HOW TO REALIZE A GIVEN NUMBER OF TANGENTS TO FOUR UNIT BALLS IN $\mathbb{R}^3$

THORSTEN THEOBALD

Abstract. By a recent result, the number of common tangent lines to four unit balls in $\mathbb{R}^3$ is bounded by 12 unless the four centers are collinear. In the present paper, we complement this result by showing that indeed every number of tangents $k \in \{0, \ldots, 12\}$ can be established in real space. Our constructions combine geometric and algebraic aspects of the tangent problem.

1. Introduction

Several applications in geometric modeling [3], visibility computations [14], and the computation of smallest enclosing cylinders of a point set [12, 1] require to find the common tangent lines to four given (not necessarily disjoint) unit balls in $\mathbb{R}^3$. In these scenarios the tangent lines can be seen as finite characterization of certain extreme situations. However, already the questions of finiteness (under what conditions do there exist only finitely many common tangents?) and the number of solutions show that the tangent problem is much more involved than its simple formulation suggests. In fact, the question on the *maximum number* of common tangent lines in the finite case was first formulated by David Larman [6]. It was answered by the following theorem in [7].

Proposition 1. *Four unit balls in $\mathbb{R}^3$ have at most 12 common tangent lines unless their centers are located on the same line. Furthermore, there exists a configuration with 12 tangents, i.e., the upper bound is tight.*

In particular, the second part of this theorem positively answers a question of Karger [5], who asked for a configuration of four points in $\mathbb{R}^3$ with more than 8 (but finitely many) unit cylinders of revolution whose surfaces pass through the four points. In the present paper, we complement the result of Proposition 1 by asking:

> For which numbers $k \in \{0, \ldots, 12\}$ does there exist a configuration with exactly k different common tangents in *real* space?

The motivation for studying this question comes from several quite different aspects. Firstly, any knowledge on the subset $K \subset \{0, \ldots, 12\}$ of realizable numbers gives important information for the mentioned applications. In order to find the common tangents we can either start from a system of polynomial equations, or we can construct a univariate polynomial equation whose solutions encode the tangents. For both approaches the numerical computation of the tangents may become instable, especially for configurations of centers which are close to singular configurations (e.g.,

Key words and phrases. tangents, balls, enumerative geometry, real solutions, computational geometry .

2000 Mathematics Subject Classification 14N10, 68U05, 51N20, 14P99, 52A15 .

configurations corresponding to *reducible* polynomials in an algebraic description). If not all numbers $k \in \{0, \dots, 12\}$ can be established in real space this offers the possibility of strong and valuable consistency checks within a program. If, however, all numbers can be realized then this would prove the non-existence of such a control mechanism.

Secondly, the set of realizable numbers gives important insights into the algebraic, geometric and combinatorial structure of a core problem in computational geometry. Observe that the 4-ball problem can be seen as a purely geometric problem. In contrast to this, the proof of Proposition 1 is mainly of algebraic nature and therefore does not fit well together with additional purely geometric constraints (e.g., disjointness) on the balls. Here, the difficulties in the geometric construction of concrete configurations might be seen as an indication of the difficulty to establish a purely geometric proof.

Thirdly, exploring the realizable numbers allows to relate the 4-ball-problem (which arose from recent applications) to some well-studied problems in classical and enumerative geometry (which mainly arose from their natural formulations). Concerning one of the most famous problem from enumerative geometry, namely the number of 27 lines on a smooth cubic surface, the question of real solutions has already been studied long time ago ([11, 13], see also [9], p. 188). In particular, for a cubic surface in $\mathbb{R}^3$ only the numbers 3, 7, 15, and 27 can be established with *real* lines. Another famous example in geometry is Apollonius' problem which asks for the circles tangent to three given circles. For this problem, there exist configurations with $k \in \{0, 1, \dots, 6, 8\}$ real tangent circles but provably no configuration with 7 real tangent circles [8].

In the present paper, we show that the situation of the 4-ball-problem is different from these situations. Namely, we prove:

Theorem 2. *For any number $k \in \{0, \dots, 12\}$ there exists a configuration of 4 unit balls in $\mathbb{R}^3$ which have exactly k different common tangents in $\mathbb{R}^3$.*

For any $k \in \{0, \dots, 12\}$ we give geometric constructions leading to this number of common tangents (of course, some values of k are trivial). It turns out that for some constructions a purely geometric correctness proof seems to be out of reach. However, in these cases the algebraic framework of [10, 7] helps to establish a rigorous proof. This leads to nice and effective interactions between the geometry and the algebra of the problem.

Before giving an outline of the paper, we remark that the cases with 0, 1, 2, or ∞ tangents are trivial. For the unit balls centered in $c_1 = (0, 0, 0)$, $c_2 = (2, 0, 0)$, $c_3 = (4, 0, 0)$, $c_4 = (6, t, 0)$, the values $t = 0$, $t = 1$, $t = 2$, and $t = 3$ lead to ∞, 2, 1, and 0 tangents, respectively.

