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LINE PROBLEMS IN NONLINEAR COMPUTATIONAL GEOMETRY

FRANK SOTTILE AND THORSTEN THEOBALD

Abstract. We first review some topics in the classical computational geometry of lines, in particular the $O(n^{3+\epsilon})$ bounds for the combinatorial complexity of the set of lines in $\mathbb{R}^3$ interacting with n objects of fixed description complexity. The main part of this survey is recent work on a core algebraic problem—studying the lines tangent to k spheres that also meet $4-k$ fixed lines.

1. Introduction

While classical computational geometry is often concerned with (piecewise) linear objects such as polyhedra, current applications and interests include the study of algorithms and computations with curved (non-linear) objects [4, 19, 31]. In contrast to classical computational geometry, whose algorithms often have linear or sub-linear complexity, this *nonlinear computational geometry* is often intractable as many natural problems have intrinsic single- or double-exponential complexity [3].

Algorithmic questions involving lines in $\mathbb{R}^3$ and $\mathbb{R}^d$ (which are fundamental problems in computational geometry [7, 20, 28]) are at the boundary between classical and nonlinear computational geometry. Although a line in $\mathbb{R}^3$ is a *polyhedral* set in $\mathbb{R}^3$ and collections of lines have a rich combinatorial structure, the interaction of lines with other objects is inherently nonlinear, since the space of lines constitutes a curved submanifold (Grassmannian) in natural global Plücker coordinates. However, many line problems have low algebraic degree and dimension and are therefore computationally tractable, with solutions involving both combinatorial and nonlinear techniques.

The investigation of line problems in computational geometry arose from applications such as hidden surface removal, motion planning, and configurations of mechanisms. Early work typically studied interactions of lines with lines and other polyhedral objects (see [7, 21]). In the last decade, several series of papers have studied problems of line transversals to more general objects (such as spheres) from various viewpoints. From the combinatorial point of view, tight bounds on the combinatorial complexity were obtained, while nearly complete solutions to the underlying (real) algebraic geometric problems were given.

We survey these developments in the computational geometry of lines. Our particular goal is to relate the different viewpoints and to explain some techniques (e.g., from computational algebraic geometry) which are not common in discrete and computational geometry, but which have been fruitful in the study of line problems. These techniques

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provide useful tools for other problems in nonlinear computational geometry. We also exhibit geometric configurations which prove that certain bounds on the number of real solutions are tight. Some of these constructions are new, in particular a configuration with four disjoint spheres in $\mathbb{R}^3$ having 12 distinct common tangent lines and 6 geometric permutations (see Figure 8).

The paper is structured as follows. Section 2 presents the classical computational geometry of lines in $\mathbb{R}^3$, including the fundamental bound of $O(n^{3+\epsilon})$ for the combinatorial complexity of the set of lines interacting with n objects in $\mathbb{R}^3$. The point of such bounds is that naive arguments give a bound of $O(n^4)$. Section 3 contains the heart of this survey, where we study the algebraic core problem of lines tangent to k spheres and transversal to $4-k$ lines in $\mathbb{R}^3$. In Section 4 we discuss some open problems.

2. Classical computational geometry of lines

2.1. Plücker coordinates for lines. A source of non-linearity in the computational geometry of lines is that while lines are objects of linear algebra, the set of all lines is naturally a curved submanifold of projective space. If we represent a line in (projective) d-space $\mathbb{P}^d$ as the affine span of two points $x^T = (x_0, x_1, \ldots, x_d)$ and $y^T = (y_0, y_1, \ldots, y_d)$, then its *Plücker coordinates* [22] are

$$p_{ij} \;:=\; x_i y_j - x_j y_i \;=\; \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \qquad \text{for} \quad 0 \le i < j \le d\,,$$

which define a point in D-dimensional projective space $\mathbb{P}^D$, where $D := \binom{d+1}{2} - 1$.

