $\mathcal{S}_n$. The only non-empty invariant affine subspaces which are invariant under the symmetric group are: a single point on the diagonal, the diagonal or an affine hyperplane $\{x : \sum_{i=1}^n x_i = \tau\}$ for some $\tau \in \mathbb{R}$. $\square$

Example 3.2. All three cases in Lemma 3.1 can occur. Let $g(x) = e^x + e^{-x} - 2$. The signomial g is a univariate AGE form, whose only zero is $x = 0$. Then

$$\begin{cases} \sum_{i=1}^n g(x_i - \gamma) & \text{vanishes only in } (\gamma, \ldots, \gamma), \gamma \in \mathbb{R}, \\ \sum_{i,j=1}^n g(x_i - x_j) & \text{vanishes on the diagonal}, \\ g((\sum_{i=1}^n x_i) - \tau) & \text{vanishes on } \{x : \sum_{i=1}^n x_i = \tau\}, \tau \in \mathbb{R}, \end{cases}$$

and they are all symmetric SAGE forms.

Following Section 2.4, for $\omega \in \{-1, 1\}^n$, the zeros of p in the open orthant $\{x \in \mathbb{R}^n : \omega_i x_i > 0, 1 \leqslant i \leqslant n\}$ correspond with the zeros of p^ω in the positive orthant. Denote by $V_{>0}(p)$ the zero set of a polynomial p in the positive orthant.

Corollary 3.3. *Let p be a non-constant SONC polynomial in n variables that is $\mathcal{S}_n$-invariant. For $\omega \in \{\pm 1\}^n$:*

(1) If $p^\omega \neq \tilde{p}$, then $V_{>0}(p^\omega)$ is empty.
(2) If $p^\omega = \tilde{p}$, then $V_{>0}(p^\omega) = V_{>0}(\tilde{p})$, which can be, if non-empty,
   (a) a singleton on the diagonal in $\mathbb{R}^n_{>0}$,
   (b) the diagonal in $\mathbb{R}^n_{>0}$,
   (c) an hypersurface of the form $\{x : \prod_{i=1}^n x_i = \tau\}$ intersected with $\mathbb{R}^n_{>0}$, for some constant $\tau > 0$.

Proof. For $\omega \in \{\pm 1\}^n$, write $p^\omega = \sum_{\alpha\in\mathcal{A}} c_\alpha x^\alpha$. If $p^\omega \neq \tilde{p}$, then there is $\kappa \in \mathcal{A} \setminus (2\mathbb{N})^n$, such that $c_\kappa > 0$. Then we have, for every $x \in \mathbb{R}^n_{>0}$,

$$p^\omega(x) - \tilde{p}(x) = \sum_{\gamma\in\mathcal{A}\setminus(2\mathbb{N})^n} (c_\gamma + |c_\gamma|)x^\gamma \geqslant (c_\kappa + |c_\kappa|)x^\kappa > 0.$$

Now recall that if p is SONC, then $\tilde{p}$ is a SONC as well, which implies that $\tilde{p}(x) \geqslant 0$, and therefore $p^\omega(x) > 0$, proving the first part of the statement. Since p is $\mathcal{S}_n$-invariant, then so is $\tilde{p}$, and the second part follows from Lemma 3.1, after the exponential change of variable. ☐

The previous results give an understanding of the zeroes of SONC polynomials in the open orthants, but we might have zeroes on the coordinate hyperplanes. Any zero of p on a hyperplane and which is not the origin itself can be viewed (by permuting the coordinates) as a zero in $(\mathbb{R}_{\neq 0})^k \times \{0\}^{n-k}$ for some $k \in \{1, \dots, n-1\}$. The characterization of all zeroes in $(\mathbb{R}_{\neq 0})^k \times \{0\}^{n-k}$ can be done by considering

$$q(x_1, \dots, x_k) = p(x_1, \dots, x_k, 0, \dots, 0)$$

and applying the SAGE version in Lemma 3.1.

Lemma 3.1 as well as Corollary 3.3 can be used as a criterion to show that certain signomials cannot be SAGE signomials or certain polynomials cannot be SONC polynomials.

Example 3.4. Consider the nonnegative, symmetric polynomial $p = (1 - x_1^2 - x_2^2)^2 \in \mathbb{R}[x_1, x_2]$. Its zero set is $\{x \in \mathbb{R}^2 : 1 - x_1^2 - x_2^2 = 0\}$, which does not fall into any of the classes in Corollary 3.3. Hence, p cannot be a SONC polynomial.

As a next question, it is natural to wonder whether in general, when the minimum is not zero, the set of minimizers of a SAGE and SONC form still offers a strong structure. However, the next example shows that there is no reason for the diagonal to contain minimizers of such forms:

Example 3.5. If f is an even, univariate SAGE signomials with several minimizers different from the origin, like

$$f(x) \;=\; 4e^x - 4e^{2x} + e^{3x} + (4e^{-x} - 4e^{-2x} + e^{-3x}),$$

which has two minimizers outside of the origin. Then the function $g(x, y) := f(x - y)$ is a symmetric SAGE signomial with several minimizers and the minimizers of g are outside of the diagonal.

Even if this example is degenerated in the sense that it has no isolated minimizers and that the negative support points are contained in the boundary of the Newton polytope, Section 4.2 will provide non-degenerate examples.

4. Comparison of the symmetric cones with the symmetric nonnegative cone

We come to the main topic of the paper: in a symmetric situation, how far is the notion of being SAGE or SONC from being nonnegative? This evaluation can be formulated with several questions of slightly different flavors: Are there cases in which the two notions are equivalent? When this is not the case, how far is the relaxation bound from the infimum of the function? Can we evaluate precisely the difference between the two cones?

After providing a new case where SAGE and SONC methods give the infimum of a function, we will focus on two cases in which Sums Of Squares coincide with nonnegative polynomials and see that this is not the case for SONC polynomials, even in the symmetric case. Finally, we open a discussion on the comparison between the symmetric SONC cone and the cone of nonnegative polynomials in the setup of symmetric inequalities and symmetric mean inequalities. This line of questions had been studied, for example, in the context of Sums Of Squares in [6, 1]. In the normalized setup, we study the classical Muirhead inequality and give a generalization and show that in the non-normalized setup SONC certificates are not always able to recover a full dimensional fraction of valid inequalities.

4.1. **A case of exactness.** As described in the introduction, there are several situations in which SAGE and SONC methods provide the infimum of a function, like in the work of Wang [38] (see also [20, 27]). In this section, we provide a new class of symmetric signomials, where the two values coincide, precisely when there is a unique orbit in the support corresponding with positive coefficients.