The constructions with $3, \dots, 12$ tangents are presented in the following order. In Section 2, we analyze constructions with 3, 6, and 12 tangents where the centers are the vertices of a regular tetrahedron. Based on this analysis, Section 3 deals with constructions where three centers form an equilateral triangle; this gives constructions with 3, 6, 9, and 7 tangents. Parallelogram configurations of the four centers are discussed in Section 4; in particular, this yields constructions with 4, 5, and 8 tangents. Finally, Section 5 gives constructions with 10 and 11 tangents. We close the paper with a short discussion of the relation between the algebra and the geometry of the tangent problem.

2. The Case of a Regular Tetrahedron

In [7], a specific configuration with exactly 12 common tangents is given, where the four centers constitute the vertices of a regular tetrahedron. The following complete classification of a regular tetrahedron configuration will be used within the constructions in the next sections. Let $c_1, \ldots, c_4$ be the centers of the four balls in Euclidean space, and let $B(c, r)$ denote the (closed) ball with center $c \in \mathbb{R}^3$ and radius r. By appropriate scaling, the four balls of radius r can be transformed into unit balls.

Lemma 3. *Let $c_1, \ldots, c_4$ be the vertices of a regular tetrahedron with edge length* 1.

(a) *For $1/2 < r < 3\sqrt{2}/8$ there exist exactly 12 common tangents to $B(c_1, r), \ldots, B(c_4, r)$.*

(b) *For $r = 1/2$ and $r = 3\sqrt{2}/8$ there exist exactly 3 and 6 common tangents, respectively.*

(c) *For $r < 1/2$ or $r > 3\sqrt{2}/8$ there do not exist any common tangents.*

In order to prove this theorem, we use the following framework and results of [10, 7] (see also [2]). Let $c_4 = (0, 0, 0)^T$, and let c_1, c_2, c_3 be linearly independent. Then the four centers define a tetrahedron in $\mathbb{R}^3$. Further, let $l = \{p + \mu s : \mu \in \mathbb{R}\}$ with $p, s \in \mathbb{R}^3$, $s \neq 0$, $p \perp s$, be a line tangent to the balls $B(c_i, r)$ for *some* radius $r > 0$. Any valid direction vector s of such a tangent uniquely determines p and (since $||p|| = r$) also r. Setting $M := (c_1, c_2, c_3)^T$, the corresponding equation is

$$r = \frac{1}{2s^2} \left\| M^{-1} \begin{pmatrix} (c_1 \times s)^2 \\ (c_2 \times s)^2 \\ (c_3 \times s)^2 \end{pmatrix} \right\| . \tag{1}$$

Let A_i denote the surface area of the face opposite to c_i, i.e., $A_1 = ||c_2 \times c_3||/2$, $A_2 = ||c_3 \times c_1||/2$, $A_3 = ||c_1 \times c_2||/2$, $A_4 = ||(c_1 - c_2) \times (c_2 - c_3)||/2$, and let

$$F := (A_1^2 + A_2^2 + A_3^2 - A_4^2)/2.$$

Further, let $t = (t_1, t_2, t_3)^T$ denote the coefficient vector expressing s in the basis c_1, c_2, c_3. In particular, both s and t are homogeneous vectors, i.e., multiplying s or t by a non-zero constant still gives the same direction. Then the direction vectors of the lines equidistant to $c_1, \ldots, c_4$ are given by the non-zero solutions to the homogeneous cubic equation

$$A_1^2 t_2 t_3 (t_2 + t_3) + A_2^2 t_3 t_1 (t_3 + t_1) + A_3^2 t_1 t_2 (t_1 + t_2) + 2F t_1 t_2 t_3 = 0. \tag{2}$$

Based on this framework we prove Lemma 3.

Proof. Let $c_4 = (0, 0, 0)^T$, $c_1 = (1, 0, 0)^T$, $c_2 = (1/2, \sqrt{3}/2, 0)^T$, $c_3 = (1/2, \sqrt{3}/6, \sqrt{6}/3)^T$ be the vertices of a regular tetrahedron with edge length 1. In particular, we have $A = B = C = D = F$. In this situation, the cubic (2) is reducible and can be decomposed into

$$(t_1 + t_2)(t_2 + t_3)(t_3 + t_1) = 0. \tag{3}$$

By symmetry of this equation it suffices to consider the factor $t_1 + t_2 = 0$. This linear equation can be parametrized by $(t_1, t_2, t_3)^T = (1, -1, \lambda)^T$, $-\infty < \lambda \leq \infty$. Here, the

