ON THE COMPLEXITY OF VISIBILITY PROBLEMS WITH MOVING VIEWPOINTS

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ABSTRACT. We investigate visibility problems with moving viewpoints in n-dimensional space. We show that these problems are $\mathbb{NP}$-hard if the underlying bodies are balls, $\mathcal{H}$-polytopes, or $\mathcal{V}$-polytopes. This is contrasted by polynomial time solvability results for fixed dimension. We relate the computational complexity to existing algebraic-geometric aspects of the visibility problems, to the theory of packing and covering, and to the view obstruction problem from diophantine approximation.

1. Introduction

Computer graphics and visualization deal with preparing data in order to show ("visualize") these data on a (two-dimensional) computer screen. In computer graphics, the original data typically stem from the three-dimensional Euclidean space $\mathbb{R}^3$, whereas in scientific visualization the data might originate from spaces of much higher dimension (e.g., in information visualization or high-dimensional sphere models in statistical mechanics) [28].

In these scenarios, *visibility computations* play a central role [24]. In the simplest case, we are given a fixed viewpoint $v \in \mathbb{R}^n$, and the scene consists of a set $\mathcal{B}$ of bodies. Now the task is to compute a suitable two-dimensional projection of the scene ("to render the scene") that reflects which part of the scene is *visible* from the viewpoint v. In a more dynamic setting, the viewpoint can be moved interactively (see, e.g., [3, 21]). However, in general, after each movement of the viewpoint a new rendering process is necessary. In order to speed up this process, commercial renderers apply caching techniques [32].

From the algorithmic and geometric point of view it is desirable to establish a more global view of the scene in advance and answer questions like: Which of these bodies can be seen (at least partially) from *some* viewpoint within a given viewpoint area? The bodies which are not even partially visible from any of these viewpoints can be removed from the whole visualization process in advance. In the case of dense scenes (like in the visualization of dense crystals, consisting of many atoms) this can reduce the time consumption of the rendering processes significantly. In n-dimensional space invisibility of a body is a sufficient criterion for its invisibility in any low-dimensional projection.

As yet, algorithmic treatment of visibility computations with moving viewpoints in dimension at least three still bears many challenges (see the recent papers [11, 13, 14, 31]). A main reason for this can be seen in various intrinsic difficulties in the underlying complexity-theoretical, geometric and algebraic questions.

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In the present paper, we analyze the *binary Turing machine complexity* of visibility computations in spaces of variable dimension. The classes of geometric bodies under consideration are that of balls, that of polytopes represented as the convex hull of finitely many points ("$\mathcal{V}$-polytopes"), and that of polytopes represented by an intersection of finitely many halfspaces ("$\mathcal{H}$-polytopes"). Roughly speaking, we show the following results that characterize the borderline between tractable and hard. If the dimension of the space is part of the input, then checking visibility of a given body B in the scene is $\mathbb{NP}$-hard for all three classes. Moreover, these hardness results persist even for very restricted classes of polytopes. In the case where the given body B degenerates to a single point, we can prove also membership in $\mathbb{NP}$ for the two classes of polytopes. If however, the dimension is fixed then the visibility problem becomes solvable in polynomial time for all three classes. (For precise statements of the results see Theorems 2 and 3.)

Moreover, we relate these complexity results to existing results from several other perspectives. From the *algebraic-geometric point of view*, visibility computations with moving viewpoints require the study of the interaction of the geometric bodies with lines. In particular, it is essential to characterize certain extreme situations which correspond to common tangent lines to a given set of bodies. In dimension 2, the resulting geometric questions typically remain rather elementary (see [24, 25]). However, in dimension 3 already, and even for simple types of bodies, such as balls, the underlying geometric problems have a high algebraic degree and hence give rise to difficult questions of real algebraic geometry [22, 29]. See Section 4 for details.

We also relate our complexity results to Hornich and Fejes Tóth configurations from the theory of packing and covering. Our results imply that already the test whether a given visibility configuration is a Hornich or Fejes Tóth configuration is an $\mathbb{NP}$-hard problem.

Finally, we establish a link between our hardness results and the *view obstruction* or *lonely runner* conjecture from diophantine approximation [33, 9, 4]. Let $||x||_I$ denote the distance of a real number x to a nearest integer. Then, for each positive integer n, let

$$\kappa(n) = \inf_{v_1,\ldots,v_n \in \mathbb{N}} \sup_{\tau \in [0,1]} \min_{1 \le i \le n} ||\tau v_i||_I \, ,$$

a measure for simultaneous homogeneous diophantine approximation. Wills [33] and later Cusick [9] conjectured that $\kappa(n) = \frac{1}{n+1}$. Although this conjecture has been investigated in a series of papers in the last 30 years (see the list of references in [7]), the exact value of $\kappa(n)$ is known only for values up to 5. Our hardness results can be seen as a complexity-theoretical indication why the number-theoretical view obstruction problem is hard.

The present paper is organized as follows. In Section 2, we introduce the necessary notation and review known algorithmic results in dimension 3. In Section 3, we determine the computational complexity of the considered visibility problems. Finally, in Section 4, we establish connections between our complexity-theoretical results and the other mentioned fields.

2. Preliminaries and Known Results

Throughout this paper let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. In particular, let $\langle \cdot, \cdot \rangle$ and $|| \cdot ||$ denote the Euclidean scalar product and the Euclidean norm, respectively, and let $\mathbb{B}^n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ and $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : ||x|| = 1\}$ denote the unit ball und unit sphere.

2.1. **Geometric objects and the model of computation.** The geometric objects under consideration are particular convex bodies. A *convex body* (or simply *body*) is a bounded, closed, and convex set which contains interior points. Our model of computation is the binary Turing machine model: all relevant convex bodies are presented by certain rational numbers, and the size of the input is defined as the length of the binary encoding of the input data (see, e.g., [16, 18, 19]).

Specifically, a $\mathcal{B}$-*ball* B is a ball that is represented by a triple $(n; c, \rho)$ with $n \in \mathbb{N}$, $c \in \mathbb{Q}^n$, and $\rho^2 \in (0, \infty) \cap \mathbb{Q}$ such that $B = c + \rho\mathbb{B}^n$. Let $\mathcal{B}^n$ denote the class of all $\mathcal{B}$-balls in $\mathbb{R}^n$, and set $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}^n$.

For rational polytopes we distinguish between $\mathcal{H}$- and $\mathcal{V}$-presentations [18]. A $\mathcal{V}$-*polytope* is a polytope P which is represented by a tuple $(n; k; v_1, \ldots, v_k)$ with $n, k \in \mathbb{N}$, and $v_1, \ldots, v_k \in \mathbb{Q}^n$ such that $P = \text{conv}\{v_1, \ldots, v_k\}$, i.e., P is the convex hull of $v_1, \ldots, v_k$. An $\mathcal{H}$-*polytope* is a polytope P represented by a tuple $(n; k; A; b)$ with $n, k \in \mathbb{N}$, a rational $k \times n$-matrix A, and $b \in \mathbb{Q}^k$ such that $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. Let $\mathcal{P}^n_{\mathcal{H}}$ and $\mathcal{P}^n_{\mathcal{V}}$ denote the classes of $\mathcal{H}$- and $\mathcal{V}$-polytopes in $\mathbb{R}^n$, respectively, and set $\mathcal{P}_{\mathcal{H}} = \bigcup_{n\in\mathbb{N}} \mathcal{P}^n_{\mathcal{H}}$, $\mathcal{P}_{\mathcal{V}} = \bigcup_{n\in\mathbb{N}} \mathcal{P}^n_{\mathcal{V}}$.

For fixed dimension $\mathcal{H}$- and $\mathcal{V}$-presentations of a polytope can be transformed into each other in polynomial time. If, however, the dimension is part of the input then the size of one presentation may be exponential in the size of the other [23].

