# Homotopy Techniques for Real-Time Visualization of Geometric Tangent Problems

Daniel Kotzor* Thorsten Theobald†
Technische Universität München

## ABSTRACT

This note accompanies a video presentation on the use of homotopy methods for real-time visualizing a class of geometric tangent problems in $\mathbb{R}^3$.

## 1. INTRODUCTION

Several applications in three-dimensional computational geometry can be reduced to computing the *lines tangent to four given spheres in* $\mathbb{R}^3$. This set of applications includes visibility computations with moving viewpoints [11, 4], computing smallest enclosing cylinders of point sets in $\mathbb{R}^3$ [1, 9], and placement problems in geometric modeling [6].

The aim of the video is to visualize the geometry and the algebra of this fundamental problem as well as the algebraic-geometric techniques to analyze it. From the algebraic-geometric point of view, the tangent problem is of degree 12.

Since many properties and constructions concerning this problem can best be understood in terms of *dynamic* configurations, *homotopy continuation techniques* seem to be particularly suited. In the last years, homotopy techniques have been improved and fruitfully applied for solving systems of polynomial equations (see, e.g., [3, 13]). However, for a dynamic visualization of our algebraic problem of degree 12, we do not only have to solve a single system, but instead have to solve several systems per second. The concrete aim of this project was to investigate in how far dynamic configurations of the tangent problem can be visualized in *real time.* We have implemented a prototype of a visualization tool, and the video provides experimental proof that visualization of algebraic-geometric problems of this degree in real time is indeed possible.

*daniel.kotzor@stud.tu-muenchen.de
†Corresponding author. Zentrum Mathematik, Technische Universität München, D–80290 München. theobald@ma.tum.de


*SoCG'02,* June 5-7, 2002, Barcelona, Spain.
ACM 1-58113-504-1/02/0006.

## 2. TANGENT PROBLEMS FOR SPHERES

The following theorem in [7] shows that the problem to compute the common tangents to four unit spheres in $\mathbb{R}^3$ is an algebraic problem of degree 12.

THEOREM 1. *Four (not necessarily disjoint) unit spheres in* $\mathbb{R}^3$ *have at most 12 common tangent lines unless their centers are located on the same line. Furthermore, there exists a configuration with 12 different real tangents, i.e., the upper bound is tight.*

If four spheres with arbitrary radii have only finitely many common tangent lines, then this number is also bounded by 12 (see [11]). However, for this case of general radii it is not yet known if there exist situations with non-collinear centers leading to an infinite number of common tangents.

With regard to possible consistency checks in programs and with regard to the relation of the tangent problem to classical problems in enumerative geometry, the following realization result has been shown in [12].

THEOREM 2. *For any number* $k \in \{0, \ldots, 12\}$ *there exists a configuration of four unit spheres in* $\mathbb{R}^3$ *which have exactly* $k$ *different common tangents in* $\mathbb{R}^3$.

This result contrasts the tangent problem to the famous problem of 27 lines on a smooth cubic surface, where for a cubic surface in $\mathbb{R}^3$ only the numbers 3, 7, 15, and 27 can be established with *real* lines. Another famous example in geometry is Apollonius' problem which asks for the circles tangent to three given circles. Here, there exist configurations with $k \in \{0, 1, \ldots, 6, 8\}$ real tangent circles but provably no configuration with 7 real tangent circles.

In order to visualize some aspects of these theorems, we use the following framework and results of [7]. Let $c_4 = (0, 0, 0)^T$, and let $c_1$, $c_2$, $c_3$ be linearly independent. Then the four centers define a tetrahedron in $\mathbb{R}^3$. Further, let $\ell = \{p + \mu s \, : \, \mu \in \mathbb{R}\}$ with $p, s \in \mathbb{R}^3$, $s \neq 0$, $p \perp s$, be a line tangent to the spheres $S(c_i, r)$ for *some* radius $r > 0$. Any valid direction vector $s$ of such a tangent uniquely determines $p$ and (since $||p|| = r$) also $r$. Setting $M := (c_1, c_2, c_3)^T$, the corresponding equation is

$$r = \frac{1}{2s^2} \left\| M^{-1} \begin{pmatrix} (c_1 \times s)^2 \\ (c_2 \times s)^2 \\ (c_3 \times s)^2 \end{pmatrix} \right\|. \qquad (1)$$

Let $A_i$ denote the surface area of the face opposite to $c_i$, i.e., $A_1 = ||c_2 \times c_3||/2$, $A_2 = ||c_3 \times c_1||/2$, $A_3 = ||c_1 \times c_2||/2$, $A_4 = ||(c_1 - c_2) \times (c_2 - c_3)||/2$, and let $F := (A_1^2 + A_2^2 + A_3^2 - A_4^2)/2$.

Further, let $t = (t_1, t_2, t_3)^T$ denote the coefficient vector expressing $s$ in the basis $c_1$, $c_2$, $c_3$. In particular, both $s$ and $t$ are homogeneous vectors. Then the direction vectors of the lines equidistant to $c_1, \ldots, c_4$ are given by the non-zero solutions to the homogeneous cubic equation

$$A_1^2 t_2 t_3 (t_2+t_3) + A_2^2 t_3 t_1 (t_3+t_1) + A_3^2 t_1 t_2 (t_1+t_2) + 2F t_1 t_2 t_3 = 0.$$

Since the radius condition (1) gives a quartic equation, the common tangents are given by the intersection points of a cubic and a quartic curve in projective plane.