The set $\mathbb{G}_{1,d}$ of all lines in $\mathbb{P}^d$ is called the *Grassmannian of lines* in $\mathbb{P}^d$. The necessary and sufficient conditions for a point $(\,p_{ij} \mid 0 \le i < j \le d\,) \in \mathbb{P}^D$ to represent a line and hence lie in $\mathbb{G}_{1,d}$ are furnished by the quadratic *Plücker equations*,

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \;=\; 0 \qquad \text{for} \quad 0 \le i < j < k < l \le d\,.$$

When $d = 3$, the Grassmannian $\mathbb{G}_{1,3}$ is the hypersurface in $\mathbb{P}^5$ cut out by the single equation

$$p_{01}p_{23} \;-\; p_{02}p_{13} \;+\; p_{03}p_{12} \;=\; 0\,. \tag{1}$$

Geometric conditions such as incidence or tangency are naturally expressed in terms of Plücker coordinates. For example, two lines in $\mathbb{P}^3$ meet if and only if their Plücker coordinates (p_{ij}) and (p'_{ij}) satisfy the linear equation

$$p_{01}p'_{23} \;-\; p_{02}p'_{13} \;+\; p_{03}p'_{12} \;+\; p_{12}p'_{03} \;-\; p_{13}p'_{02} \;+\; p_{23}p'_{01} \;=\; 0\,. \tag{2}$$

Indeed, two lines spanned by points x, y and x', y' in $\mathbb{P}^3$ meet if and only if the four points are affinely dependent, which is expressed by $\det(x, y, x', y') = 0$. Laplace expansion of this determinant along the first two columns gives the linear equation (2).

A sphere in $\mathbb{R}^3$ with radius r and center c has equation $x^T Q x = 0$, where

$$x^T \;=\; (1, x_1, x_2, x_3) \qquad \text{and} \qquad Q \;=\; \begin{pmatrix} c_1^2 + c_2^2 + c_3^2 - r^2 & -c_1 & -c_2 & -c_3 \\ -c_1 & 1 & 0 & 0 \\ -c_2 & 0 & 1 & 0 \\ -c_3 & 0 & 0 & 1 \end{pmatrix}.$$

A line ℓ spanned by two points x and y in $\mathbb{P}^3$ corresponds to a 2-dimensional linear subspace H in $\mathbb{R}^4$ spanned by the vectors x and y. We may also consider H to be a matrix with columns x and y.

The line ℓ is tangent to the sphere if and only if the restriction of the quadratic form Q to H is singular. This restriction is represented by the 2×2 symmetric matrix H^TQH. Taking its determinant, we obtain

$$\left(\wedge^2 H\right)^T (\wedge^2 Q)\ \wedge^2 H \;=\; 0\,,$$

where $\wedge^2 H$ is the vector of Plücker coordinates for the line ℓ and $\wedge^2 Q$ is the 6×6 matrix whose entries are the 2×2 minors of Q. This is a quadratic equation in the Plücker coordinates of ℓ which cuts out the lines tangent to the sphere defined by Q.

A key tool for us is the following estimate.

Bézout's Theorem. *The number of isolated solutions to n polynomial equations in n-space is bounded from above by the product of the degrees of the polynomials.*

2.2. **Combinatorial complexity.** We consider the combinatorial complexity of the set $\mathcal{T}(\mathcal{S})$ of lines interacting (in a specified way) with a given set $\mathcal{S}$ of lines or other objects in $\mathbb{R}^3$. If the sets in $\mathcal{S}$ are *semi-algebraic* in that they are defined by equations and inequalities, then $\mathcal{T}(\mathcal{S})$ will be a semi-algebraic set in $\mathbb{G}_{1,3}$. Its boundary $\partial\mathcal{T}(\mathcal{S})$ is also semi-algebraic and consists of objects which are tangent to at least one set in $\mathcal{S}$.

For example, fix a line $\ell \subset \mathbb{R}^3$ and let $\mathcal{T}(\ell)$ be the set of lines which pass above ℓ. The boundary $\partial\mathcal{T}(\ell)$ consists of lines that meet ℓ, and we have already seen that this boundary is defined by a linear equation in the Plücker coordinates (2).