2.2. **Visibility problems.** A *ray issuing from* x is a set of the form $x + [0, \infty)w$ with some $w \in \mathbb{R}^n \setminus \{0\}$. If a ray issues from the origin then it is also called a *0-ray.* For $m + 1$ bodies $B_0, B_1, \ldots, B_m$ from a class $\mathcal{X}$ we call B_0 *visible (with respect to* $B_1, \ldots, B_m$*)* if there exists a visibility ray r for B_0, i.e., a ray issuing from some point $p \in B_0$ satisfying $r \cap B_i = \emptyset$ for all $1 \leq i \leq m$.

Our definition of algorithmic visibility problems depends on the class $\mathcal{X}$ of geometric bodies. Note that the dimension of the ambient space is part of the input.

Problem VISIBILITY$_{\mathcal{X}}$**:**

Instance: m, n, bodies $B_0, B_1, \ldots, B_m \subset \mathbb{R}^n$ from the class $\mathcal{X}$.
Question: Decide whether B_0 is visible with respect to $B_1, \ldots, B_m$.

A visibility problem is called *anchored* if B_0 is a single point located at the origin. With regard to a more restricted viewing region, we call B_0 *visible from the positive orthant* (with respect to $B_1, \ldots, B_m$) if there exists a visibility ray for B_0 contained in the (closed) positive orthant.

Problem QUADRANT VISIBILITY$_{\mathcal{X}}$**:**

Instance: m, n, bodies $B_0, B_1, \ldots, B_m \subset \mathbb{R}^n$ from the class $\mathcal{X}$.
Question: Decide whether B_0 is visible from the positive orthant with respect to $B_1, \ldots, B_m$.

In the basic form we do not require the bodies to be disjoint. We add the index $\emptyset$ if the input bodies $B_0, \ldots, B_m$ are required to be disjoint (e.g., VISIBILITY$_{\mathcal{B},\emptyset}$). If $\mathcal{X} = \mathcal{P}_{\mathcal{H}}$ or $\mathcal{X} = \mathcal{P}_{\mathcal{V}}$, we will usually denote the bodies by $P_0, \ldots, P_m$.

Remark 1. Using the techniques presented in the treatment of QUADRANT VISIBILITY, it is also possible to prove hardness results for many other classes of restricted viewing regions. For the sake of simplicity, we restrict ourselves to the one example of that type that is relevant for the view obstruction problem.

Let e_i be the i-th unit vector in $\mathbb{R}^n$. For $c \in \mathbb{R}^n$ and $\rho_1, \ldots, \rho_n > 0$, $\operatorname{conv}\{c \pm \rho_i e_i : 1 \le i \le n\}$ is called a *cross polytope* in $\mathbb{R}^n$. A *parallelotope* is a polytope $c + \sum_{i=1}^n [-1,1] z_i$ with $c \in \mathbb{R}^n$ and linearly independent $z_1, \ldots, z_n \in \mathbb{R}^n$.

For a set $A \subset \mathbb{R}^n$ let $\operatorname{pos} A = \{\sum_{i=1}^k \lambda_i x_i : k \in \mathbb{N},\, x_1, \ldots, x_k \in A,\, \lambda_1, \ldots, \lambda_k \ge 0\}$ denote the *positive hull* of A.

For $c \in \mathbb{R}^n$ and a j-flat F, let $d(c,F)$ denote the Euclidean distance of c from F.

3. COMPLEXITY RESULTS FOR VARIABLE DIMENSION

3.1. **Main results.** We analyze the binary Turing machine complexity of the visibility problems for the case of variable dimension. Our main intractability results are summarized in the following theorem.

Theorem 2. *(a) For $\mathcal{X} \in \{\mathcal{B}, \mathcal{P}_{\mathcal{H}}, \mathcal{P}_{\mathcal{V}}\}$ the problems* VISIBILITY$_{\mathcal{X}}$ *and* QUADRANT VISIBILITY$_{\mathcal{X}}$ *are $\mathbb{NP}$-hard. The hardness persists even if the instances are restricted to those for which the bodies are disjoint. Moreover, in case of $\mathcal{H}$-polytopes the hardness persists if all bodies are axis-aligned cubes, and in case of $\mathcal{V}$-polytopes if all bodies are axis-aligned cross polytopes.*
(b) For $\mathcal{X} \in \{\mathcal{P}_{\mathcal{H}}, \mathcal{P}_{\mathcal{V}}\}$ the anchored versions of VISIBILITY$_{\mathcal{X}}$ *and* QUADRANT VISIBILITY$_{\mathcal{X}}$ *are $\mathbb{NP}$-complete.*

These hardness results are contrasted by the following positive results for *fixed* dimension.

Theorem 3. *Let $\mathcal{X} \in \{\mathcal{B}, \mathcal{P}_{\mathcal{H}}, \mathcal{P}_{\mathcal{V}}\}$, and the dimension n be fixed. Then* VISIBILITY$_{\mathcal{X}}$ *and* QUADRANT VISIBILITY$_{\mathcal{X}}$ *can be solved in polynomial time.*

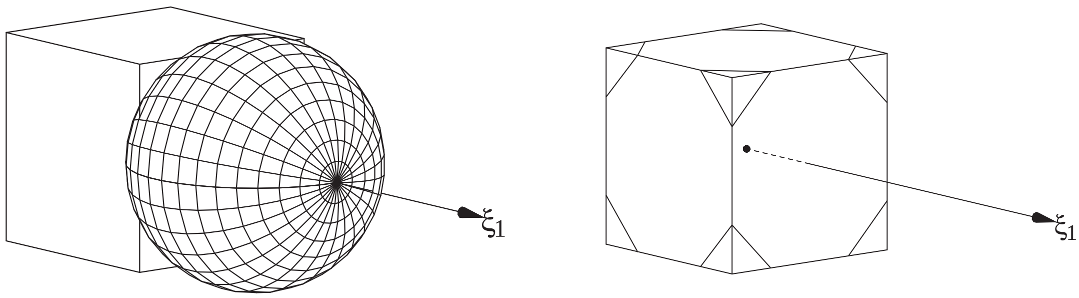
3.2. **Informal description of the hardness proofs.** Let us consider an anchored visibility problem.

In order to show $\mathbb{NP}$-hardness, we provide reductions from the well-known $\mathbb{NP}$-complete 3-satisfiability (3-SAT) problem [16]. Let $\mathcal{C} = \mathcal{C}_1 \wedge \ldots \wedge \mathcal{C}_k$ denote a 3-SAT formula with clauses $\mathcal{C}_1, \ldots, \mathcal{C}_k$ in the variables $\eta_1, \ldots, \eta_n$. Further, let $\overline{\eta_i}$ denote the complement of a variable η_i, and let the literals η_i^1 and η_i^{-1} be defined by $\eta_i^1 = \eta_i$, $\eta_i^{-1} = \overline{\eta_i}$. Let the clause $\mathcal{C}_i$ be of the form

$$\mathcal{C}_i = \eta_{i_1}^{\tau_{i_1}} \vee \eta_{i_2}^{\tau_{i_2}} \vee \eta_{i_3}^{\tau_{i_3}}, \tag{1}$$

where $\tau_{i_1}, \tau_{i_2}, \tau_{i_3} \in \{-1, 1\}$ and $1 \le i_1, i_2, i_3 \le n$ are pairwise different indices.

In our reduction, we construct an anchored visibility problem in $\mathbb{R}^n$. The reduction consists of two ingredients. First we enforce that any visibility 0-ray has a direction which is close to a direction in the set $\{-1,1\}^n$. For this purpose, consider the cube $[-1,1]^n$. For each of the $2n$ facets of the cube we construct a suitable body (a ball or a polytope) whose positive hull covers the whole facet with the exception of "regions near the vertices". We call these bodies *structural bodies.* Figure 1(a) shows this situation for the 3-dimensional case of a ball. Any visibility 0-ray can then be naturally associated with a 0-ray in one of the directions $\{-1,1\}^n$; this imposes a discrete structure on the problem. The $2n$ structural bodies are always part of the



(a) Placing structural bodies (b) Vertex simplices

Figure 1. Imposing discrete structure

construction, independent of the specific 3-Sat formula. The positions of each of these $2n$ bodies will depend linearly on some positive parameter γ. In fact, all bodies can be moved radially and their size be appropriately adjusted so that the crucial covering properties persist. The parameters will be used later to make the bodies disjoint. In order to define the "region near a vertex" we consider Figure 1(b). For every vertex v of $[-1,1]^n$ let the *vertex simplex of* v be defined as the convex hull of v and those n points which result by dividing exactly one component of v by 2. The construction will be such that any point in the boundary of $[-1,1]^n$ that is not contained in the positive hull of a structural body will be contained in some vertex simplex.