Theorem 4.1. *Let* $\hat{\alpha} \in \mathbb{R}^n$ *and* $\hat{\beta}_i \in \mathbb{R}^n$ *for* $1 \leqslant i \leqslant m$ *be such that* $\hat{\beta}_i \in \operatorname{int}(\operatorname{conv}(G \cdot \hat{\alpha} \cup \{0\}))$ *and* $\hat{\alpha}, \hat{\beta}_1, \ldots, \hat{\beta}_m$ *are in distinct orbits under* $G = \mathcal{S}_n$. *Let* $\mathcal{A} = G \cdot \hat{\alpha}$, $\mathcal{B}_i = G \cdot \hat{\beta}_i$ *and*

$$f(x) = c \sum_{\alpha \in \mathcal{A}} \exp(\langle \alpha, x \rangle) - \sum_{i=1}^{m} d_i \sum_{\beta \in \mathcal{B}_i} \exp(\langle \beta, x \rangle) + w \tag{4.1}$$

with $c, d_i > 0$ *and* $w \in \mathbb{R}$. *Then* $f^* = f^{\mathrm{SAGE}}$.

Remark 4.2. Even in the restriction to nonnegative integer exponents, Theorem 4.1 covers situations which are not covered by Wang's result [38, Theorem 4.1] (which is stated in the language of polynomials). This happens as soon as there are hyperplanes H determined by positive support points, for which both corresponding halfspaces contain interior points of the Newton polytope of f. Moreover, this result generalizes [20, Corollary 7.5], where the outer orbit had to be a simplex.

Proof. Without loss of generality, we may assume that $c = 1$. Set $a := \sum_{j=1}^{n} \alpha_j$ and for $i \in \{1, \ldots, n\}$ set $b_i := \sum_{j=1}^{n} \beta_j$ for any arbitrarily chosen $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}_i$. We have $a \neq 0$, since otherwise $\mathcal{A}$ is contained in the linear hyperplane with normal vector $(1, \ldots, 1)^T$ and thus $\operatorname{int}(\operatorname{conv}(G \cdot \hat{\alpha} \cup \{0\}))$ would be empty. Further, we have $b \neq 0$, since otherwise β cannot be contained in $\operatorname{int}(\operatorname{conv}(G \cdot \hat{\alpha} \cup \{0\}))$. Let $g(t)$ be the univariate signomial describing the restriction of f to the diagonal,

$$g(t) = f(t, \ldots, t) = \sum_{\alpha \in \mathcal{A}} e^{ta} - \sum_{i=1}^{m} d_i \sum_{\beta \in \mathcal{B}_i} e^{tb_i} = |\mathcal{A}| e^{ta} - \sum_{i=1}^{m} d_i |\mathcal{B}_i| e^{tb_i} + w. \tag{4.2}$$

By Descartes' rule of signs for signomials [8] applied to the derivative g', we see that g' has a unique root t_0. Let $f_{\mathrm{diag}} = g(t_0)$. We show that t_0 is a global minimizer for f by showing that $f - f_{\mathrm{diag}}$ is a SAGE signomial.

First, since $\hat{\beta}_i \in \operatorname{int}(\operatorname{conv}(G\hat{\alpha} \cup \{0\}))$, there exists $\lambda_0^{(i)} \geqslant 0$, and $\lambda_\alpha^{(i)} \geqslant 0$ for every $\alpha \in \mathcal{A}$ that satisfy $\lambda_0^{(i)} + \sum_{\alpha \in \mathcal{A}} \lambda_\alpha^{(i)} = 1$ and $\sum_{\alpha \in \mathcal{A}} \lambda_\alpha^{(i)} \alpha = \hat{\beta}_i$. We can even assume that these $\lambda_\alpha^{(i)}$ are invariant under the action of $\operatorname{Stab} \hat{\beta}_i$, by taking if necessary $\mu_\alpha^{(i)} = \frac{1}{|\operatorname{Stab} \hat{\beta}_i|} \sum_{\sigma \in \operatorname{Stab} \hat{\beta}_i} \lambda_{\sigma(\alpha)}^{(i)}$.

We observe that $\lambda_0^{(i)} = \frac{a - b_i}{a}$, because summing over the n coordinate equations gives

$$b_i = \sum_{j=1}^{n} \hat{\beta}_{i,j} = \sum_{j=1}^{n} \sum_{\alpha \in \mathcal{A}} \lambda_\alpha^{(i)} \alpha_j = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha^{(i)} a = (1 - \lambda_0^{(i)}) a.$$

We introduce some notation. Let $m_i = \frac{a|\mathcal{A}|}{b_i|\mathcal{B}_i|} = \frac{|\operatorname{Stab}\hat{\beta}_i|}{|\operatorname{Stab}\hat{\alpha}|(1-\lambda_0^{(i)})}$, and $u_i = \frac{d_i}{m_i}e^{t_0(b_i-a)}$, and set

$$\begin{cases} \nu_0^{(i)} = d_i\lambda_0^{(i)} & \text{and} \quad \nu_\alpha^{(i)} = d_i\lambda_\alpha^{(i)}, \\ c_0^{(i)} = \nu_0^{(i)}e^{t_0 b_i} & \text{and} \quad c_\alpha^{(i)} = u_i m_i \lambda_\alpha^{(i)}. \end{cases}$$

Finally, consider for any $1 \leqslant i \leqslant m$,

$$h_i = c_0^{(i)} + \sum_{\alpha\in\mathcal{A}} c_\alpha^{(i)} \exp(\langle \alpha, x\rangle) - d_i \exp(\langle \hat{\beta}_i, x\rangle).$$

It is clear that $\nu_0^{(i)}$, $\nu_\alpha^{(i)}$, $c_0^{(i)}$ and $c_\alpha^{(i)}$ are all nonnegative. We claim that

$$f - f_{\text{diag}} = \sum_{i=1}^{m} \sum_{\sigma\in G/\operatorname{Stab}\hat{\beta}_i} \sigma h_i$$

is a SAGE decomposition of $f - f_{\text{diag}}$. In order to prove it, we show that the relative entropy characterization in [25, Theorem 4.1] applies. The equation (4.1) therein is trivially verified by definition of $\nu^{(i)}$. For equation (4.2), compute the relative entropy expression

$$\begin{aligned} D(\nu^{(i)}, e\cdot c^{(i)}) &= \nu_0^{(i)} \ln \frac{\nu_0^{(i)}}{e\cdot c_0^{(i)}} + \sum_{\alpha\in\mathcal{A}} \nu_\alpha^{(i)} \ln \frac{\nu_\alpha^{(i)}}{e\cdot c_\alpha^{(i)}} \\ &= d_i\lambda_0^{(i)} \ln \frac{\nu_0^{(i)}}{e\nu_0^{(i)}e^{t_0 b_i}} + d_i \sum_{\alpha\in\mathcal{A}} \lambda_\alpha^{(i)} \ln \frac{d_i\lambda_\alpha^{(i)}}{eu_i m_i \lambda_\alpha^{(i)}} \\ &= -d_i\lambda_0^{(i)} - d_i\lambda_0^{(i)} b_i t_0 - d_i \sum_{\alpha\in cA} \lambda_\alpha^{(i)} + d_i \sum_{\alpha\in\mathcal{A}} \lambda_\alpha^{(i)}(a-b_i)t_0 \\ &= -d_i\lambda_0^{(i)} - d_i\lambda_0^{(i)} b_i t_0 - d_i(1-\lambda_0^{(i)}) + d_i(1-\lambda_0^{(i)})(a-b_i)t_0 \\ &= -d_i + d_i\left((1-\lambda_0^{(i)})(a-b_i) - \lambda_0^{(i)} b_i\right) t_0 \\ &= -d_i. \end{aligned}$$