For another example, let $\mathcal{T}(C)$ be the set of lines which intersect a fixed convex body C. Its boundary $\partial\mathcal{T}(C)$ consists of lines which are tangent to C. When C is a ball in $\mathbb{R}^3$ given by an equation of the form $x^TQx = 0$, then $\mathcal{T}(C)$ and $\partial\mathcal{T}(C)$ are defined by the conditions

$$p^T\left(\wedge^2 Q\right)p \;\geq\; 0 \qquad \text{and} \qquad p^T\left(\wedge^2 Q\right)p \;=\; 0\,, \tag{3}$$

respectively.

A *face* of $\mathcal{T}(\mathcal{S})$ is a connected component of the set of lines in $\partial\mathcal{T}(\mathcal{S})$ which are tangent to a fixed subset of $\mathcal{S}$. The *combinatorial complexity* of the set $\mathcal{T}(\mathcal{S})$ is the total number of its faces. In this combinatorial analysis, we assume that the objects are in general position (which is discussed in Section 3).

Suppose that we have n lines $\mathcal{L} = \{\ell_1, \ldots, \ell_n\} \subset \mathbb{R}^3$. The *upper envelope of $\mathcal{L}$* is the set of lines which pass above all elements of $\mathcal{L}$. The following result was proved in [7].

Theorem 2.1. *The maximum combinatorial complexity of the entire upper envelope of n lines in space is $\Theta(n^3)$.*

The significance of this cubic upper bound is that every four lines could have two common transversals. This gives possibly $O(n^4)$ zero-dimensional faces of the upper envelope. Only those transversals which pass above the other lines in $\mathcal{L}$ are zero-dimensional faces, and Theorem 2.1 says that there are relatively few of these.

Here, Θ expresses that there is also a cubic lower bound, that is, there exists a construction whose upper envelope has exactly cubic combinatorial complexity. Consider three families $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{L}$ of lines, each of cardinality $N := \lfloor n/3 \rfloor$. The lines of the two families $\mathcal{G}$ and $\mathcal{H}$ are

$$\begin{aligned} g_i &= (0, i, 0)^T + \mathbb{R}(1, 0, i)^T , \quad 1 \le i \le N , \\ h_j &= (j, 0, 0)^T + \mathbb{R}(0, 1, j)^T , \quad 1 \le j \le N , \end{aligned}$$

which form a grid of lines on the hyperbolic paraboloid $z = xy$. Each line of $\mathcal{G}$ is parallel to the xz-plane and each line of $\mathcal{H}$ is parallel to the yz-plane (see Figure 1). The third

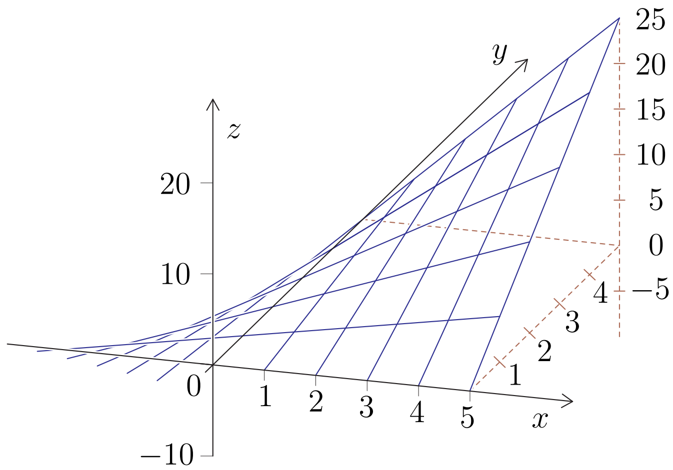


FIGURE 1. Families $\mathcal{G}$ and $\mathcal{H}$ of lines on the hyperbolic paraboloid $z = xy$.

family of lines $\mathcal{L}$ is given by

$$\ell_k = (0, 0, -n^5)^T + \mathbb{R}\Big(1, \frac{k}{n^2} - 1, n\Big)^T , \quad 1 \le k \le N .$$