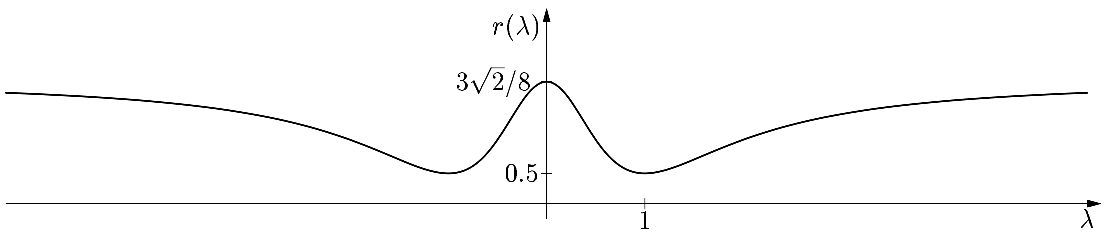


FIGURE 1. The function $r(\lambda) = \left(\frac{9\lambda^4+14\lambda^2+9}{32(\lambda^2+1)^2}\right)^{1/2}$

case $\lambda = \infty$ refers to the homogeneous vector $t = (0, 0, 1)^T$. Using (1), $r^2(\lambda)$ can be expressed by

$$r^2(\lambda) = \frac{9\lambda^4 + 14\lambda^2 + 9\lambda}{32(\lambda^2 + 1)^2}$$

with nominator of degree 4 and strictly positive denominator. The function graph of $r(\lambda)$ is depicted in Figure 1. Elementary calculus yields

$$\left[r^2(\lambda)\right]' = \frac{\lambda(\lambda^2 - 1)}{4(\lambda^2 + 1)}$$

with strictly positive denominator. Hence,

$$\begin{aligned} \min r(\lambda) &= r(1) = r(-1) = 1/2, \\ \max r(\lambda) &= r(0) = \lim_{\lambda \to -\infty} r(\lambda) = \lim_{\lambda \to \infty} r(\lambda) = 3\sqrt{2}/8 \approx 0.5303. \end{aligned}$$

Note that the difference between $\min r(\lambda)$ and $\max r(\lambda)$ is rather small. The extreme values and the strict monotony of $r^2(\lambda)$ between these values show: for $1/2 < r < 3\sqrt{2}/8$ there are four different solutions of λ and hence four different tangents. Considering all three factors of (3), there are exactly 12 different tangents altogether.

In case $r = 1/2$ these 12 tangents collapse to 3 tangents. The direction vectors in t-coordinates are $(1, 1, -1)$, $(1, -1, 1)$, and $(-1, 1, 1)$, respectively. In case $r = 3\sqrt{2}/8$ the 12 tangents collapse to 6 tangents; the direction vectors are the direction vectors of the 6 tetrahedron edges. □

Figure 2 shows a regular tetrahedron configuration with edge length 1 and radius $r = 53/100$. A tangent to $B(c_1, r), \ldots, B(c_4, r)$ can also be interpreted as axis of a circular cylinder with radius r circumscribing the tetrahedron with vertices $c_1, \ldots, c_4$. Hence, the following statement concerning circumscribing cylinders can be deduced immediately.

Corollary 4. *Let T be a regular tetrahedron with edge length $a > 0$. Then the smallest and largest circular cylinder circumscribing T have radius $a/2$ and $3\sqrt{2}a/8$, respectively.*

Remark. The lower bound $a/2$ in Corollary 4 can also be deduced from the fact that a minimal circular cylinder *containing* a regular tetrahedron with edge length a has radius $a/2$ [15].

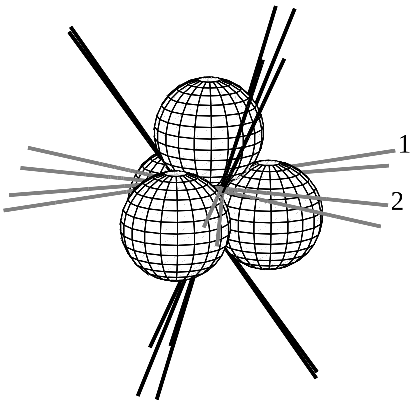
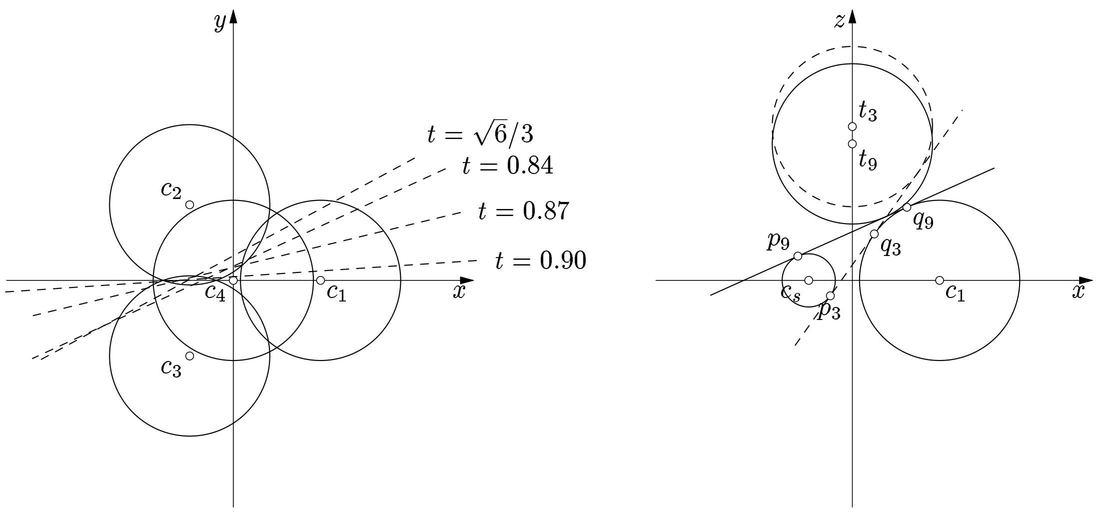


FIGURE 2. Construction of four (non-disjoint) balls with 12 common tangents. Here, if the coordinates of $c_1, \ldots, c_4$ are those of Section 3 then there are exactly 6 tangents which touch all balls with positive z-coordinate. These tangents are drawn in grey color.