In the second step, we relate satisfying assignments of a clause (1) to certain visibility 0-rays. Let $t : \{\text{True}, \text{False}\} \to \{-1, 1\}$ be defined by $t(\text{True}) = 1$ and $t(\text{False}) = -1$. We utilize the correspondence between a truth assignment $a = (\alpha_1, \ldots, \alpha_n)^T \in \{\text{True}, \text{False}\}^n$ to the variables $\eta_1, \ldots, \eta_n$ and the 0-ray with direction $(t(\alpha_1), \ldots, t(\alpha_n))^T \in \{-1, 1\}^n$.

For each clause (1), we construct a body whose positive hull contains the set

$$\{x = (\xi_1, \ldots, \xi_n)^T \in \{-1,1\}^n \, : \, \xi_{i_1} = -t(\tau_{i_1}) \wedge \xi_{i_2} = -t(\tau_{i_2}) \wedge \xi_{i_3} = -t(\tau_{i_3})\}$$

as well as the corresponding vertex simplices, but which does not contain the set

$$\{x \in \{-1,1\}^n \, : \, \xi_{i_1} = t(\tau_{i_1}) \vee \xi_{i_2} = t(\tau_{i_2}) \vee \xi_{i_3} = t(\tau_{i_3})\}.$$

Again, the position of each body depends linearly on some positive parameter δ, which will be used to achieve disjointness of the bodies.

The construction will guarantee that a truth assignment a satisfies the given 3-Sat formula $\mathcal{C}$ if and only if there exists a visibility 0-ray.

3.3. **The case of balls.** The following simple and well-known distance formula is needed in the subsequent constructions. Here, for $x \in \mathbb{R}^n$ let $x^2 := \langle x, x \rangle$.

Remark 4. *Let $c \in \mathbb{R}^n$, $p \in \mathbb{R}^n$ and $q \in \mathbb{R}^n \setminus \{0\}$. Then the distance from c to the line $p + \mathbb{R}q$ is given by*

$$d(c, p + \mathbb{R}q)^2 \;=\; (p-c)^2 - \frac{\langle q, (p-c)\rangle^2}{q^2} .$$

Proof. The line $p + \mathbb{R}q$ has distance ρ from c if and only if the quadratic equation $(p + \lambda q - c)^2 = \rho^2$ in λ has a solution of multiplicity two. This gives the equation

$$\frac{\langle q, (p-c)\rangle^2}{q^2} - (p-c)^2 + \rho^2 \;=\; 0 .$$

□

Lemma 5. Anchored Visibility$_{\mathcal{B},\emptyset}$ *is* $\mathbb{NP}$*-hard. Also,* Anchored Visibility$_{\mathcal{B}}$ *is* $\mathbb{NP}$*-hard even if the instances are restricted to (not necessarily disjoint) balls of the same radius.*

Proof. We complete the construction outlined in Section 3.2 so as to provide a polynomial time reduction from 3-Sat to Anchored Visibility$_{\mathcal{B},\emptyset}$. Without loss of generality let $n \geq 4$.

Let us consider the $2n$ structural balls $S_i(\gamma_i) = (n; s_i(\gamma_i), \sigma_i(\gamma_i))$, $1 \leq i \leq 2n$, where γ_i is the scaling parameter of S_i as described above. Naturally, we place these balls symmetrically so that their centers lie on coordinate axes, i.e., let

$$s_i(\gamma_i) = \gamma_i e_i \text{ and } s_{n+i}(\gamma_{n+i}) = -\gamma_{n+i} e_i\,, \quad 1 \leq i \leq n.$$

In order to specify the radii $\sigma_i(\gamma_i)$ of the structural balls, let us consider $S_1(\gamma_1)$; see Figure 1(a). (The construction of the other balls is done analogously.) For convenience of notation, we shortly write $S = (n; s, \sigma)$.

In order to impose the discrete structure we will satisfy the following two conditions. Firstly, $\text{pos}(S)$ must not contain the vertices $\{1\} \times \{-1,1\}^{n-1}$. Secondly, $\text{pos}(S)$ must contain those points which result from the vertices of the facet $\{1\} \times [-1,1]^{n-1}$ after dividing exactly one of the last $n-1$ components by 2. The two conditions will yield an upper and a lower bound for σ.

We start with the first condition. Since any of the 0-rays $\{1\} \times \{-1,1\}^{n-1}$ has the same distance from the center s, it suffices to consider $[0,\infty)q$ with $q = (1,1,\ldots,1)^T$. By Remark 4,

$$d(s,[0,\infty)q)^2 \;=\; \gamma^2 \frac{n-1}{n}\,.$$

Consequently, we have to choose $\sigma^2 < \gamma^2(n-1)/n$. For the second condition, consider the point $q = (1,\ldots,1,1/2)^T$. Then Remark 4 yields

$$d(s,[0,\infty)q)^2 \;=\; \gamma^2 \frac{4n-7}{4n-3}\,.$$

Therefore, a ball centered in s with radius σ satisfying

$$\gamma^2 \frac{4n-7}{4n-3} \;<\; \sigma^2 \;<\; \gamma^2 \frac{n-1}{n} \tag{2}$$

guarantees the two conditions. The construction of structural balls for all $2n$ facets guarantees that any point in a facet of $[-1,1]^n$ that is not contained in the positive hull of a structural ball is contained in a facet of some vertex simplex.

Now we turn to the balls $C_i(\delta_i) = (c_i(\delta_i), \rho_i(\delta_i))$, $1 \leq i \leq k$, representing the k clauses. For notational convenience we describe the construction for the clause

$$\mathcal{C} = \eta_1^{-1} \vee \eta_2^{1} \vee \eta_3^{-1}.$$

It should of course be clear that the construction works just as well for other clauses. We abbreviate the ball for the clause $\mathcal{C}$ by $C = (n; c, \rho)$ (without referring explicitly to the dependence on the parameter $\delta := \delta_i$). We set $c = \delta(1,-1,1,0,\ldots,0)^T$, hence all the Boolean variables $\eta_4,\ldots,\eta_n$ are treated similarly.

In order to represent the given clause by the ball C we guarantee the following two properties. First, none of the 0-rays spanned by $\{-1,1\}^n \setminus (1,-1,1) \times \{-1,1\}^{n-3}$ must be contained in $\text{pos}(C)$. Within this set of rays, the ray $[0,\infty)q$ with $q =$

$(1,-1,-1,1,\ldots,1)^T$ (among others) leads to the smallest distance from C. Remark 4 implies

$$d(c,[0,\infty)q)^2 = \delta^2\frac{3n-1}{n}$$

which yields the condition $\rho^2 < \delta^2(3n-1)/n$.

Moreover, we guarantee the following second property. The positive hull of C must contain all the points in $(1,-1,1)\times\{-1,1\}^{n-3}$ as well as their vertex simplices. Among all these points and among the vertices of the vertex simplices, the vector $q=(1,-1,1/2,1,\ldots,1)^T$ leads to the ray with the largest distance from c. Remark 4 implies

$$d(c,[0,\infty)q)^2 = \delta^2\frac{12n-34}{4n-3}\,.$$

Hence, a ball centered in c with radius ρ satisfying

$$\delta^2\frac{12n-34}{4n-3} \;<\; \rho^2 \;<\; \delta^2\frac{3n-1}{n}$$

guarantees the two conditions for the clause ball. Note that the upper bound implies that the origin is not contained in the ball.