All lines in $\mathcal{L}$ pass through the point $(0, 0, -n^5)^T$ below the hyperboloic paraboloid and have a steep z slope. For each triple of lines $(g_i, h_j, \ell_k) \in \mathcal{G} \times \mathcal{H} \times \mathcal{L}$, there exists a line connecting some point of ℓ_k to the intersection point $g_i \cap h_j$ which lies above all the other lines in $\mathcal{G}, \mathcal{H}, \mathcal{L}$ (see [7] for details). This implies the lower bound $\Omega(n^3)$, and a perturbation of this construction brings it into general position.

2.3. **Line transversals to balls and semialgebraic sets.** Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a set of balls and $\mathcal{T}(\mathcal{B})$ be the set of lines which meet every ball in $\mathcal{B}$. Suppose that the balls are in general position (which is discussed in Section 3). Then, for each $j = 0, 1, 2, 3$, a j-dimensional face of $\mathcal{T}(\mathcal{B})$ is a connected component of the set of lines in $\mathcal{T}(\mathcal{B})$ which are tangent to a fixed set of $4-j$ balls in $\mathcal{B}$. Agarwal, Aronov, and Sharir [2] have shown a near-cubic bound on the complexity of $\mathcal{T}(\mathcal{B})$. This result is striking—again there is a natural quartic upper bound.

Theorem 2.2. *Let $\mathcal{B}$ be a set of n balls in $\mathbb{R}^3$. Then the complexity of $\mathcal{T}(\mathcal{B})$ is $O(n^{3+\varepsilon})$ for any $\varepsilon > 0$.*

Note that the constant in the O-notation depends on ε. For balls of general radius, the upper bound is essentially tight, as there is a construction with complexity $\Omega(n^3)$. For unit balls the tightness of the lower bound is open. The best known construction has only quadratic complexity [2].

Theorem 2.2 is now understood as a special case of a more general result. The *lower envelope* of a collection of functions is their pointwise minimum, and their *upper envelope* is their pointwise maximum. A *sandwich region* is a region which lies above an upper envelope for one collection of functions and below a lower envelope for a different collection of functions.

The set of line transversals to $\mathcal{B}$ is a sandwich region given by two sets of trivariate functions (see [2, 7]). If we exclude lines parallel to the yz-plane (which is no loss of generality as the balls are in general position), a line ℓ in $\mathbb{R}^3$ can be uniquely represented by the quadruple $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \mathbb{R}^4$ parametrizing its projections to the xy- and xz-planes: $y = \sigma_1 x + \sigma_2$ and $z = \sigma_3 x + \sigma_4$.

Let B be a ball in $\mathbb{R}^3$. For fixed $\sigma_1, \sigma_2, \sigma_3$, the set of lines $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ that intersect B is obtained by translating a line in the z-direction between two extreme values $(\sigma_1, \sigma_2, \sigma_3, \phi_B^-(\sigma_1, \sigma_2, \sigma_3))$ and $(\sigma_1, \sigma_2, \sigma_3, \phi_B^+(\sigma_1, \sigma_2, \sigma_3))$, which represent lines tangent to B from below and from above, respectively. Hence, the set of line transversals to $\mathcal{B}$ is

$$\left\{ (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \ : \ \max_{B \in \mathcal{B}} \phi_B^-(\sigma_1, \sigma_2, \sigma_3) \le \sigma_4 \le \min_{B \in \mathcal{B}} \phi_B^+(\sigma_1, \sigma_2, \sigma_3) \right\},$$

which is a sandwich region of two sets of trivariate functions. As the balls in $\mathcal{B}$ are in general position, the vertices of the boundary of this region are lines which are tangent to four of the balls in $\mathcal{B}$.

Recently, Koltun and Sharir have shown the following result for trivariate functions [14]. Together with the preceeding discussion, this implies Theorem 2.2.