(a) Projection on the xy-plane (b) Section through the xz-plane

FIGURE 3. Different views of the constructions with 3, 6, and 9 tangents. The common radius of the balls is 0.53.

3. Equilateral Triangle Constructions

In this section, we give configurations with 3, 6, 7, and 9 tangents. We start from a regular tetrahedron configuration with edge length 1. However, in order to stress symmetries, we now use the coordinates $c_1 = (\sqrt{3}/3, 0, 0)^T$, $c_2 = (-\sqrt{3}/6, 1/2, 0)^T$, $c_3 = (-\sqrt{3}/6, -1/2, 0)^T$, $c_4 = (0, 0, \sqrt{6}/3)^T$. Further, let $1/2 < r < 3\sqrt{2}/8$. Figure 3(a) shows the parallel projection of this configuration on the xy-plane. Note that $c_1, \ldots, c_3$ form an equilateral triangle in the xy-plane with center in the origin. By Lemma 3, the balls $B(c_i, r)$, $1 \le i \le 4$, have 12 common tangents.

r	t_9	t_3
0.51	0.8463	0.8478
0.52	0.8760	0.9293
0.53	0.9028	1.0172

TABLE 1. Some values of the radius r and the resulting coordinates t_9 and t_3 leading to 9 and 3 common tangents, respectively.

In this configuration with 12 tangents, 6 of the tangents touch all four balls with positive z-coordinate, and 6 tangents touch exactly two balls with negative z-coordinates (see Figure 2). We call these tangents the upper and the lower tangents, respectively.

Now observe what happens when replacing the z-coordinate in c_4 by increasing values $t > \sqrt{6}/3$. The geometry of this process implies: the z-coordinate $s_3/||s||$ of the unit direction vector increases, until eventually – for some value $t = t_9$ – the tangent touches two of the balls $B(c_1, r)$, $B(c_2, r)$, $B(c_3, r)$ at the same point (see Figure 3(a) for an illustration of the xy-projection). In the latter situation, the 6 upper tangents collapse to 3 tangents. Figure 3 depicts the section of this constellation through the xz-plane. One of these 3 remaining upper tangents touches $B(c_2, r)$ and $B(c_3, r)$ in the same point, namely on the circle where the boundaries of $B(c_2, r)$ and $B(c_3, r)$ intersect; this circle of intersection is located in the hyperplane $y = 0$. By symmetry of the equilateral triangle, the other 4 upper tangents collapse to 2 tangents in the same way. Since for $t = t_9$ the lower tangents neither have vanished nor collapsed (see below), the four balls have exactly 9 different common tangents.

In order to compute t_9, let $c_s = (-\sqrt{3}/6, 0, 0)^T$ and $r_s = \sqrt{r^2 - 1/4}$ denote the center and the radius of the circle of intersection. Then, setting $b = ||c_s - c_1||$ and $z_9 = ((\sqrt{3}/2)^2 - (r - r_s)^2)^{1/2}$, a straightforward geometric computation yields the two points on the tangent p_9, q_9,

$$p_9 = (-\sqrt{3}/6 - r_s(r - r_s)/b, 0, r_s z_9/b)^T, \qquad q_9 = (\sqrt{3}/3 - r(r - r_s)/b, 0, r z_9/b)^T.$$

p_9 is located on the circle of intersection, and q_9 is located on the boundary of $B(c_3, r)$ (see Figure 3(b)). Now the tangent condition for the ball $B((0, 0, t_9)^T, r)$ implies a quadratic equation for t_9. The larger one of the two solutions gives the desired value of t_9.

For values $t > t_9$ there exist at most 6 tangents. Analogous to the critical case with 9 tangents there exists some value t_3 where the 6 lower tangents collapse to three tangents. The dashed lines in Figure 3(b) show the section of this situation through the xz-plane. The tangent in the xz-plane is given by the two points

$$p_3 = (-\sqrt{3}/6 + r_s(r + r_s)/b, 0, -r_s z_3/b)^T, \qquad q_3 = (\sqrt{3}/3 - r(r + r_s)/b, 0, r z_3/b)^T,$$

where $z_3 = ((\sqrt{3}/2)^2 - (r + r_s)^2)^{1/2}$. For values $t > t_3$ there does not exist any common tangent to the four balls.

In particular, for any given r satisfying $1/2 < r < 3\sqrt{2}/8$ the two values t_3 and t_9 can be computed exactly. However, since the resulting expressions are quite long, we only give some numerical values to illustrate the relationships in size. Table 1 contains some values of r together with the resulting numerical values of t_3 and t_9. Figure 4 illustrates the construction.