It remains to show that (4.3) are satisfied. For $i \in \{1,\dots,m\}$, we have

$$\sum_{\sigma\in\operatorname{Stab}\hat{\beta}_i\backslash G} c_{\sigma(0)}^{(i)} = |\mathcal{B}_i| d_i \lambda_0^{(i)} e^{t_0 b_i} = d_i|\mathcal{B}_i| \frac{a-b_i}{a} e^{t_0 b_i} = d_i|\mathcal{B}_i|e^{t_0 b_i} - \frac{d_i|\mathcal{B}_i|b_i e^{t_0 b_i}}{a}.$$

Since t_0 is a root of $g'(t) = a|\mathcal{A}|e^{ta} - \sum_{i=1}^m d_i b_i |\mathcal{B}_i| e^{tb_i}$, we obtain

$$\sum_{i=1}^{m} \sum_{\sigma\in\operatorname{Stab}\hat{\beta}_i\backslash G} c_{\sigma(0)}^{(i)} = \sum_{i=1}^{m} d_i|\mathcal{B}_i|e^{t_0 b_i} - |\mathcal{A}|e^{t_0 a} = w - g(t_0) = w - f_{\text{diag}}. \tag{4.3}$$

Now let $\alpha \in \mathcal{A}$. For $i \in \{1, \dots, m\}$, we have

$$\begin{aligned}\sum_{\sigma \in \operatorname{Stab}\hat{\beta}_i \backslash G} c^{(i)}_{\sigma(\alpha)} &= \frac{1}{|\operatorname{Stab}\hat{\beta}_i|} \sum_{\tau \in \operatorname{Stab}\hat{\beta}_i} \sum_{\sigma \in \operatorname{Stab}\hat{\beta}_i \backslash G} c^{(i)}_{\tau\sigma(\alpha)} = \frac{1}{|\operatorname{Stab}\hat{\beta}_i|} \sum_{\rho \in S_n} c^{(i)}_{\rho(\alpha)} \\ &= \frac{|\operatorname{Stab}\hat{\alpha}|}{|\operatorname{Stab}\hat{\beta}_i|} \sum_{\sigma \in S_n/\operatorname{Stab}\hat{\alpha}} c^{(i)}_{\sigma(\alpha)} = \frac{|\operatorname{Stab}\hat{\alpha}|}{|\operatorname{Stab}\hat{\beta}_i|} \sum_{\alpha \in \mathcal{A}} c^{(i)}_{\alpha} = \frac{|\mathcal{B}_i|}{|\mathcal{A}|} \sum_{\alpha \in \mathcal{A}} u_i m_i \lambda^{(i)}_{\alpha} \\ &= \frac{a}{b_i} u_i \sum_{\alpha \in \mathcal{A}} \lambda^{(i)}_{\alpha} = \frac{a}{b_i} u_i (1 - \lambda^{(i)}_0) = u_i.\end{aligned}$$

Here, we used the bijections $\operatorname{Stab}\hat{\alpha} \times G/\operatorname{Stab}\hat{\alpha} \to G$ and $\operatorname{Stab}\hat{\beta} \backslash G \times \operatorname{Stab}\hat{\beta} \to G$, combined with the fact that $(c^{(i)}_{\alpha})_{\alpha}$ is stable under the action of $\operatorname{Stab}\hat{\beta}_i$. Hence, for $\alpha \in \mathcal{A}$,

$$\sum_{i=1}^{m} \sum_{\sigma \in \operatorname{Stab}\hat{\beta}_i \backslash G} c^{(i)}_{\sigma(\alpha)} = \sum_{i=1}^{m} u_i = \frac{1}{a|\mathcal{A}|e^{t_0 a}} \sum_{i=1}^{m} d_i b_i |\mathcal{B}_i| e^{t_0 b_i} = 1. \tag{4.4}$$

Equations (4.3) and (4.4) show (4.3), which completes the proof. $\square$

Remark 4.3. In the proof of Theorem 4.1, we could define the same quantities even if $\hat{\beta}_i$ was on the boundary of the convex polytope $\operatorname{conv}(G \cdot \hat{\alpha} \cup \{0\})$, except the vertices. Moreover, the Descartes rule of signs would still apply if $|\mathcal{A}| - \sum_{i, b_i = a} d_i |\mathcal{B}_i| > 0$. So, under some additional conditions on the coefficients d_i for those $\hat{\beta}_i \in \operatorname{conv}(G \cdot \hat{\alpha})$, we can relax the condition of $\hat{\beta}_i$ being in the interior of the convex hull, and the theorem would still be true.

One can notice the connection between Theorem 4.1 and Section 3: we show that $f^* = f^{\mathrm{SAGE}}$ by showing that there is a point x_0 such that $f - f(x_0)$ is SAGE. This implies in particular that x_0 is a zero of a SAGE form, and Section 3 encourages to look for such a point on the diagonal. The assumptions of the theorem lead to a unique candidate for x_0, and we can show that it is indeed the minimum of f. We can reformulate Theorem 4.1 in the following way: it gives a large class of signomials whose nonnegativity can be detected via SAGE certificates.

Corollary 4.4. *For a G-symmetric signomial f of the form* (4.1)*, the following are equivalent:*

(1) f is nonnegative.
(2) $f \in C_{\mathrm{SAGE}}(\mathcal{A}, \mathcal{B})$.
(3) $w \geqslant -h(t_0)$, where $h(t) := |\mathcal{A}| e^{t \sum_{j=1}^{n} \hat{\alpha}_j} - \sum_{i=1}^{m} d_i |\mathcal{B}_i| e^{t \sum_{j=1}^{n} (\hat{\beta}_i)_j}$ and t_0 is the unique real zero of the derivative $h'(t)$.

Condition (3) in (4.4) can be viewed as a symmetric analog of the circuit number condition.