As yet, the definitions of the $2n$ structural balls and the k clause balls depend on the positive parameters $\gamma_1,\ldots,\gamma_{2n}$ and $\delta_1,\ldots,\delta_k$, respectively. By choosing these parameters appropriately, we make the balls disjoint. Since $\sigma_i < \gamma_i\sqrt{(n-1)/n}$ for the structural balls, we choose the parameter γ_i of the i-th structural ball successively so that

$$\gamma_i - \gamma_i\sqrt{\frac{n-1}{n}} > \gamma_{i-1} + \gamma_{i-1}\sqrt{\frac{n-1}{n}}\,.$$

Then

$$(\gamma_i e_i - \gamma_j e_j)^2 > (\sigma_i+\sigma_j)^2 \quad \text{for } i>j.$$

Setting $\gamma_0 = 1$, this leads to the condition

$$\begin{aligned}\gamma_i \;&>\; \left(\frac{1+\sqrt{\frac{n-1}{n}}}{1-\sqrt{\frac{n-1}{n}}}\right)^i \\ &=\; \left(2n-1+2\sqrt{n\cdot(n-1)}\right)^i\,.\end{aligned}$$

Hence, choosing $\gamma_i = (4n-1)^i$ for $1\le i\le 2n$ guarantees that the structural balls are pairwise disjoint. Note that the binary logarithm of these numbers grows only polynomially in the number of balls, i.e., we can choose rational centers and (squares of) radii of the structural balls of polynomial size. Similarly, the parameters $\delta_1,\ldots,\delta_k$ of the clause balls can be chosen. In particular, when also choosing δ_1 sufficiently large, then the clause balls are disjoint from the structural balls.

Now it is easy to show that the given 3-Sat formula $\mathcal{C}$ can be satisfied if and only if the single point B_0 is visible. Let $(\alpha_1,\ldots,\alpha_n)^T$ be a satisfying assignment of $\mathcal{C}$. Then there does not exist any ball B in the construction whose positive hull contains the 0-ray in direction $(t(\alpha_1),\ldots,t(\alpha_n))^T$. Hence, B_0 is visible. Conversely, let q be a visibility ray for B_0. Due to the structural balls the ray q intersects with the vertex simplex of some vector $v=(\nu_1,\ldots,\nu_n)^T\in\{-1,1\}^n$. Consequently, the truth assignment $(t^{-1}(\nu_1),\ldots,t^{-1}(\nu_n))^T$ is a satisfying assignment because otherwise the positive hull of some clause ball would contain the vertex simplex of v. Hence, $\mathcal{C}$ can be satisfied.

In order to achieve the result for balls of the same size, the role of σ and γ (respectively, ρ and δ) in (2) is interchanged in the sense that the radius σ is now given and a condition on γ is imposed. Clearly, these conditions for $\gamma_1, \ldots, \gamma_{2n}$ can be satisfied in the same way as the conditions on the radius before. $\square$

Corollary 6. VISIBILITY$_{\mathcal{B},\emptyset}$ *is* $\mathbb{NP}$*-hard.*

Proof. It is obvious that the proof for the case that B_0 is a single point generalizes to the case of a non-degenerated ball centered in 0 with some sufficiently small radius $\sigma_0 > 0$. In the following we will outline the precise argument.

Let $S_i = (n; s_i, \sigma_i)$, $1 \le i \le 2n$, and $C_j = (n; c_j, \rho_j)$, $1 \le j \le k$, be the structural balls and the clause balls in the proof of Lemma 5, and set $\hat{B}_0 = (n; 0, \sigma_0)$, where σ_0 is such that the inequalities given in the proofs of Lemma 5 hold for both σ_i, ρ_j and for $\sigma_i' := \sigma_i - \sigma_0$, $\rho_j' := \rho_j - \sigma_0$, $1 \le i \le 2n$, $1 \le j \le k$. Further, let $S_i' = (n; s_i, \sigma_i - \sigma_0)$, $1 \le i \le 2n$, and $C_j' = (n; c_j, \rho_j - \sigma_0)$, $1 \le j \le k$. Then it follows from the fact that $(B_0, S_1, \ldots, S_{2n}, C_1, \ldots, C_k)$ and $(B_0, S_1', \ldots, S_{2n}', C_1', \ldots, C_k')$ are Yes-instances of the visibility problem if and only if the given Boolean expression is satisfiable that the same holds for $(\hat{B}_0, S_1, \ldots, S_{2n}, C_1, \ldots, C_k)$. $\square$

3.4. **The case of $\mathcal{V}$-polytopes.**

Lemma 7. ANCHORED VISIBILITY$_{\mathcal{P}_\mathcal{V},\emptyset}$ *is* $\mathbb{NP}$*-hard. This result persists if the instance are restricted to axes-aligned cross polytopes.*

Proof. We establish a polynomial time reduction from 3-SAT to ANCHORED VISIBILITY$_{\mathcal{P}_\mathcal{V},\emptyset}$ based on the framework in Section 3.2. Again we assume $n \ge 4$.

This time, we choose the $2n$ structural bodies as cross polytopes of the form $S_i(\gamma_i) = \text{conv}\{s_i(\gamma_i) + \sigma_{ij}(\gamma_i)e_j \, : \, 1 \le j \le n\}$ with rational coefficients $s_i(\gamma_i)$, $\sigma_{ij}(\gamma_i)$ depending on the scaling parameter γ_i. The centers of the cross polytopes are defined by

$$s_i(\gamma_i) = \gamma_i e_i \text{ and } s_{n+i}(\gamma_{n+i}) = -\gamma_{n+i} e_i \, , \quad 1 \le i \le 2n \, .$$

Now we specify the coefficients σ_{ij}. We describe the construction of $S_1(\gamma_1)$ which for simplicity will be abbreviated by $S = \text{conv}\{s + \sigma_j e_j \, : \, 1 \le j \le n\}$. The construction of the other structural bodies is then similar.

For any choice of the parameters $\sigma_2, \ldots, \sigma_n > 0$, the $(n-1)$-dimensional cross polytope $S' = \text{conv}\{s + \sigma_j e_j \, : \, 2 \le j \le n\}$ is contained in the hyperplane $\xi_1 = \gamma$. Similar to the case of the balls, two conditions are imposed on the choice of $\sigma_2, \ldots, \sigma_n$. Firstly, the positive hull of S' must not contain the vertices $\{1\} \times \{-1, 1\}^n$. Secondly, the positive hull of S' must contain those points resulting from the vertices of the facet $\{1\} \times [-1, 1]^{n-1}$ by dividing exactly one of the last $n - 1$ components by 2.

We choose $\sigma_2 = \ldots = \sigma_n$. The necessary upper and lower bounds for σ_2 result as follows. Without loss of generality we consider the 0-ray $[0, \infty)(1, \ldots, 1)^T$. The vertex $\gamma(1, \ldots, 1)^T$ of $\gamma[-1, 1]^n$ is contained in a facet of the $(n-1)$-dimensional cross polytope $\text{conv}\{s \pm \gamma(n-1)e_j \, : \, 2 \le j \le n\}$. On the other hand, the point $\gamma(1, 1, 1, \ldots, 1, 1/2)^T$ is contained in a facet of the $(n-1)$-dimensional cross polytope with vertices $\text{conv}\{s \pm \gamma(n - 3/2)e_j\}$, $2 \le j \le n$. Hence, if σ_2 satisfies

$$\gamma \left(n - \frac{3}{2} \right) < \sigma_2 < \gamma \, (n-1)$$

then the two conditions enforcing the discrete structure are satisfied.

In order to make the $(n-1)$-dimensional polytope S' full-dimensional we consider some ε with $0 < \varepsilon < \gamma$. Then $s - \varepsilon e_1 \in \text{pos } S'$. Hence, by adding the vertices $s \pm \varepsilon e_1$ we obtain an n-dimensional cross polytope S with $\text{pos}(S) = \text{pos}(S')$.