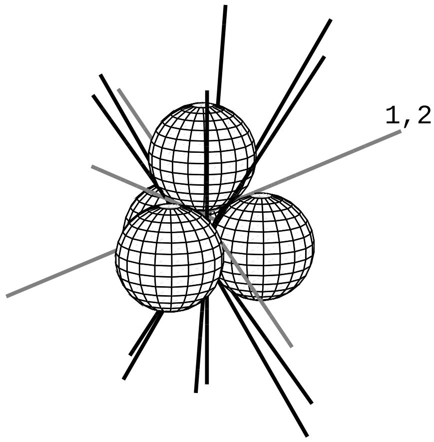


FIGURE 4. In this construction with 9 tangents, the remaining 3 upper tangents are drawn in grey color. The tangent labeled by 1,2 stems from the tangents labeled by 1 and 2 in Figure 2.

For a construction with 7 tangents, we start from the above configuration with 9 tangents. In this configuration, the remaining 3 upper tangents are critical in the sense that for any additional increment of the z-value of c_4 these tangents vanish. Now we move the fourth center $(0,0,t_9)^T$ along the line $(0,0,t_9)^T + \lambda(q_9 - p_9)$, $\lambda \in \mathbb{R}$. For any $\lambda > 0$, the line through p_9 and q_9 is still tangent to the four balls. However, the other two upper tangents from the situation $\lambda = 0$ immediately vanish for $\lambda > 0$. Hence, there exists some $\epsilon > 0$ such that any configuration with $0 < \lambda < \epsilon$ leads to exactly 7 common tangents. As an example, for $r = 0.53$ we can choose $0 < \lambda < 1/10$.

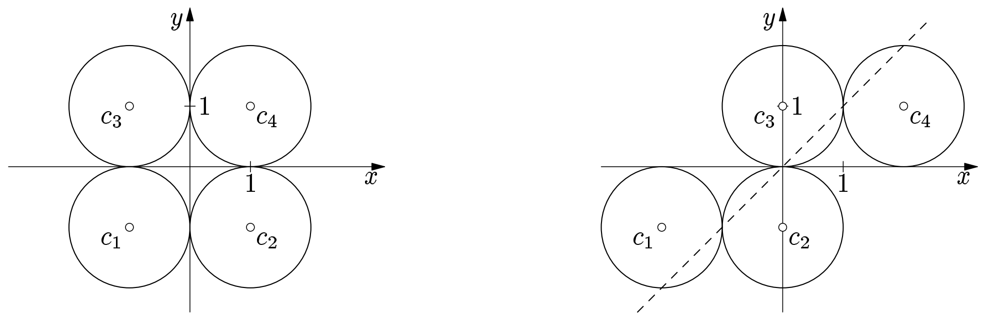
4. PARALLELOGRAM CONSTRUCTIONS

In order to give constructions with 4, 5, and 8 tangents, we start from the following situation depending on some parameter $a \in \mathbb{R}$. Let $c_1 = (-a-1,-1,0)^T$, $c_2 = (-a+1,-1,0)^T$, $c_3 = (a-1,1,0^T)$, $c_4 = (a+1,1,0)^T$ define a parallelogram in the xy-plane, and let $r = 1$. It was shown in [7] that a parallelogram configuration gives at most 8 common tangents.

As illustrated in Figure 5(a), the special case $a = 0$ yields a square. Obviously, these four balls have two common tangents, namely the lines $x = z = 0$ and $y = z = 0$.

Now observe what happens for parameter values $0 < a < 1$. For $0 < a < 1$, there exist exactly 5 tangents. As before, one of the tangents is the line defined by $y = z = 0$. However, the tangent $x = z = 0$ from the case $a = 0$ splits for $a > 0$ into four tangents. More precisely, for $0 < a < \infty$ there are two tangents parallel to the xy-plane (see the dotted line in Figure 5(b)); these two tangents are symmetric with respect to the xy-plane.

For $0 < a < 1$, there exist two tangents passing through the origin. These two tangents are symmetric with respect to the xz-plane, too. Here, we have to compute the lines which pass through the origin and which are tangent to $B(c_3,1)$ and $B(c_4,1)$. For $0 < a < 1$, there exist two lines with this property. By symmetry, these lines are also tangent to $B(c_1,1)$ and $B(c_2,1)$. For $a = 1$, these two lines collapse to the line $y = z = 0$. Obviously, if $0 < a < 1$ then multiplying the y-coordinates of all four



(a) $a = 0$ gives two common tangents. (b) $a = 1$ gives three common tangents.

FIGURE 5. Initial configurations for constructions with 5 and 8 tangents. In the right figure the dotted line shows the two tangents with z-coordinate $\sqrt{2}$ and $-\sqrt{2}$, respectively.

centers by a factor μ slightly larger than 1 yields a configuration with 4 instead of 5 common tangents.