Proof. The equivalence of (1) and (2) follows immediately from Theorem 4.1. For the equivalence to (3), observe that the function h coincides with the function g defined in (4.2) in the proof of Theorem 4.1 up to the constant w. In the critical situation $w := -h(t_0)$, the signomial f has a zero at the diagonal point $(t_0, \dots, t_0)^T$. $\square$

Example 4.5. Let

$$f = e^{4x_1} + e^{4x_2} + e^{4x_3} - 5(e^{x_1+x_2} + e^{x_1+x_3} + e^{x_2+x_3}) - 6e^{x_1+x_2+x_3} + w$$

with some constant w. In Corollary 4.4, we have $h(t) = 3e^{4t} - 15 \cdot e^{2t} - 6 \cdot e^{3t}$ and $t_0 = \ln \frac{5}{2}$. The minimum of f is taken at the diagonal point $(\ln \frac{5}{2}, \ln \frac{5}{2}, \ln \frac{5}{2})^T$. Hence, f is nonnegative if and only if $w \geqslant -(3e^{4t_0} - 15 \cdot e^{2t_0} - 6 \cdot e^{3t_0})$, i.e., if and only if $w \geqslant 1125/16$.

We cannot directly transfer Theorem 4.1 and Corollary 4.4 to an equality of cones, because of the assumption on the sign of the coefficients d_i. Our result is true only when these coefficients are negative, while both in $C_{\mathrm{SAGE}}(\mathcal{T})$ and $C_{\mathrm{SAGE}}(\mathcal{A},\mathcal{B})$, coefficients corresponding with $\mathcal{B}$ might be positive.

This discussion remains valid when going to the SONC situation. Theorem 4.1 and Corollary 4.4 have natural analogues when $\hat{\alpha}$ is required to be in $(2\mathbb{N})^n$ and $\hat{\beta}_i \in \mathbb{N}^n$, still with the assumption that the coefficients corresponding with $\mathcal{B}$ are negative. However, this assumption is very natural when we hope for an equivalence between nonnegativity and SONC, since a polynomial f is SONC if and only if $\tilde{f}$ is SAGE, see the discussion in Section 2.4.

4.2. **Study of the Hilbert cases.** Following the previous discussion, if it is hard to provide new general conditions on the support of a form to detect its nonnegativity through SAGE and SONC certificates, additional conditions on the coefficients can be sufficient to get new criteria.

Here, we focus on two natural cases: we restrict our attention to polynomials, and look at the cases where nonnegativity can be decided with Sums Of Squares certificates: quadratic forms, and degree 4 polynomials in 2 variables. We show that in these two situations, even for symmetric polynomials, nonnegativity cannot always be certified by SONC methods. We provide a precise comparison between f^* and f^{SONC} depending on the coefficients of the polynomials.

We start by the case of symmetric quadratic forms. For studying the difference between f^* and f^{SONC}, it is enough to consider polynomials of the form

$$f(x) = \sum_{i=1}^{n} x_i^2 + a\sum_{i=1}^{n} x_i + b\sum_{i<j} x_i x_j,$$

where $a, b \in \mathbb{R}$. We then have:

Proposition 4.6. *Let $f(x) = \sum_{i=1}^{n} x_i^2 + a\sum_{i=1}^{n} x_i + b\sum_{i<j} x_i x_j$ with $a, b \in \mathbb{R}$. Then*

(1) If $b > 2$ or $b < \frac{-2}{n-1}$, then $f^* = f^{\mathrm{SONC}} = -\infty$.
(2) If $\frac{-2}{n-1} \leqslant b \leqslant 0$, then $f^* = f^{\mathrm{SONC}} = \frac{-a^2 n}{4+2b(n-1)}$.
(3) If $0 \leqslant b \leqslant \frac{2}{n-1}$, then $f^* = \frac{-a^2 n}{4+2b(n-1)}$ and $f^{\mathrm{SONC}} = \frac{-a^2 n}{4-2b(n-1)}$.
(4) If $\frac{2}{n-1} < b \leqslant 2$, then $f^* = \frac{-a^2 n}{4+2b(n-1)}$ and $f^{\mathrm{SONC}} = -\infty$.

Proof. We have the decomposition

$$f(x) = \frac{2-b}{2n}\sum_{i<j}(x_i - x_j)^2 + \frac{2+b(n-1)}{2n}\left(\sum_{i=1}^{n} x_i + \frac{na}{2+b(n-1)}\right)^2 - \frac{a^2 n}{4+2b(n-1)},$$

which directly shows that if $b > 2$, then $f(t, -t, 0, \ldots, 0)$ goes to $-\infty$ when t grows, while if $b < \frac{-2}{n-1}$, then $f(t, t, t, \ldots, t)$ goes to $-\infty$ when t grows, proving the first assertion. Moreover, when $\frac{-2}{n-1} \leqslant b \leqslant 2$, this decomposition shows that $f^* = \frac{-a^2 n}{4+2b(n-1)}$, achieved for $x = (t_0, \ldots, t_0)$, where $t_0 = \frac{-a}{2+b(n-1)}$.

It remains to understand f^{SONC}, by looking for the maximal λ such that $f - \lambda$ admits a SONC decomposition. Since f is symmetric, according to Theorem 2.7 and Proposition 2.2, we have a SONC decomposition

$$f(x) = \sum_{i=1}^{n}(\tau x_i^2 + a x_i + \delta) + \sum_{i<j}\left(\sigma(x_i^2 + x_j^2) + b x_i x_j\right) + R(x),$$

where σ, τ, δ are positive, $R(x)$ can only contain squares of variables and a constant term, and the inequalities

$$\tau + (n-1)\sigma \leqslant 1 \quad \text{and } n\delta \leqslant -\lambda \tag{4.5}$$

are satisfied. Moreover, the second term is a SONC if and only if $4\sigma^2 \geqslant b^2$, that is $\sigma \geqslant \frac{|b|}{2}$. Then (4.5) forces $\frac{(n-1)|b|}{2} \leqslant 1$, so that if $\frac{2}{n-1} < b \leqslant 2$, then f cannot be a SONC, proving the last case.

Finally, assume that $\frac{(n-1)|b|}{2} \leqslant 1$. Since we want to maximize λ (which corresponds to minimizing δ), the best decomposition will be given by the smallest σ, that is $\frac{|b|}{2}$. Then, the largest τ we can have is $1 - \frac{(n-1)|b|}{2}$. Furthermore, the first term is a SONC if and only if $a^2 \leqslant 4\tau\delta$, which yields

$$\delta \geqslant \frac{a^2}{4 - 2(n-1)|b|}$$

and the second part of (4.5) gives

$$f^{\mathrm{SONC}} = \frac{na^2}{4 - 2(n-1)|b|},$$

proving the second and the third statement. □

We initiate a similar study for symmetric polynomials of degree 4 in two variables, depending on their support. The possible coefficients lie in the simplex whose vertices are $(0,0), (4,0)$ and $(0,4)$. In particular, there are only three possible interior points $(1,1)$, $(1,2)$, $(2,1)$. If the only interior point is $(1,1)$, then we know that there is equivalence between being SONC and being nonnegative [38]. The next case to consider is then when the support of our polynomial f contains the orbit made of $(1,2)$ and $(2,1)$. If the positive support is only $(0,0)$, $(4,0)$, $(0,4)$, then we can apply Theorem 4.1: we also have equivalence in this case. Then the most natural next case is to add the diagonal point $(2,2)$ to the positive support. In other words, we are considering polynomials of the form

$$f = (x^4 + y^4) + ax^2y^2 - b(x^2y + xy^2)$$

and we want to understand and compare, depending on the coefficients a and b, the minimum f^* of f and the value f^{SONC}. As intuited by Example 3.5, these values do not always agree.