Now we show how to represent a clause by a cross polytope. Again, we describe the construction for the clause $\eta_1^{-1} \vee \eta_2^1 \vee \eta_3^{-1}$. The associated cross polytope will be of the form $C = \text{conv}\{c \pm \rho_j e_j : 2 \leq j \leq n\}$ with $c = \delta(1, -1, 1, 0, \ldots, 0)^T$ and coefficients ρ_j (also depending on the parameter δ). For any choice of parameters $\rho_4, \ldots, \rho_n > 0$, the $(n-3)$-dimensional cross polytope $C' = \text{conv}\{c \pm \rho_j e_j : 4 \leq j \leq n\}$ is contained in the $(n-3)$-dimensional flat $\xi_1 = \delta$, $\xi_2 = -\delta$, $\xi_3 = \delta$. We choose $\rho_4 = \ldots = \rho_n$. As before, we add the vertices $c \pm \varepsilon e_j$, $1 \leq j \leq 3$, for some parameter $0 < \varepsilon < \delta$ to obtain a full-dimensional cross polytope. If $\rho_4 = 2(n-3)$ then the point $\delta(1, -1, 1/2, 1, \ldots, 1)^T$ is contained in the n-dimensional cross polytope. Hence, by choosing $\rho_4 > 2(n-3)$ the positive hull of C contains all the points in $(1, -1, 1) \times \{-1, 1\}^{n-3}$ as well as their vertex simplices. Moreover, since $\text{pos}(C)$ is contained in the cone defined by $\xi_1 \geq 0$, $\xi_2 \leq 0$, $\xi_3 \geq 0$, none of the vectors in $\{-1, 1\}^n \setminus (1, -1, 1) \times \{-1, 1\}^{n-3}$ is contained in the positive hull of the cross polytope.

Similarly to the proof of Lemma 5, we can choose the parameters $\gamma_1, \ldots, \gamma_{2n}$, $\delta_1, \ldots, \delta_k$, and ε (for making the bodies n-dimensional) in such a way that the bodies are pairwise disjoint and that their encoding lengths remain polynomially bounded. Hence, the polynomial time reduction from 3-SAT follows in the same way as in the proof of Theorem 5. □

Using an inclusion technique like in Corollary 6 we readily obtain the following corollary.

Corollary 8. VISIBILITY$_{\mathcal{P}_\mathcal{V}, \emptyset}$ *is* $\mathbb{NP}$*-hard even for axis-aligned cross polytopes.*

Lemma 9. ANCHORED VISIBILITY$_{\mathcal{P}_\mathcal{V}}$ *is contained in* $\mathbb{NP}$.

Proof. Let $(m; n; P_0, \ldots, P_m)$ be an instance of ANCHORED VISIBILITY$_{\mathcal{P}_\mathcal{V}}$ with $P_0 = \{0\}$ and $\mathcal{V}$-polytopes $P_1, \ldots, P_m$, and let $\mathcal{F}_{n-2}(P_i)$ denote the set of all $(n-2)$-dimensional faces of P_i, $1 \leq i \leq m$. The set of all linear subspaces lin F, $F \in \mathcal{F}_{n-2}(P_i)$, naturally decomposes the unit sphere $\mathbb{S}^{n-1}$ into $(n-1)$-dimensional sectors. For two 0-rays belonging to the (relative) interior (w.r.t. $\mathbb{S}^{n-1}$) of the same sector either both of them are visibility rays or none of them is. Each 0-ray through a vertex of a sector can be computed in polynomial time. In particular, two such vertices have a distance that is bounded below by a polynomial in the input. Hence for each sector there does indeed exist a polynomial size vector specifying a ray that meets the sector in its (relative) interior (w.r.t. $\mathbb{S}^{n-1}$). Hence there exists a polynomial size certificates for candidates for visibility rays.

It remains to show that it can be verified in polynomial time that a given witness ray does not intersect any of the polytopes P_i. Since the number of polytopes is bounded by the input length of the instance, it suffices to explain this polynomial verification method for a single polytope $P \in \{P_1, \ldots, P_m\}$. Let the $\mathcal{V}$-presentation of P be $P = \text{conv}\{v_1, \ldots, v_k\}$. P does not intersect the ray $[0, \infty)q$ if and only if the

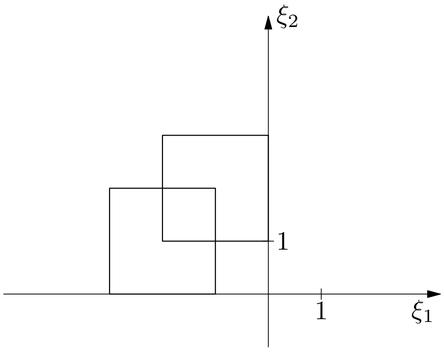


FIGURE 2. In order to represent the 2-clause $y_1^1 \vee y_2^{-1}$, all visibility rays in the orthant $\xi_1 \leq 0$, $\xi_2 \geq 0$ have to be blocked. This can be achieved by placing two unit squares centered at $(-2,1)^T$ and $(-1,2)^T$.

system

$$\begin{array}{rcll} \sum_{i=1}^k \mu_i v_i & = & \lambda q, & \\ \sum_{i=1}^k \mu_i & = & 1, & \\ \mu_i & \geq & 0, & 1 \leq i \leq k, \\ \lambda & \geq & 0 & \end{array}$$

does not have a solution. This can be checked in polynomial time by linear programming. □

3.5. **The case of $\mathcal{H}$-polytopes.**

Lemma 10. ANCHORED VISIBILITY$_{\mathcal{P}_\mathcal{H}}$ *and* VISIBILITY$_{\mathcal{P}_\mathcal{H}}$ *are* $\mathbb{NP}$*-hard. These statements persist if we restrict the polytopes to be axis-aligned n-dimensional unit cubes. The hardness also persists if we restrict the polytopes to be disjoint axis-aligned n-dimensional cubes.*

Proof. We give a polynomial time reduction from 3-SAT, but this time the proof differs from the framework in Section 3.2. We begin with the anchored version, in which P_0 is a single point located in the origin.

Let $\mathcal{C} = \mathcal{C}_1 \wedge \ldots \wedge \mathcal{C}_k$ be an instance of 3-SAT with clauses $\mathcal{C}_1, \ldots, \mathcal{C}_k$ in the variables $\eta_1, \ldots, \eta_n$. Let

$$\mathcal{C}_i = \eta_{i_1}^{\tau_{i_1}} \vee \eta_{i_2}^{\tau_{i_2}} \vee \eta_{i_3}^{\tau_{i_3}}.$$

We construct a set of axis-aligned unit cubes ensuring that 0-rays spanned by any (non zero) vector $b = (\beta_1, \ldots, \beta_n)^T$ with $\operatorname{sgn}(\beta_{i_1}) \in \{-\tau_{i_1}, 0\}$, $\operatorname{sgn}(\beta_{i_2}) \in \{-\tau_{i_2}, 0\}$, $\operatorname{sgn}(\beta_{i_3}) \in \{-\tau_{i_3}, 0\}$ cannot be visibility rays. Figure 2 depicts the idea of the construction for two variables y_1 and y_2 and the 2-clause $y_1^1 \vee y_2^{-1}$. Define the $2n-3$ axis-aligned unit cubes

$$\begin{array}{rcll} P_1 & = & -2\tau_{i_1} e_{i_1} - \tau_{i_2} e_{i_2} - \tau_{i_3} e_{i_3} + [-1,1]^n \,, & \\ P_2 & = & -\tau_{i_1} e_{i_1} - 2\tau_{i_2} e_{i_2} - \tau_{i_3} e_{i_3} + [-1,1]^n \,, & \\ P_3 & = & -\tau_{i_1} e_{i_1} - \tau_{i_2} e_{i_2} - 2\tau_{i_3} e_{i_3} + [-1,1]^n \,. & \\ P_j' & = & -\tau_{i_1} e_{i_1} - \tau_{i_2} e_{i_2} - \tau_{i_3} e_{i_3} + 2e_j + [-1,1]^n \,, & j \in \{1, \ldots, n\} \setminus \{i_1, i_2, i_3\} \,, \\ P_j'' & = & -\tau_{i_1} e_{i_1} - \tau_{i_2} e_{i_2} - \tau_{i_3} e_{i_3} - 2e_j + [-1,1]^n \,, & j \in \{1, \ldots, n\} \setminus \{i_1, i_2, i_3\} \,. \end{array}$$