Now we turn towards a construction with 8 tangents. For $0 < a \leq 1/2$, we multiply the y-coordinates of all four centers by some $0 < \mu \leq 1$ such that $||c_1 - c_3|| = ||c_2 - c_4|| = 2$. Geometrically, the upper balls "roll" on top of the lower balls (see Figure 6(a)). Elementary geometry yields $\mu = \sqrt{1 - a^2}/2$. Compared to the situation $a = 0$, for $0 < a < 1/2$ the tangent $y = z = 0$ is split into 4 tangents in the same way as in the transition from 2 to 5 tangents.

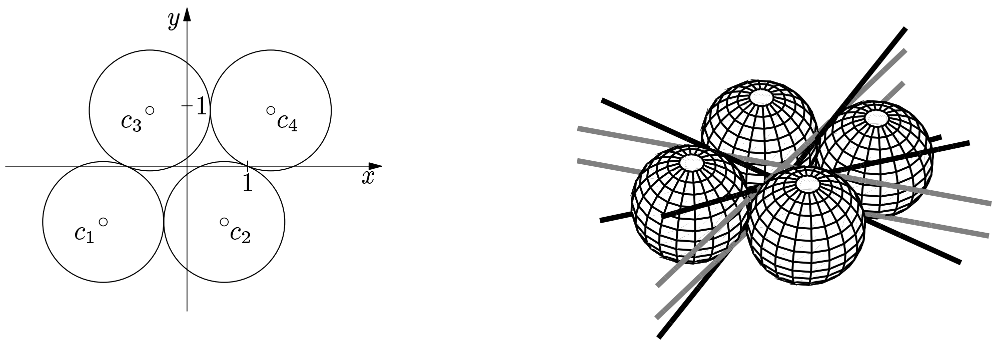
In particular, since $5^2 + 12^2 = 13^2$, the choice $a = 5/13$ yields the rational coordinates $c_1 = (-18, -12, 0)^T/13$, $c_2 = (8, -12, 0)^T/13$, $c_3 = (-8, -12, 0)^T/13$, $c_4 = (18, -12, 0)^T/13$. This configuration is depicted in Figure 6(b). For $a = 1/2$ the 4 tangents passing through the origin collapse to 2 tangents; hence, this yields another configuration with 6 tangents.

Note that in the configuration with 8 tangents there are 4 points which belong to more than one ball. However, the radius can be slightly decreased without altering the number of common tangents. After rescaling these disjont balls we obtain a configuration of 4 *disjoint* unit balls with 8 common tangents.

5. CONSTRUCTIONS WITH 10 AND 11 TANGENTS

In order to give constructions with 10 and 11 tangents, we start from the initial regular tetrahedron in Section 3 (see Figure 3(a)). However, for notational convenience, we exchange the centers c_3 and c_4. By Lemma 3, the radius $r = 3\sqrt{2}/8$ leads to 6 common tangents, whose directions are the directions of the six tetrahedron edges. Figure 7 shows the projection of this situation in the direction of the edge c_2c_4. Note that the lower left disc in this figure refers to the balls $B(c_2, r)$ and $B(c_4, r)$.

In this situation, we move the balls $B(c_2, r)$ and $B(c_4, r)$ slightly in opposite directions along the edge connecting their centers. This movement does not change the position of the tangent with direction c_2c_4. However, the movement will give some "freedom" to any of the five other tangents, and hence any of these edges will split



(a) Parallel projection on the xz-plane (b) Three-dimensional view

FIGURE 6. Construction with 8 tangents. In the right picture, tangents which are parallel to the xy-plane are drawn in grey color.

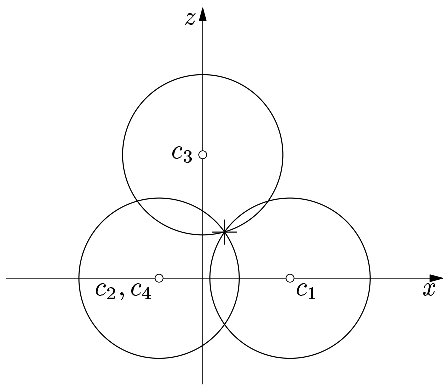


FIGURE 7. Parallel projection of $B(c_1, r), \ldots, B(c_4, r)$ in the xz-plane with $r = 3\sqrt{2}/8$. This is the projection along the edge with direction c_2c_4. The position of the common tangent in this direction is marked by the cross.

into two edges. Intuitively, this situation leads to 11 tangents; by increasing the radius slightly the tangent with direction c_2c_3 vanishes.