Proposition 4.7. *Let $f = (x^4 + y^4) + ax^2y^2 - b(x^2y + xy^2)$ with $a, b \in \mathbb{R}$. Then:*

(1) If $a \geqslant 22$, then $f^* = -\dfrac{(a^2 + 14a + 22 + 2(2a+5)\delta)b^4}{64(a-2)^3(a+2)}$, $f^{\mathrm{SONC}} = -\dfrac{b^4}{8a^2}$, where $\delta = \sqrt{5+2a}$.

(2) If $4 \leqslant a \leqslant 22$, then $f^* = -\dfrac{27b^4}{16(a+2)^3}$, $f^{\mathrm{SONC}} = -\dfrac{b^4}{8a^2}$.

(3) If $-2 < a \leqslant 4$, then $f^* = f^{\mathrm{SONC}} = \frac{27b^4}{16(a+2)^3}$.

(4) If $a \leqslant -2$, then $f^* = f^{\mathrm{SONC}} = -\infty$, except for $a = -2$ and $b = 0$, where $f^* = f^{\mathrm{SONC}} = 0$.

Note that for the two particular cases $a = 4$ and $a = 22$, the corresponding values agree.

Proof. To verify the claimed values for f^*, the idea is to decompose $f - f^*$ as a sum of squares, and show that $f - f^*$ attains 0 in some point. We treat the case $a \geqslant 22$ first. In this case, since

$$-f^* = \frac{(a^2 + 14a + 22 + 2(2a+5)\delta)b^4}{64(a-2)^3(a+2)} \geqslant 0,$$

we can set $\mu = \sqrt{-f^*}$. Defining the polynomials

$$P_1 = \mu\left(1 + \frac{8(3\delta - a - 7)}{b^2}(x+y)^2\right),$$

$$P_2 = \sqrt{\frac{1+a+\delta}{4(a^2-4)}}\,(b(x+y) + 2(\delta - 1 - a)xy)\,,$$

$$P_3 = \sqrt{\frac{2(\delta-1)}{a+2}}\left(x^2 + y^2 + \frac{3-\delta}{2}xy\right),$$

one can check the decomposition $f - f^* = P_1^2 + P_2^2 + P_3^2$, which vanishes in (x_0, y_0) and (y_0, x_0) for the real values

$$x_0 = \frac{b}{8(a-2)}\left(3 + \delta + \sqrt{\frac{2a^2 - 36 - 22a - (20+2a)\delta}{a+2}}\right),$$

$$y_0 = \frac{b}{8(a-2)}\left(3 + \delta - \sqrt{\frac{2a^2 - 36 - 22a - (20+2a)\delta}{a+2}}\right).$$

For the case $-2 < a < 22$, by considering

$$g(x,y) = f(x,y) - \left(\sqrt{\frac{22-a}{24}}(x^2 - y^2)\right)^2$$

we can write

$$g(x,y) = \frac{a+2}{24}\left((x^4 + y^4) + 22x^2y^2 - \frac{24b}{a+2}(x^2y + xy^2)\right),$$

and from the previous case, we know that

$$g^* = -\frac{27b^4}{16(a+2)^3},$$

achieved for $x_0 = y_0 = \frac{3b}{2(a+2)}$, so that we have decomposed f into a sum of 4 squares, which attains a zero on the diagonal.

For the case $a \leqslant -2$,

$$f(x,x) = (2+a)x^4 - 2bx^3.$$

When $a < -2$, $f(x,x)$ goes to $-\infty$ when x grows. When $a = -2$ and $b \neq 0$, $f(x,x)$ tends to $-\infty$ whenever $x \to \pm\infty$, depending on the sign of b. Thus $f^* = -\infty$ and therefore $f^{\text{SONC}} = -\infty$. In the special case $a = -2$ and $b = 0$, $f(x,y) = (x^2 - y^2)^2$, with $f^* = f^{\text{SONC}} = 0$.

Now we look at f^{SONC} for the remaining cases. From Theorem 2.7, $f - \lambda$ is a SONC polynomial if and only if there exists $0 \leqslant t \leqslant 1$ such that the polynomial

$$tx^4 + (1-t)y^4 + \frac{a}{2}x^2y^2 - bx^2y - \frac{\lambda}{2}$$

is a SONC polynomial. And now, since we just have one interior point, this function is a SONC polynomial if and only if it is nonnegative. Hence,

$$\begin{aligned} f^{\text{SONC}} &= \max\{\lambda \;:\; f - \lambda \text{ is SONC}\} \\ &= 2\max\{\rho \;:\; tx^4 + (1-t)y^4 + \frac{a}{2}x^2y^2 - bx^2y - \rho \geqslant 0 \text{ for some } 0 \leqslant t \leqslant 1\}. \end{aligned}$$

Let $j_t(x,y) = tx^4 + (1-t)y^4 + \frac{a}{2}x^2y^2 - bx^2y - \frac{1}{2}\omega$, where ω is the value of f^{SONC} claimed in the statement. The strategy is to exhibit a $0 \leqslant t_0 \leqslant 1$ and a decomposition into sum of squares and circuit polynomials of j_{t_0}, such that it attains a zero, and such that for all other $0 \leqslant t \leqslant 1$, j_t has a negative value.

We start with the case $0 \leqslant a \leqslant 4$, where $\omega = -\frac{27b^4}{16(a+2)^3}$. Let $x_0 = \frac{3b}{2(a+2)}$. Then $j_t(x_0,x_0) = \frac{1}{2}(f(x_0,x_0) - f^{\mathrm{SONC}}) = 0$ for every $0 \leqslant t \leqslant 1$. Let $t_0 = \frac{a+8}{12}$. Then we have

$$j_{t_0}(x,y) = \left(\frac{4-a}{12}(x^2-y^2)^2\right) + \left(\frac{a+2}{6}x^4 + 2\frac{a+2}{6}x^2y^2 - bx^2y + \frac{27b^4}{32(a+2)^3}\right).$$

The second summand $\frac{a+2}{6}x^4 + 2\frac{a+2}{6}x^2y^2 - bx^2y + \frac{27b^4}{32(a+2)^3}$ is a circuit polynomial, whose circuit number is precisely $|b|$, it is therefore nonnegative. Because $a \leqslant 4$, it follows that j_{t_0} is nonnegative, with a zero in (x_0,x_0). Now, let $t \in [0,1]\backslash\{t_0\}$. We show that the value 0 achieved at (x_0,x_0) is not a local minimum. The partial derivatives at that point are given by

$$\frac{\partial j_t}{\partial x}(x_0,x_0) = -\frac{\partial j_t}{\partial y}(x_0,x_0) = -\frac{9b^3(a+8-12t)}{8(a+2)^3},$$

and one of them is strictly negative as soon as $t \neq t_0$, proving that j_t takes negative values.