Theorem 2.3. *For any $\varepsilon > 0$, the combinatorial complexity of the sandwich region between two families of n trivariate functions of constant description complexity is $O(n^{3+\varepsilon})$.*

This is a central achievement in a sequence of papers (including [12, 14, 23]). The basic idea is to reduce the geometric problem to a combinatorial problem involving Davenport-Schinzel sequences.

Given a word $u = u_1 u_2 \dots u_m$ over an alphabet with n symbols, a subsequence of u is a word $u_{i_1} \dots u_{i_k}$ for some indices $1 \le i_1 < \cdots < i_k \le m$. An *$(n, s)$-Davenport-Schinzel sequence* is a word u over an alphabet with n symbols such that

(1) two consecutive symbols of u are distinct, and
(2) for any two symbols a, b in the alphabet, the alternating sequence $abab\dots$ of length $s + 2$ is not a subsequence of u.

The lower envelope of n continuous univariate functions $f_1, \dots, f_n$ is formed by a sequence of curved edges, where each edge is a maximal connected subset of the envelope that belongs to the graph of a single function f_i. This is illustrated in Figure 2.

We label each edge by the index of the corresponding function. Reading these labels left-to-right gives a sequence of labels. If the graphs of the functions have pairwise at most s intersection points, then this sequence is an (n, s)-Davenport-Schinzel sequence.

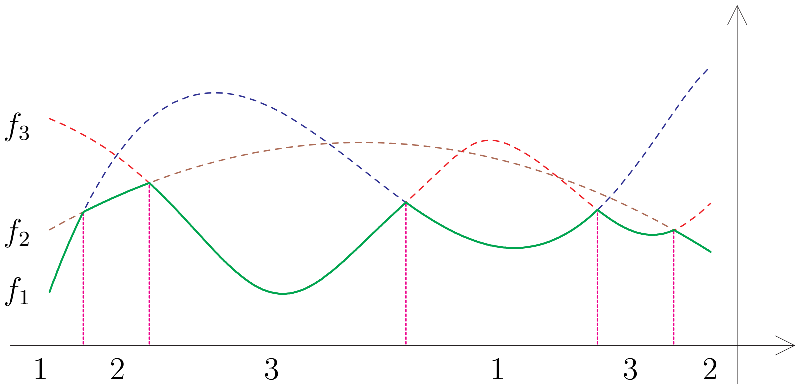


FIGURE 2. The lower envelope of three functions f_1, f_2, f_3 with label sequence $(1, 2, 3, 1, 3, 2)$.

Indeed, suppose that we have intervals I and J with distinct labels i and j so that f_i is the minimum of all functions over I and f_j is the minimum of all functions over J. Then f_i and f_j must meet at some point that lies weakly between I and J. Thus an alternating subsequence of length $s+2$ for two symbols i and j implies the existence of $s+1$ intersection points between the graphs of f_i and f_j, which is a contradiction. A (sharp) upper bound for the number of (n, s)-Davenport-Schinzel sequences is only slightly super-linear in n (rather than n^2). See, e.g., [24].

Recently, Agarwal, Aronov, Koltun, and Sharir [1] proved an upper bound on the combinatorial complexity of a set of lines defined with respect to a collection of balls which does not fall in the framework of Theorem 2.3, since the corresponding set cannot be described as a sandwich region. A line in $\mathbb{R}^3$ is *free* (with respect to a set of balls $\mathcal{B}$) if it does not intersect the interior of any ball in $\mathcal{B}$.

Theorem 2.4. *For any $\varepsilon > 0$ the combinatorial complexity of the space of lines free with respect to a set of n unit balls in $\mathbb{R}^3$ is $O(n^{3+\varepsilon})$.*

For polyhedra, the following result is known [21]. Here, the *total complexity* of a set of polyhedra is the total number of their faces.