To formalize this idea, we consider the four centers $c_1 = (\sqrt{3}/3, 0, 0)^T$, $c_2 = (-\sqrt{3}/6, 1/2 + a, 0)^T$, $c_3 = (0, 0, \sqrt{6}/3)^T$, $c_4 = (-\sqrt{3}/6, -1/2 - a, 0)^T$ for some $a > 0$. In order to apply the algebraic framework from Section 2, we translate all centers by $-c_4$; this translation moves c_4 into the origin. Since the two faces $c_1c_2c_3$ and $c_1c_3c_4$ have the same area, and the two faces $c_1c_2c_4$ and $c_2c_3c_4$ have the same area, we have $A_1 = A_3$ and $A_2 = A_4$. Hence, the cubic (2) specializes to

$$(A_2^2 t_1 + A_1^2 t_3)(A_1^2(t_1t_2 + t_2^2 + t_2t_3) + A_2^2 t_1 t_3) = 0.$$

In particular, the cubic is reducible. Following the reducible case of [10, 7], the set of all tangents to the four balls $B(c_i, r)$ for *some* radius $r > 0$ can be parametrized by the line

$$(t_1, t_2, t_3)^T = (A_1^2, A_2^2\lambda, -A_1^2)^T, \qquad -\infty < \lambda \leq \infty \tag{4}$$

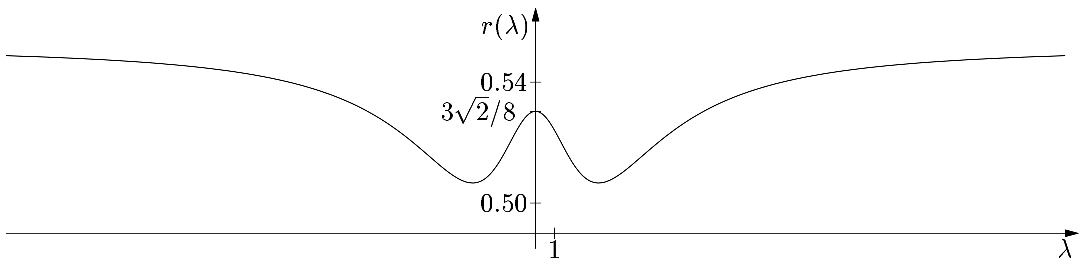


FIGURE 8. In the parametrization of the linear factor, the square of the radius function $r(\lambda)$ is a rational function in λ.

and the conic section

$$(5) \qquad (t_1, t_2, t_3)^T = (-A_1^2(\lambda - 1) - A_2^2, A_2^2\lambda, A_1^2(\lambda - 1)\lambda)^T, \qquad -\infty < \lambda \le \infty.$$

For a given radius, the linear function gives at most 4 common tangents and the conic section gives at most 8 common tangents. Analogous to Section 2, for both parametrizations the square of the radius function $r(\lambda)$ is a rational function in λ.

A suitable choice of a which will have the desired properties and which leads to rational values of A_1^2, A_2^2 is $a = (\sqrt{112/100} - 1)/2$. Then $A_1^2 = 78/400$, $A_2^2 = 84/400$, and the parametrization of the linear factor yields

$$r^2(\lambda) = \frac{169(1764\lambda^4 + 2492\lambda^2 + 1521)}{32(175\lambda^2 + 169)^2}.$$

The graph of $r(\lambda)$ is shown in Figure 8. The derivative of $r^2(\lambda)$ is

$$\left[r^2(\lambda)\right]' = \frac{1183\lambda(11438\lambda^2 - 7943)}{8(169\lambda^2 + 175)^3}$$

with nominator of degree 3 and strictly positive denominator. In particular, $r(0) = 3\sqrt{2}/8 \approx 0.5303$ is a local maximum, and

$$\lim_{\lambda \to -\infty} r(\lambda) = \lim_{\lambda \to \infty} r(\lambda) = \sqrt{\frac{169 \cdot 1764}{32 \cdot 175^2}} > 0.54.$$

Consequently, there exist exactly three different values of λ with $r(\lambda) = 3\sqrt{2}/8$; and for slightly larger radii r than $3\sqrt{2}/8$, say $r_1 < r \le r_2$ with $r_1 := 3\sqrt{2}/8$, $r_2 := 0.54$, we only obtain two such values of λ.

It remains to show: for a given radius $r \in [r_1; r_2]$, the parametrization of the conic section contains exactly 8 values of λ with $r(\lambda) = r$. Figure 9 illustrates the function graph of $r(\lambda)$. By (5), the λ-values $-\infty$, $-A_2^2/A_1^2 + 1$, 0, 1, ∞ represent the t-vectors $(0,0,1)^T$, $(0,1,-1)^T$, $(1,0,0)^T$, $(1,-1,0)^T$, and $(0,0,1)^T$, respectively. For all these λ-values we obtain $r(\lambda) = 3\sqrt{1378}/206 > 0.54$. These 5 values decompose the real axis into 4 intervals. If any of these intervals contains some value λ with $r(\lambda) < 3\sqrt{2}/8$, then for a given $r \in [r_1; r_2]$, there are at least 8 solutions with $r(\lambda) = r$. We can choose, e.g., the following values of λ: $-3/10$, $-5/100$, $2/10$, and 2. For any of these 4 values we obtain $r(\lambda) < 0.52$ which implies the desired result. Since there cannot be more than 8 solutions, there are exactly 8 solutions.