All these cubes are contained in the set $\{x = (\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n : \operatorname{sgn}(\xi_{i_1}) \in \{-\tau_{i_1}, 0\}, \operatorname{sgn}(\xi_{i_2}) \in \{-\tau_{i_2}, 0\}, \operatorname{sgn}(\xi_{i_3}) \in \{-\tau_{i_3}, 0\}\}$, and none of the cubes contains the origin. The union of the $2n-3$ cubes contains all facets of the cube

$$-\tau_{i_1} e_{i_1} - \tau_{i_2} e_{i_2} - \tau_{i_3} e_{i_3} + [-1, 1]^n$$

except the three facets which are contained in one of the hyperplanes $\xi_{i_1} = 0$, $\xi_{i_2} = 0$, or $\xi_{i_3} = 0$. Namely, P_1, P_2, and P_3 contain the facets in the hyperplanes $\xi_{i_1} = -2\tau_{i_1}$, $\xi_{i_2} = -2\tau_{i_2}$, and $\xi_{i_3} = -2\tau_{i_3}$, respectively, and for $j \in \{1, \ldots, n\} \setminus \{i_1, i_2, i_3\}$ the cubes P_j and P'_j contain the facets in the hyperplanes $\xi_j = 1$ and $\xi_j = -1$. Hence, a ray $[0, \infty)b$ intersects one of the $2n-3$ cubes if and only if $\operatorname{sgn}(\beta_{i_1}) \in \{-\tau_{i_1}, 0\}$, $\operatorname{sgn}(\beta_{i_2}) \in \{-\tau_{i_2}, 0\}$, and $\operatorname{sgn}(\beta_{i_3}) \in \{-\tau_{i_3}, 0\}$.

Altogether, a ray $[0, \infty)b$ is a visibility ray for P_0 if and only if C can be satisfied. Hence, ANCHORED VISIBILITY$_{\mathcal{P}_{\mathcal{H}}}$ is $\mathbb{NP}$-hard even if we restrict the polytopes to be axis-aligned n-dimensional unit cubes. Note that if the instance cannot be satisfied then the union of the polytopes in our construction contains the boundary of the cube $[-1, 1]^n$. Hence, the single point P_0 can be replaced by the cube $[-1, 1]^n$, which shows that VISIBILITY$_{\mathcal{P}_{\mathcal{H}}}$ is $\mathbb{NP}$-hard even if the polytopes are restricted to be axis-aligned n-dimensional unit cubes.

In order to show that ANCHORED VISIBILITY$_{\mathcal{P}_{\mathcal{H}},\emptyset}$ and VISIBILITY$_{\mathcal{P}_{\mathcal{H}},\emptyset}$ are $\mathbb{NP}$-hard, we can scale the cubes as in the earlier proofs. □

Lemma 11. ANCHORED VISIBILITY$_{\mathcal{P}_{\mathcal{H}}}$ *is contained in* $\mathbb{NP}$.

Proof. The proof is analogous to that of Lemma 9. □

3.6. **Polynomial solvability results for fixed dimension.** In order to prove the polynomial solvability results for fixed dimension, we use the fact that the theory of real closed fields can be decided in polynomial time [2, 8]. More precisely, for rational polynomials $p_1(\xi_1, \ldots, \xi_n), \ldots, p_l(\xi_1, \ldots, \xi_n)$ in the variables $\xi_1, \ldots, \xi_n$, a *Boolean formula over* $p_1, \ldots, p_l$ is defined as a Boolean combination (allowing the operators $\wedge$, $\vee$, $\neg$) of polynomial equations and inequalities of the type $p_i(\xi_1, \ldots, \xi_n) = 0$ or $p_i(\xi_1, \ldots, \xi_n) \leq 0$. We consider the following decision problem for quantified Boolean formulas over the real numbers.

Problem REAL QUANTIFIER ELIMINATION:

Instance: n, l, rational polynomials $p_1(\xi_1, \ldots, \xi_n), \ldots, p_l(\xi_1, \ldots, \xi_n)$, a Boolean formula $\varphi(\xi_1, \ldots, \xi_n)$ over $p_1, \ldots, p_l$, and quantifiers $Q_1, \ldots, Q_n \in \{\forall, \exists\}$.

Question: Decide the truth of the statement

$$Q_1(\xi_1 \in \mathbb{R}) \ldots Q_n(\xi_n \in \mathbb{R}) \quad \varphi(\xi_1, \ldots, \xi_n).$$

In [2, 8] it was shown:

Proposition 12. *For fixed dimension n,* REAL QUANTIFIER ELIMINATION *can be decided in polynomial time.*

Remark 13. Despite of this polynomial solvability result for fixed dimension, current implementations are only capable of dealing with very small dimensions. Generally, there are two approaches towards practical solutions of decision problems over the reals. One is based on Collins' cylindrical algebraic decomposition (CAD) [8], and the other is the critical point method ([17]; for the state of the art see [1]).

In order to prove polynomial solvability of VISIBILITY$_\mathcal{B}$ for fixed dimension, we formulate the problem algebraically. We represent a ray $p + \lambda q$, $\lambda \geq 0$, by its initial vector $p \in \mathbb{R}^n$ and a direction vector $q \in \mathbb{R}^n$ with $||q|| = 1$. B_0 is visible with respect to $B_1 = (n; c_1, \rho_1), \ldots, B_m = (n; c_m, \rho_m)$ if and only if there exist $p, q \in \mathbb{R}^n$ such that for all $\lambda \in \mathbb{R}$ the following formula holds:

$$\begin{array}{lrcll} & ||q||^2 & = & 1\,, & \\ \text{and} & ||p - c_0||^2 & \leq & \rho_0^2\,, & \\ \text{and} & (\lambda < 0 \ \text{ or } \ ||p + \lambda q - c_i||^2 & \geq & \rho_i^2), & 1 \leq i \leq m\,. \end{array}$$

Hence, we have to decide the truth of the following statement:

$$\exists p \in \mathbb{R}^n \quad \exists q \in \mathbb{R}^n \quad \forall \lambda \in \mathbb{R}$$
$$||q||^2 = 1 \ \wedge ||p - c_0||^2 \leq \rho_0^2 \ \wedge \big((\lambda < 0 \ \vee \ ||p + \lambda q - c_i||^2 \geq \rho_i^2), \qquad 1 \leq i \leq m\big) \ .$$

After expanding the Euclidean norm and applying some trivial transformations (such as establishing the mentioned normal form $p_i(\xi_1, \ldots, \xi_n) \leq 0$ for the polynomial inequalities), this is a quantified Boolean formula of the required form. Hence, Proposition 12 implies the following statement.

Lemma 14. *For fixed dimension n,* VISIBILITY$_\mathcal{B}$ *can be solved in polynomial time.*

For the case of $\mathcal{H}$-polytopes, let $P_i = \{x \in \mathbb{R}^n \ : \ A_i x \leq b_i\}$ with $A_i \in \mathbb{Q}^{k_i \times n}$, $b_i \in \mathbb{Q}^{k_i}$, $0 \leq i \leq m$. P_0 is visible if and only if there exist $p, q \in \mathbb{R}^n$ such that for all $\lambda \in \mathbb{R}$ we have

$$\begin{array}{lrcll} & ||q||^2 & = & 1\,, & \\ \text{and} & A_0 p & \leq & b_0\,, & \\ \text{and} & \neg(\, A_i(p + \lambda q) & \leq & b_i\,), & 1 \leq i \leq m. \end{array}$$

Applying Proposition 12 on this formulation we can conclude:

Lemma 15. *For fixed dimension n,* VISIBILITY$_{\mathcal{P}_\mathcal{H}}$ *can be solved in polynomial time.*

Since for fixed dimension n, a $\mathcal{V}$-polytope can be transformed into a $\mathcal{H}$-polytope in polynomial time [15], this also implies

Corollary 16. *For fixed dimension n,* VISIBILITY$_{\mathcal{P}_\mathcal{V}}$ *can be solved in polynomial time.*

Similarly, by small modifications of the proofs, the polynomial time solvability results for VISIBILITY can also be transferred to QUADRANT VISIBILITY.