Consider now the case $a > 4$, where $\omega = -\frac{b^4}{8a^2}$. Setting $x_1 = \frac{b}{2\sqrt{a}}$, $y_1 = \frac{b}{a}$, we obtain

$$j_t(x_1,y_1) = -\frac{(a-4)(a+4)(1-t)}{16a^4}.$$

Consequently, $j_t(x_1,y_1) = 0$ if and only if $t = 1$, and $j_t(x_1,y_1) < 0$ when $t < 1$. Moreover, the polynomial j_1 is a circuit polynomial, since its circuit number is also $\mid b \mid$.

Finally, the case $-2 < a < 0$ is a direct application of Remark 4.3 and Proposition 2.5, since f is orthant dominated. Thus $f^{\mathrm{SONC}} = f^*$. □

Proposition 4.6 already shows that there might be a large gap between f^{SONC} and f^*: for b growing towards $\frac{2}{n-1}$, the bound f^{SONC} goes to $-\infty$, while f^* does not, and for $\frac{2}{n-1} < b \leqslant 2$, the SONC method does not provide any bound. This can happen because the negative term corresponds to a point on the boundary of the Newton polytope. However, even with a negative term corresponding with a point in the relative interior of the Newton polytope, Proposition 4.7 implies that the gap between f^{SONC} and f^* can be arbitrarily large:

Corollary 4.8. *For $f \in \mathbb{R}[x,y]$, denote by $\|f\|$ the supremum of the absolute values of the coefficients of f. There exists a sequence of polynomials $(f_k(x,y))_k$ of degree 4 such that*

$$\lim_{k\to+\infty} \frac{f_k^* - f_k^{\mathrm{SONC}}}{\|f_k\|} = -\infty.$$

Proof. Take
$$f_k(x,y) = \frac{1}{k}\left((x^4+y^4) + 8x^2y^2 - k(x^2y + xy^2)\right).$$
As soon as $k \geqslant 8$, we have $\|f_k\| = 1$. Moreover, according to Proposition 4.7,

$$f^* = \frac{1}{k}\frac{-27k^4}{16000} = \frac{-27k^3}{16000}, \quad f^{\mathrm{SONC}} = \frac{1}{k}\frac{-k^4}{512} = \frac{-k^3}{512}$$

and the difference $\frac{-17k^3}{64000}$ goes to $-\infty$ with k. □

4.3. **Monomial symmetric inequalities and mean inequalities.** A key feature of symmetric polynomials is their stable behavior with respect to increasing the number of variables. In the context of an increasing number of variables, the concept of nonnegativity in symmetric polynomials can be linked to so called symmetric polynomial inequalities. For example, the simple polynomial identity $x_1^2 + x_2^2 + \cdots + x_n^2 \geqslant 0$ is clearly valid for all number of variables. Related to such symmetric inequalities are inequalities of symmetric means, which arise when the polynomial identity is normalized, to ensure it takes, for every n, the same value at the point $(1, \ldots, 1)$. Classical examples of such inequalities are attributed to renowned mathematicians like Muirhead, Maclaurin, and Newton (see [9]) and are still an active area of research [36]. Furthermore, the well-known inequality of the arithmetic and geometric mean also falls into this category of symmetric inequalities. It is interesting to notice that Hurwitz [19] demonstrated that this fundamental inequality could be established through representation in terms of sums of squares. Recent works have further shed light on the potency of the sum of squares approach as a tool for establishing or disproving such inequalities [17, 6, 1, 2, 10]. Thus, our work on symmetric SAGE/SONC certificates naturally leads to the question whether these certificates can be used to prove symmetric inequalities for arbitrary or a large number of variables. We will focus here on the following setup with monomial symmetric functions, as these naturally fit into the framework of controlled (sparse) support of SONC polynomials.

Definition 4.9. For $\alpha, \beta \in \mathbb{N}^n$ we write $\alpha \sim \beta$ if β is obtained from α through a permutation. For a fixed $\alpha \in \mathbb{N}_0^n$ we define the associated *monomial symmetric polynomial* by

$$M_\alpha^{(n)} := \sum_{\beta \sim \alpha} x_1^{\beta_1} \cdot x_2^{\beta_2} \cdots x_n^{\beta_n}$$

and the *monomial mean* by

$$m_\alpha^{(n)} := \frac{M_\alpha^{(n)}}{M_\alpha^{(n)}(1, \ldots, 1)}.$$

Remark 4.10. Noticing that the value of $M_\alpha^{(n)}(1, \ldots, 1)$ equals the number of monomials in $M_\alpha^{(n)}$, which is given by $\frac{n!}{|\operatorname{Stab}(\alpha)|}$, we obtain directly the following identity for the normalized monomial symmetric polynomial:

$$m_\alpha^{(n)} = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \sigma(x^\alpha). \tag{4.6}$$

Let $\mathcal{T} \subset \mathbb{N}^n$ be an $\mathcal{S}_n$-invariant support, with $\widehat{\mathcal{T}}$ a set of representatives. Then, clearly the sets

$$\{M_\lambda^{(n)} : \lambda \in \widehat{\mathcal{T}}\} \text{ and } \{m_\lambda^{(n)} : \lambda \in \widehat{\mathcal{T}}\}$$

are bases of the space $V^n(\mathcal{T})$ of symmetric polynomials supported on $\mathcal{T}$. By extending λ by a 0 one obtains a natural identification between $\mathcal{T} \subset \mathbb{N}^n$ and its induced support $\tilde{\mathcal{T}} \subset \mathbb{N}^{n+1}$, and we can naturally define, for $k \geqslant n$, the space

$$V^k(\mathcal{T}) = \operatorname{span}(\{M_\lambda^{(k)} : \lambda \in \widehat{\mathcal{T}}\}) = \operatorname{span}(\{m_\lambda^{(k)} : \lambda \in \widehat{\mathcal{T}}\}),$$

and we define, for $k \geqslant n$, the cones

$$\begin{aligned} C_{\geqslant 0}^k(\mathcal{T}) &= \{f \in V^k(\mathcal{T}) : f \geqslant 0\}, \\ \text{and } C_{\mathrm{SONC}}^k(\mathcal{T}) &= \{f \in V^k(\mathcal{T}) : f \text{ is SONC}\}. \end{aligned}$$

Since for $k \geqslant n$, the resulting vector spaces are isomorphic, we can identify them with $\mathbb{R}^{|\widehat{\mathcal{T}}|}$. This identification depends on the chosen basis and depending on this choice, we obtain a sequence of cones in $\mathbb{R}^{|\widehat{\mathcal{T}}|}$, which we will denote by $C^{M,k}_{\geqslant 0}(\mathcal{T})$ respectively $C^{M,k}_{\mathrm{SONC}}(\mathcal{T})$ for the non-normalized monomial symmetric cones and $C^{m,k}_{\geqslant 0}(\mathcal{T})$ respectively $C^{m,k}_{\mathrm{SONC}}(\mathcal{T})$ for their counterparts in the normalized setup.