Finally, it can be easily checked that for $A_2 > A_1$ the line (4) and the conic section (5) do not have real intersection points; so the tangents stemming from the line

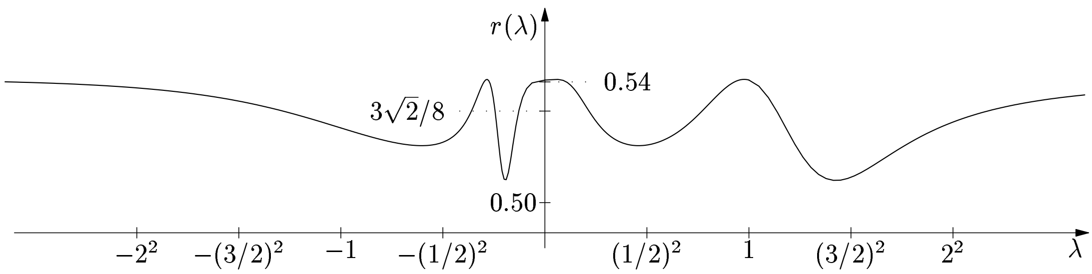


FIGURE 9. $r(\lambda)$ for the parametrization of the conic section. For better illustration of the region near $\lambda = 0$ the λ-axis is scaled quadratically.

and the tangents stemming from the conic section are indeed different. This completes the proof of the constructions with 10 and 11 tangents.

6. DISCUSSION

We have shown that for any $k \in \{0, \ldots, 12\}$ there exists a configuration with 4 unit balls and exactly k different common tangents. Although we have motivated every construction by purely geometric arguments, the rigorous proofs of some constructions (in particular 10, 11 tangents) heavily depend on the algebraic description of the problem. We interpret this observation as an indication why a purely geometric proof of Proposition 1 should be quite hard to establish.

Furthermore, observe that all constructions with more than 8 tangents are based on non-disjoint ball configurations. In fact, we conjecture that in case of disjoint balls the maximum number of common tangents is bounded by 8. The difficulty in treating this problem is the same one as above. Namely, it is difficult to exploit the condition of disjointness in the algebraic setting; but we do not know how to handle these situations from a purely geometric point of view.

Finally, the following open problem plays an important role in the interplay between the algebra and the geometry of the 4-ball problem. For some famous problems in enumerative geometry (flexes and bitangents of plane curves, lines on hypersurfaces, conics tangent to five given conics), the resulting Galois groups in the generic case are non-solvable [4], i.e., the solutions of these problems cannot be expressed in terms of roots. This situation reflects the difficulty of purely geometric methods to handle these problems. It is an open problem to characterize the (non-)solvability of the Galois groups for the 4-ball problem.

ACKNOWLEDGMENT

The author would like to thank David Cox for bringing the issue of realization to his attention.

REFERENCES

[1] P.K. Agarwal, B. Aronov, and M Sharir. Line transversals of balls and smallest enclosing cylinders in three dimensions. *Discrete Comput. Geom.*, 21:373–388, 1999.
[2] O. Bottema and G.R. Veldkamp. On the lines in space with equal distances to n given points. *Geom. Dedicata*, 6:121–129, 1977.

[3] C. Durand. *Symbolic and Numerical Techniques for Constraint Solving.* PhD thesis, Purdue University, 1998.
[4] J. Harris. Galois groups of enumerative problems. *Duke Math. Journal*, 46(4):685–724, 1979.
[5] A. Karger. Classical geometry and computers. *J. Geom. Graph.*, 2:7–15, 1998.
[6] D. Larman. Problem posed in the Problem Session of the DIMACS Workshop on Arrangements, Rutgers University, New Brunswick, NJ, 1990.
[7] I.G. Macdonald, J. Pach, and T. Theobald. Common tangents to four unit balls in $\mathbb{R}^3$. To appear in *Discrete and Computational Geometry.* Available from http://www-m9.mathematik.tu-muenchen.de/~theobald/ps/tangents.ps .
[8] D. Pedoe. The missing seventh circle. *Elem. Math.*, 25:14–15, 1970.
[9] G. Salmon. *A Treatise on the Analytic Geometry of Three Dimensions*, volume 1. Longmans and Green, London, 7th edition, 1928. Reprinted by Chelsea Publishing Company, New York, 1958.
[10] H. Schaal. Ein geometrisches Problem der metrischen Getriebesynthese. In *Sitzungsber., Abt. II, Österr. Akad. Wiss.*, volume 194, pages 39–53, 1985.
[11] L. Schläfli. On the distributions of surfaces of the third order into species, in reference to the absence of presence of singular points, and the reality of their lines. *Phil. Trans. Royal Society of London*, 153:193–241, 1863.
[12] E. Schömer, J. Sellen, M. Teichmann, and C.K. Yap. Efficient algorithms for the smallest enclosing cylinder problem. In *Proc. 8th Canad. Conf. Comput. Geom.*, pages 264–269, 1996.
[13] B. Segre. *The Non-Singular Cubic Surfaces.* Oxford University Press, 1942.
[14] T. Theobald. Work in preparation.
[15] B. Weißbach. Über die senkrechten Projektionen regulärer Simplexe. *Beitr. Algebra Geom.*, 15:35–41, 1983.

Thorsten Theobald, Zentrum Mathematik, Technische Universität München, D–80290 München, Germany
E-mail address: `theobald@ma.tum.de`