4. ON THE FRONTIERS OF THE RESULTS AND THEIR RELATIONS TO OUR OTHER FIELDS

4.1. **Relations to algebraic geometry.** Theorems 2 and 3 do not guarantee membership of VISIBILITY$_\mathcal{B}$ in $\mathbb{NP}$. Let us illuminate this situation from the algebraic point of view. First note that even though quantifier elimination methods can decide ANCHORED VISIBILITY$_\mathcal{B}$ or VISIBILITY$_\mathcal{B}$ for fixed dimension in polynomial time (see Lemma 14), it is not known how to compute a short witness of a positive solution with these methods (see [2]).

For "Yes" instances of ANCHORED VISIBILITY$_\mathcal{B}$ or VISIBILITY$_\mathcal{B}$ there always exists a ray in the closure of all visibility rays whose underlying line is simultaneously tangent to several balls. Hence, the question of membership in $\mathbb{NP}$ is tightly connected to the algebraic characterization of the lines simultaneously tangent to a given set of balls in $\mathbb{R}^n$. In particular, it is essential to characterize the lines tangent to $2n - 2$ balls, since the Grassmannian of lines in n-space has dimension $2n-2$ (i.e., a line in $\mathbb{R}^n$ has $2n-2$ degrees of freedom). In [27] it was shown that for $n \geq 3$, $2n-2$ balls in general position in $\mathbb{R}^n$ have $3 \cdot 2^{n-1}$ (complex) common tangent lines. Hence, the visibility problem in dimension n is tightly connected to an algebraic problem of degree $3 \cdot 2^{n-1}$.

Similarly, Theorems 2 and 3 do not guarantee membership of $\text{VISIBILITY}_{\mathcal{P}_{\mathcal{H}}}$ or $\text{VISIBILITY}_{\mathcal{P}_{\mathcal{V}}}$ in $\mathbb{NP}$. These questions are tightly connected to the common transversals to $2n-2$ given $(n-2)$-dimensional flats in $\mathbb{R}^n$. The generic number of (complex) transversals to $2n-2$ given $(n-2)$-flats in $\mathbb{R}^n$ is $\frac{1}{n}\binom{2n-2}{n-1}$; (see, e.g., [20, 26]).

In both cases (balls and polytopes), the algebraic degree is reflected by our hardness results in the Turing machine model.

4.2. **Relations to the theory of packing and covering.** Concerning $\mathbb{NP}$-hardness, Theorem 2 does not include a result for $\text{ANCHORED VISIBILITY}_{\mathcal{B},\emptyset}$ or $\text{VISIBILITY}_{\mathcal{B},\emptyset}$ if the balls are unit balls. However, the following statement shows that in "No"-instances of $\text{VISIBILITY}_{\mathcal{B},\emptyset}$ the number of balls necessarily grows exponentially in the input dimension n. Even if this does not rule out the existence of a polynomial time algorithm (since the running time of the algorithm is not measured in terms of the dimension n but in the overall length of the input size which in this case is exponential in n), it might give a useful sufficient criterion for large input dimensions.

Lemma 17. *Let $n \geq 6$, $m \in \mathbb{N}$, and let $B_0, B_1, \ldots, B_m$ be disjoint unit balls in $\mathbb{R}^n$. If $m < \sqrt{3n}\, e^{\frac{3}{8}(n-1)}$ then B_0 is visible with respect to $B_1, \ldots, B_m$.*

Proof. Without loss of generality we can assume that B_0 is the unit ball centered at the origin. Let $0 < r < 1$ and H be a hyperplane in $\mathbb{R}^n$ at distance r from the origin. Then the set of points on the unit sphere separated from the origin by H is called an *r-cap*. Since any ball B_i, $1 \leq i \leq m$, is disjoint from B_0, an elementary geometric computation shows that $\text{pos}(B_i)$ intersects the unit sphere in an r-cap with $\sqrt{3}/2 < r < 1$. A necessary condition for B_0 being invisible is that these r-caps cover the unit sphere. Let $\tau(n, r)$ denote the minimum number of r-caps covering the unit sphere. By Lemma 5.2 in [6], we have for $r > 2/\sqrt{n}$

$$\tau(n, r) \geq 2r\sqrt{n}e^{r^2(n-1)/2}\,.$$

Substituting the value $r = \sqrt{3}/2$ into this formula yields the desired estimation. □

Moreover, the problem VISIBILITY is closely related to difficult problems in the theory of packing and covering (see [30] or [35, Chapter 12]). A *Hornich configuration* in $\mathbb{R}^n$ is a set $\{B_1, \ldots, B_m\}$ of disjoint unit balls with $\{B_1, \ldots, B_m\} \cap \mathbb{B}^n = \emptyset$ such that the origin is not visible with respect to $B_1, \ldots, B_m$. The *Hornich number h_n* is the smallest number m of disjoint unit balls $B_1, \ldots, B_m$ such that $\{B_1, \ldots, B_m\}$ is a Hornich configuration. Hence, for the class of unit spheres, $\text{ANCHORED VISIBILITY}_{\mathcal{B},\emptyset}$ asks whether a given configuration is a Hornich configuration. Similarly, a *Fejes Tóth configuration* in $\mathbb{R}^n$ is a set $\{B_0, \ldots, B_m\}$ of disjoint unit balls such that B_0 is not visible with respect to $B_1, \ldots, B_m$. The *Fejes Tóth number ℓ_n* in $\mathbb{R}^n$ is the smallest number m of disjoint unit balls $B_0, \ldots, B_m$ such that there exists a Fejes Tóth configuration with m balls.

Even in dimension 3, the Hornich number h_3 is not known, and the best known bounds are $30 \leq h_3 \leq 42$. Lower and upper bounds for general dimensions n can be found in [35]. Concerning the Fejes Tóth number, Zong gave the upper bound $\ell_n \leq (8e)^n(n+1)^{n-1}n^{(n^2+n-2)/2}$ [34].

If the balls are allowed to be of different radius then Theorem 2 implies that already the test whether a given configuration is a (generalized) Hornich or Fejes Tóth configuration is $\mathbb{NP}$-hard.

4.3. **Quadrant visibility and view obstruction.** In Sections 3.2–3.5 our hardness results for VISIBILITY were based on reductions from 3-SAT in which any assignment $a \in \{\text{TRUE}, \text{FALSE}\}^n$ was identified with one of the 2^n quadrants in $\mathbb{R}^n$. For that reason, the question arises whether the hardness results still hold for more restricted viewing areas, say, for those viewing areas which are contained in a single quadrant.

In the following we prove the correponding part of Theorem 2.

Lemma 18. ANCHORED QUADRANT VISIBILITY$_{\mathcal{B},\emptyset}$ *is* $\mathbb{NP}$*-hard. Moreover,* ANCHORED QUADRANT VISIBILITY$_{\mathcal{B}}$ *is* $\mathbb{NP}$*-hard even if all balls are restricted to (not necessarily disjoint) balls of the same radius.*

Proof. Once more, we provide a reduction from 3-SAT, and therefore consider a 3-SAT formula in the variables $\eta_1, \ldots, \eta_n$. The essential idea of the reduction is to construct an instance of QUADRANT VISIBILITY in $(n+1)$-dimensional space $\mathbb{R}^{n+1}$. The 0-ray with direction $v := (1, \ldots, 1)^T$ is contained in the positive orthant Q of $\mathbb{R}^{n+1}$. By considering a hyperplane which is orthogonal to v and which intersects $(0, \infty)v$, we transfer the proof ideas of ANCHORED VISIBILITY to ANCHORED QUADRANT VISIBILITY.