The choice of identification gives raise to different behaviors in the different setups: We start our investigation in the normalized setup.

Definition 4.11. Let λ, μ be partitions of n. If

$$\lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i \text{ for all } i \geqslant 1$$

we say that λ dominates μ and write $\lambda \succeq \mu$.

With this definition, the following classical inequality due to Muirhead (see [16, Sec. 2.18, Thm. 45] falls into the setup of normalized symmetric means introduced above.

Proposition 4.12 (Muirhead inequality)**.** *Let $\lambda, \mu \vdash d$. If $\lambda \succeq \mu$, then for all $n \geqslant \mathrm{len}(\mu)$, $m^{(n)}_{\lambda}(x) - m^{(n)}_{\mu}(x) \geqslant 0$ for all $x \in \mathbb{R}^n_{>0}$.*

Example 4.13. The Muirhead inequality yields that for all $x \in \mathbb{R}^n_{>0}$, we have $m^{(n)}_3 \geqslant m^{(n)}_{1,1,1}$. We want to certify this inequality with SONC certificates. With the standard change of variable $x_i = e^{y_i}$, we can actually use SAGE certificates, by observing that for $n = 3$ we have $f_3 := \frac{1}{3}(e^{3y_1} + e^{3y_2} + e^{3y_3}) - e^{y_1+y_2+y_3}$ is indeed a SAGE certificate. Moreover, by (4.6) we find that $f_n = \sum_{\sigma \in \mathcal{S}_n} \sigma f_3$ is therefore a SAGE and in particular nonnegative. □

As for example the AM/GM inequality is a special case of Muirhead's inequality, this classical result is connected to SONC certificates. For example, the authors in [18] derive a version of the symmetric decomposition shown in [25] using a version of this inequality. We now want to show that Muirhead's inequality can in fact be seen as a symmetric SONC certificate, i.e., that indeed one can always certify this inequality with SONC certificates. To this end, we use SONC techniques to prove the following version of the inequality. Notice that a SONC certificate is defined for the whole of $\mathbb{R}^n$ whereas the Muirhead certificate is restricted to $\mathbb{R}^n_{>0}$. As seen above, this difference can be consolidated by a change of variables leading to SAGE certificates. In order to keep notation simple, we will just speak of SONC certificates on the open positive orthant without transferring to SAGE.

SONC proof of Muirhead's inequality. We can assume that $\lambda \succ \mu$ since in the case $\lambda = \mu$ nothing is to show. By a theorem of Hardy, Littlewood and Polya ([3, Thm 2.1.1] and the discussion thereafter) this is equivalent to saying that μ can be represented as a convex combination of the permutations of λ, i.e., that there exists a vector $(\zeta_\sigma)_{\sigma \in \mathcal{S}_n}$ of nonnegative reals summing to 1 which satisfies

$$\mu = \sum_{\sigma \in \mathcal{S}_n} \zeta_\sigma \sigma \lambda.$$

Consider the orbits of λ and μ under $\mathcal{S}_n$, denoted $\mathcal{A} = \mathcal{S}_n \cdot \lambda$ and $\mathcal{B} = \mathcal{S}_n \cdot \mu$. For each $\alpha \in \mathcal{A}$, we define

$$c_\alpha = \nu_\alpha = \sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma\lambda = \alpha}} \zeta_\sigma.$$

Then consider

$$f = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha - x^\mu.$$

We claim that this polynomial is SONC on the open positive orthant. Indeed, we have $\sum_{\alpha \in \mathcal{A}} \nu_\alpha \alpha = \beta$ and

$$\sum_{\alpha \in \mathcal{A}} \nu_\alpha \ln \frac{\nu_\alpha}{e \cdot c_\alpha} = -\sum_{\alpha \in \mathcal{A}} \nu_\alpha = -1.$$

Now, taking the sum $\sum_{\sigma \in \mathcal{S}_n} \sigma f$ and considering the definition of the coefficients c_α we find

$$\sum_{\sigma \in \mathcal{S}_n} \sigma f = \sum_{\sigma \in \mathcal{S}_n} \sigma x^\lambda - \sum_{\sigma \in \mathcal{S}_n} \sigma x^\mu,$$

and obtain that f is SONC and have thus shown that the Muirhead inequality can be expressed as a SONC condition. □

Remark 4.14. Note that the condition in $\lambda \succeq \mu$ is both necessary and sufficient. Indeed, if $\lambda \not\succeq \mu$ then μ is not in the convex hull of $\mathcal{S}_n \cdot \lambda$, so by the hyperplane separation theorem, we can show that the function f in the proof has $-\infty$ as infimum.

Actually, Theorem 4.1 provides a slight generalization of Muirhead's inequality in two respects: first it allows to consider exponents that are not partitions of the same integer, and second we can add coefficients in the inequality. More precisely, we can prove inequalities of the form

$$c_\lambda m_\lambda^{(n)} - c_\mu m_\mu^{(n)} \geqslant \delta \text{ on } \mathbb{R}_{>0}^n$$

where the coefficients c_λ, c_μ and δ do not depend on n. As usual, up to rescaling, we may assume that $c_\lambda = 1$.

Definition 4.15. Let λ, μ be two integer partitions, not necessarily of the same integer. We say that λ dominates μ, denoted by $\lambda \succeq_* \mu$ if μ is in the convex hull of $\{0\} \cup \{\sigma\lambda \ : \ \sigma \in \mathcal{S}_{\mathrm{len}(\mu)}\}$.