## 3. HOMOTOPY CONTINUATION

**General framework.** Homotopy continuation methods serve to numerically find all solutions of a system of polynomial equations

$$f_1(x_1, \ldots, x_n) = \ldots = f_n(x_1, \ldots, x_n) = 0\,,$$

abbreviated $f(x) = 0$ (see [3, 13]). The idea of the homotopy technique is to start from a second system $g(x) = 0$ whose solutions are known a priori. Then we consider the family of systems of equations

$$0 = h_\lambda(x) := (1 - \lambda) g(x) + \lambda f(x)$$

for $0 \le \lambda \le 1$. By successively increasing $\lambda$ in small steps from 0 to 1 we can use either Newton's method to find the solutions for the next step, or solvers of ordinary differential equations. The latter approach is based on the equation

$$J(x(\lambda), \lambda) \frac{dx(\lambda)}{d\lambda} = -\frac{\partial h}{\partial \lambda}(x(\lambda), \lambda),\ J(x, \lambda) := \left(\frac{\partial h_i}{\partial x_j}(x, \lambda)\right),$$

which is implied by the Implicit Function Theorem.

**Homotopy methods for the tangents to spheres.** If the starting system $g(x) = 0$ of a homotopy solver has more solutions than the system $f(x) = 0$, some paths necessarily diverge as $\lambda \to 1$. Therefore a main concern in the design of homotopy solvers is to find an appropriate starting system of polynomials $g(x)$, which is expected to have the same number of zeros as $f(x)$. By Bernstein's Theorem, this means that the starting polynomials $g(x)$ should have the same Newton polytope as $f(x)$ (see, e.g., [3]).

For two reasons, homotopy techniques seem to be particularly suitable for visualizing configurations of the tangents to spheres. Firstly, for the given polynomial formulation the Bézout number (= product of the degrees) agrees with the number of expected zeroes. Secondly, geometric understanding of configurations suggests also to inspect topologically neighboring configurations (cf. [10]). For two-dimensional geometric problems, the latter issue is treated in dynamic geometry software such as CINDERELLA [8].

For numerical computations of the tangents based on eigenvalue techniques see [5].

## 4. IMPLEMENTATION ASPECTS

The homotopy-based visualization of dynamic tangent configurations has been prototypically implemented in Visual C++. The input to the program is a description of the dynamic configurations. For computing and visualizing the tangents of the initial configurations, the homotopy method starts from a standard starting system. For the subsequent configurations, it starts from the preceding configuration. Both Newton's method and numerical methods for solving the differential equation are implemented.

The 3D graphics have been implemented using the OPEN GL-based COIN 3D graphics library [2]. This library provides an application programming interface based on the widely distributed OPEN INVENTOR graphics library.

**Frontiers of the implementation.** Despite an automatic adaption of the step size, numerical problems of course arise whenever we reach too close to a configuration in which the Jacobian matrix $J$ is singular. If this configuration is only an intermediate configuration on a homotopy path, this can be avoided by choosing a long way round the singularity. However, if the singular configuration is our destination, then this strategy obviously does not work.

**Examples on the video.** The video serves to illustrate the geometry and the algebra of the tangent problem as well as to demonstrate the implementation. After shortly introducing the tangent problems and homotopy techniques, the video shows an animated configuration of four unit spheres with the maximum number of 12 different real tangent lines. Exemplarily, it then illustrates how to obtain a configuration with 9 different real tangent lines. Finally, the video shows a dynamic configuration which contains several changes in the number of real common tangent lines. Although (as always when using software encoding) producing the video influences the visualization program, the speed of the output corresponds to the real speed of the program (on an 800 MHz PC with Pentium III processor).

## 5. REFERENCES


[1] P. Agarwal, B. Aronov, and M. Sharir. Line transversals of balls and smallest enclosing cylinders in 3 dim. *Discrete Comput. Geom.*, 21:373–388, 1999.

[2] http://www.coin3d.org .

[3] D. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry*, vol. 185 of *GTM*. Springer, New York, 1998.

[4] O. Devillers, V. Dujmovic, H. Everett et. al. A linear bound on the expected number of visibility events. Preprint, 2002.

[5] O. Devillers, B. Mourrain, F. Preparata, and P. Trébuchet. On circular cylinders by 4 or 5 points in space. Technical Report 4195, INRIA Lorraine, 2001.

[6] C. Hoffmann, B. Yuan. On spatial constraint solving approaches. In Proc. Automated Deduction in Geom., vol. 2061 of Springer LNCS/LNAI, 1–15, 2000.

[7] I. Macdonald, J. Pach, and T. Theobald. Common tangents to four unit balls in $\mathbb{R}^3$. *Discrete Comput. Geom.*, 26:1–17, 2001.

[8] U. Kortenkamp and J. Richter-Gebert. Dynamic geometry I: The problem of continuity. In Proc. European Workshop Comput. Geom. 1999, 51–53.

[9] E. Schömer, J. Sellen, M. Teichmann, and C. Yap. Smallest enclosing cylinders. *Algorithmica*, 27:170–186, 2000.

[10] F. Sottile, T. Theobald. Lines tangent to $2n-2$ spheres in $\mathbb{R}^n$. To appear in *Trans. Am. Math. Soc.*

[11] T. Theobald. An enumerative geometry framework for algorithmic line problems in $\mathbb{R}^3$. To appear in *SIAM J. Computing.*

[12] T. Theobald. How to realize a given number of tangents to 4 unit balls in $\mathbb{R}^3$. To app. in *Mathematika.*

[13] J. Verschelde. PHCpack: A general-purpose solver for polynomial systems by homotopy continuation. *ACM Trans. Math. Software*, 25(2):251–276, 1999.