In order to simplify the notation, we apply an orthogonal transformation to transform the diagonal ray $[0, \infty)v$ into $[0, \infty)e_{n+1}$, the non-negative part of the ξ_{n+1}-axis. By this operation, Q is transformed into a cone Q'. As in the proof of Lemma 5, we impose a discrete structure on the visibility problem. Namely, for some positive parameter $\tau > 0$ specified below, we associate the 2^n truth assignments $\{\text{TRUE}, \text{FALSE}\}^n$ with the 0-rays spanned by the vectors $\{1\} \times \{-\tau, \tau\}^n$. Note that the set $\{1\} \times [-\tau, \tau]^n$ is an n-dimensional cube in $\mathbb{R}^{n+1}$.

In order to achieve this discrete structure, we place $2n+1$ structural balls $S_i(\gamma_i, \tau) = (n; s_i(\gamma_i, \tau), \sigma_i(\gamma_i, \tau))$, $0 \le i \le 2n$, at the centers $c_0 = \gamma_0 e_{n+1}$, $c_i = \gamma_i(e_{n+1} + \tau e_i)$, $c_{n+i} = \gamma_{n+i}(e_{n+1} - \tau e_i)$, $1 \le i \le n$. In contrast to the proofs for ANCHORED VISIBILITY, the centers of the structural balls do not only depend on positive parameters γ_i, but also on the global positive parameter τ. Figure 3 shows this situation for the case $n = 2$. The parameter τ is chosen so that the n-dimensional cube $\{1\} \times [-\tau, \tau]^n$ is contained in Q'. The radii $s_i(\gamma_i, \tau)$, $1 \le i \le n$, of the structural balls can be chosen such that any visibility ray must be close to a vertex of the n-dimensional cube; this establishes the discrete structure. In a second step, the parameters γ_i can be used to scale the balls in order to make them disjoint.

Then, similarly to the proof of Lemma 5, we can construct balls representing the clauses of the 3-SAT formula in order to complete the desired polynomial time reduction. □

Clearly, the hardness result can be extended to the case of QUADRANT VISIBILITY$_{\mathcal{B},\emptyset}$, where B_0 is a proper ball. Moreover, by combining the proofs in Sections 3.4 and 3.5 with a lifting into $\mathbb{R}^{n+1}$, the hardness results can also be established for the case of $\mathcal{V}$- and $\mathcal{H}$-polytopes. (For $\mathcal{H}$-polytopes, the construction from the proof of Lemma 10 is carried out in the hyperplane given by $\xi_{n+1} = \gamma$; and – as in that lemma – the construction manages without any structural bodies.)

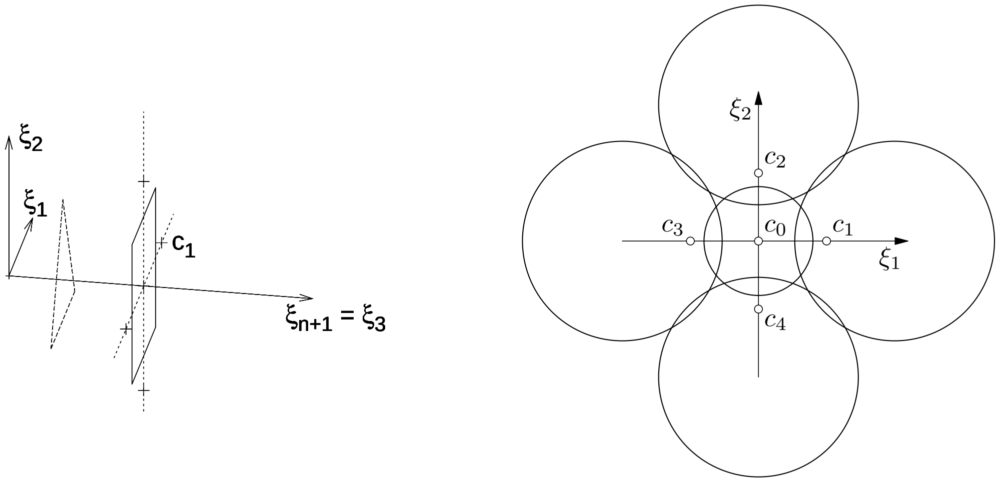


FIGURE 3. The figure shows how to impose discrete structure on ANCHORED QUADRANT VISIBILITY in case $n = 2$ and $\gamma_0 = \ldots = \gamma_{2n} =: \gamma$ (so all the centers of the structural balls are contained in the hyperplane $\xi_{n+1} = \gamma$). The positive hull of the triangle on the left represents Q', the positive orthant after the orthogonal transformation. The right figure shows the section of the balls through the hyperplane $\xi_{n+1} = \gamma$.

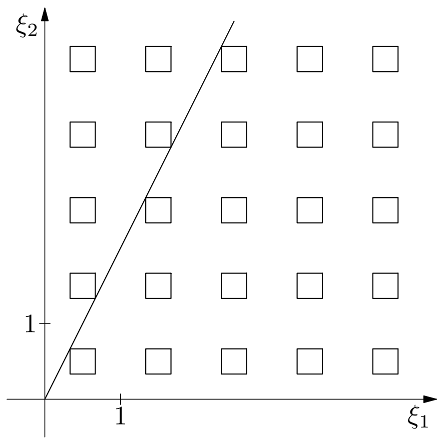


FIGURE 4. The picture shows the situation of the view obstruction problem in $\mathbb{R}^2$. In particular, $\lambda(2) = \frac{1}{3}$.

Note that the proof technique of Lemma 18 can also be generalized to establish hardness results for other classes of viewing areas.

The problem ANCHORED QUADRANT VISIBILITY is related to the problem of diophantine approximation introduced by Wills [33] of determining

$$\kappa(n) = \inf_{v_1,\ldots,v_n \in \mathbb{N}} \sup_{\tau \in [0,1]} \min_{1 \le i \le n} ||\tau v_i||_I .$$

Based on the pigeonhole principle, Wills showed $\frac{1}{2n} \le \kappa(n) \le \frac{1}{n+1}$ and conjectured $\kappa(n) = \frac{1}{n+1}$. This conjecture was later restated by Cusick [9] who interpreted it as a visibility problem called *view obstruction*. Let $C = [-\frac{1}{2}, \frac{1}{2}]^n$. For some factor $\alpha > 0$, consider the infinite set of cubes

$$\left\{ \left(\gamma_1 + \frac{1}{2}, \ldots, \gamma_n + \frac{1}{2} \right)^T + \alpha C \; : \; \gamma_1, \ldots, \gamma_n \in \mathbb{N}_0 \right\} . \tag{3}$$

Now the problem is to determine the supremum of $\alpha > 0$ such that there exists a visibility ray in the strictly positive orthant (see Figure 4). This supremum, called $\lambda(n)$, can be written as follows

$$\lambda(n) = 2 \sup_{\omega_1, \ldots, \omega_n \in (0,\infty)} \inf_{\xi \in (0,\infty)} \max_{1 \le i \le n} ||\omega_i \xi - \frac{1}{2}||_I .$$

The connection between Wills' problem and the view obstruction problem is established by the statement that for $n \geq 2$ we have $\lambda(n) = 1 - 2\kappa(n)$ (see [33, 9]).

Yet another approach to the same core problem called *lonely runner* has been given in [4]. In spite of many research efforts during the last 30 years, the exact value of $\kappa(n)$ is known only for values up to 5 ([5]). For $n \geq 5$, only upper and lower bounds have been determined. If one considers balls instead of cubes [10], then also the exact values for the view obstruction problem are known only up to dimension 5 [12].

Although, of course, the view obstruction problem involves an infinite number of bodies, our complexity results for finite instances can be seen as a certain complexity-theoretical indication for the hardness of the computation of $\lambda(n)$ for larger n. Namely, by Theorem 3, for fixed dimension Anchored Visibility or Anchored Quadrant Visibility can be solved in polynomial time. However, if the dimension is part of the input, then the problem becomes $\mathbb{NP}$-hard by Theorem 2. In a non-rigorous sense, this can be seen as a quantification of the strong influence of the dimension compared to the other input parameters.

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