Note that this dominance is a partial order on partitions of possibly different integers, that generalizes the usual dominance order on partitions from Definition 4.11. More precisely, if we denote $|\lambda| = \sum_i \lambda_i$ and $|\mu| = \sum_i \mu_i$, then λ (resp μ) is a partition of $|\lambda|$ (resp $|\mu|$). The condition $\lambda \succeq_* \mu$ then implies $|\lambda| \geqslant |\mu|$, and when $|\lambda| = |\mu|$, then $\lambda \succeq_* \mu$ precisely means $\lambda \succeq \mu$. With this notion we obtain the following generalization of Muirhead inequality, which is also a generalization of [18, Lemma 3.1]:

Theorem 4.16 (Generalized Muirhead inequality)**.** *Let λ, μ be two integer partitions such that $\lambda \succeq_* \mu$, and $c > 0$. Let*

$$\delta = -c\left(\frac{|\lambda| - |\mu|}{|\lambda|}\right)\left(c\frac{|\mu|}{|\lambda|}\right)^{\frac{|\mu|}{|\lambda|-|\mu|}}.$$

Then for any $n \geqslant |\lambda|$, the inequality

$$m_\lambda^{(n)} - c\, m_\mu^{(n)} \geqslant \delta \text{ on } \mathbb{R}_{>0}^n$$

is valid for:

(1) any $c > 0$ if $|\lambda| > |\mu|$. In this case, the inequality is an equality if and only if

$$x_1 = \cdots = x_n = \left(c\frac{|\mu|}{|\lambda|}\right)^{\frac{1}{|\lambda|-|\mu|}}.$$

(2) any $1 \geqslant c > 0$ if $|\lambda| = |\mu|$. The inequality is then always strict except if $c = 1$. In this case, equality occurs

$$\begin{cases} \text{on } \mathbb{R}^n_{>0}, & \text{if } \lambda = \mu, \\ \text{on the diagonal}, & \text{otherwise.} \end{cases}$$

Remark 4.17. In the second situation where $|\lambda| = |\mu|$, and therefore $\delta = 0$, we recover a version of [18, Lemma 3.1] with the only restriction $c \leqslant 1$ on the coefficients, necessary for the polynomial to be nonnegative.

Proof. We consider the polynomial

$$f = m_\lambda^{(n)} - m_\mu^{(n)} = \frac{|\operatorname{Stab}_n \lambda|}{n!} \sum_{\alpha \in \mathcal{A}} x^\alpha - \frac{|\operatorname{Stab}_n \mu|}{n!} \sum_{\alpha \in \mathcal{B}} x^\beta.$$

In the first case, the condition on λ and μ allow us to apply Theorem 4.1 to the signomial g associated to f to see that $\inf_{\mathbb{R}^n_{>0}} f = g^{\mathrm{SAGE}} = g^*$. Then, Corollary 4.4 gives that g^* is given by the infimum on $\mathbb{R}_{>0}$ of the polynomial

$$h(t) = t^{|\lambda|} - c\, t^{|\mu|},$$

which occurs for $t = \left(c\frac{|\mu|}{|\lambda|}\right)^{\frac{1}{|\lambda|-|\mu|}}$ with $h(t) = \delta$. This provides the claimed inequality, and the unique minimizer of f on the open positive orthant is $(t, \dots, t)$.

The second situation corresponds with Remark 4.3: In this situation we have

$$h(t) = (1-c)t^{|\lambda|},$$

which provides the result in the second situation. The case $c = 1$ being the Muirhead inequality proved above. $\square$

We see that the property of symmetrization highlighted in (4.6) gives an identification between the cones in n and $n+1$ variables, which yields an increasing sequence of cones and moreover, this symmetrization is very favorable to SONC decompositions and can in fact be used quite nicely. In contrast to this, in the non-normalized setup, there is a natural identification from $n+1$ to n variables by setting $x_{n+1} = 0$. This map sends $M_\alpha^{(n+1)}$ to $M_\alpha^{(n)}$ and maps both $C^{n+1}_{\geqslant 0}(\mathcal{T})$ and $C^{n+1}_{\mathrm{SONC}}(\mathcal{T})$ into $C^n_{\geqslant 0}(\mathcal{T})$ and $C^n_{\mathrm{SONC}}(\mathcal{T})$, respectively. Therefore, in this context we obtain a decreasing sequence of cones in $\mathbb{R}^{|\widehat{\mathcal{T}}|}$. In the setup of polynomials of fixed degrees it can be shown (see for example [10, Theorem II.2.5]) that both the sequences of cones of symmetric nonnegative forms as well as the cones of symmetric sums of squares approach a full dimensional limit. The next example shows however, that this may fail for the SONC cone:

Example 4.18. Consider the set of representatives $\mathcal{T} = \{(6), (3,3)\}$ and for $n \geqslant 6$, we take $M_{(6)}^{(n)} = \sum_{i=1}^n x_i^6$ and $M_{(3,3)}^{(n)} = \sum_{1 \leqslant i < j \leqslant n} x_i^3 x_j^3$. Defining $f_n := \alpha M_{(6)}^{(n)} + \beta M_{(3,3)}^{(n)}$ we would like to know for which values of α and β the resulting family of symmetric polynomials f_n is nonnegative for all values of n, and for which values this nonnegativity can be established by SONC certificates. Since $\left(\sum_{i=1}^k x_i^3\right)^2 = M_{(6)}^n + 2M_{(3,3)}^n$, it is clear that for all n the set of α and β such that $f_n \geqslant 0$ is two-dimensional. However, by Propositions 2.2 and 2.7 we find that $\alpha M_{(6)}^{(n)} + \beta M_{(3,3)}^{(n)} \in C^n_{\mathrm{SONC}}(\mathcal{T})$ if and only if there exists $\alpha_n > 0$ such that $\alpha_n(x_1^6 + x_2^6) + \beta x_1^3 x_2^3 \geqslant 0$ and $\sum_{1 \leqslant i < j \leqslant n} \alpha_n \leqslant \alpha$. However, the first condition implies that $\alpha_n \geqslant \frac{\beta^2}{4}$, and therefore, for

$n \gg 6$, we have $\sum_{1\leqslant i<j\leqslant n} \alpha_n \geqslant \binom{n}{2}\frac{\beta^2}{4} > \alpha$. This is, however, impossible and we can thus conclude that $\beta = 0$. So the set of all SONC certifiable symmetric inequalities in this setup is of lower dimension. □

These investigations give the following theorem.

Theorem 4.19. *Let $\mathcal{T} \subset \mathbb{N}^n$ be $\mathcal{S}_n$-invariant and $\mathcal{T}^+ = \mathcal{T} \cap (2\mathbb{N})^n$. Assume that for every $\beta \in \widehat{\mathcal{T}} \setminus \mathcal{T}^+$ we have $\beta \in \operatorname{conv}(\mathcal{T}^+ \cup \{0\})$. Then:*

(1) The sequence of cones $C^{m,k}_{\mathrm{SONC}}(\mathcal{T})$ is increasing and full-dimensional.

(2) The cones $C^{M,k}_{\mathrm{SONC}}(\mathcal{T})$ can be of strictly lower dimension than the corresponding cones $C^{M,k}_{\geqslant 0}(\mathcal{T})$.

Proof. The proof for (1) follows since we get the inclusions from (4.6) and the full-dimensionality after symmetrization from Proposition 2.6, while (2) is established by Example 4.18. □

Theorem 4.19 gives some indications that in the setup of symmetric inequalities given by monomial symmetric polynomials that are not normalized, the SONC approach may in general not be able to certify nonnegativity for a large fraction of nonnegative forms if n is large. We leave it as a future task to study the relation of the cones $C^{M,k}_{\mathrm{SONC}}(\mathcal{T})$ anc $C^{M,k}_{\geqslant 0}(\mathcal{T})$ in Theorem 4.19(2) in more detail.

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