Theorem 2.5. *Given a set $\mathcal{P}$ of polyhedra with total complexity n, the combinatorial complexity of $\mathcal{T}(\mathcal{P})$ is bounded by $O(n^3 2^{c\sqrt{\log n}})$, where c is a constant. Moreover, the set of extremal stabbing lines (i.e., those representing the vertices of $\mathcal{T}(\mathcal{P})$) can be found in time $O(n^3 2^{c\sqrt{\log n}})$.*

2.4. **Transversals, convexity, and geometric permutations.** A line transversal to a set $\mathcal{S}$ of pairwise disjoint convex bodies in $\mathbb{R}^d$ induces two linear orders on $\mathcal{S}$. These two orders are reverse to each other, and we consider them as a single *geometric permutation*. Studying the geometry of line transversals and geometric permutations has a long history in discrete and computational geometry; see the surveys [10, 32]. An obstacle is the lack of a suitable convexity structure on the space of lines. We review a recent result which connects that geometry to certain algebraic problems discussed in Section 3.

Let $\mathcal{S}$ be an (ordered) sequence of convex bodies. A *directed line transversal* to $\mathcal{S}$ is an oriented line intersecting all bodies in the order on $\mathcal{S}$. Let $\mathcal{K}(\mathcal{S})$ denote the set of directions of the directed line transversals to $\mathcal{S}$. A set $\mathcal{B}$ of balls with centers c_i and radii r_i is called *pairwise inflatable* if for every two balls $B_i, B_j \in \mathcal{B}$ we have $||c_i - c_j||^2 > 2(r_i^2 + r_j^2)$, where $||\cdot||$ denotes the Euclidean norm. In particular, any set of disjoint unit balls is pairwise inflatable. Cheong, Goaoc, Holmsen, and Petitjean [8, 9] have shown the following convexity result.

Theorem 2.6. *Let $\mathcal{B}$ be a sequence of unit balls in $\mathbb{R}^d$. If $\mathcal{K}(\mathcal{B})$ is nonempty, then it is convex. This result persists if $\mathcal{B}$ is a set of pairwise inflatable balls in $\mathbb{R}^d$.*

In dimension 3, the case $|\mathcal{B}| \leq 2$ is trivial and the case $|\mathcal{B}| = 3$ follows from an explicit case analysis (see [13]). To show convexity when $|\mathcal{B}| > 3$, it suffices to show $\mathcal{K}(B) = \bigcap_{\mathcal{B}' \subset \mathcal{B}, |\mathcal{B}'|=3} \mathcal{K}(\mathcal{B}')$. The inclusion $\subset$ is obvious; for the converse direction let $v \in \bigcap_{\mathcal{B}' \subset \mathcal{B}, |\mathcal{B}'|=3} \mathcal{K}(\mathcal{B}')$. By choice of v, the orthogonal projections of any three balls along v intersect. Hence, the classical Helly theorem implies that the projections of all balls have a common point of intersection, and so the balls of $\mathcal{B}$ have a common line transversal with direction v. Since this transversal is consistent with the ordering induced by $\mathcal{B}$ on every triple of balls, it is a directed line transversal to $\mathcal{B}$.

Using algebraic characterizations from [15, 17] discussed in the next section, Theorem 2.6 can be strengthened to strict convexity (see [8, 9]).

3. The Algebraic Core

Many problems from Section 2 share the following algebraic core.

(4) Given k spheres and $4-k$ lines in $\mathbb{R}^3$, determine which common tangents to the spheres also meet each of $4-k$ given lines.

This basic question about the geometry of lines is also motivated by problems of visibility in $\mathbb{R}^3$. When viewing a scene in a particular direction along a moving viewpoint, the objects that can be viewed may change when the line of sight becomes tangent to one or more objects. If the objects are in general position, then the most degenerate such lines of sight are the lines tangent to four objects. When the objects are polyhedra and spheres, these most degenerate lines of sight are the tangents to k spheres which also meet $4-k$ edges. Our algebraic core (4) is the relaxation where we replace the edges by their supporting lines.

We will discuss the number of lines (both real and complex) that can be solutions to (4) when the spheres and lines are in general position. We will also classify the degenerate positions of k spheres and $4-k$ lines. Identifying these degeneracies is important when devising robust algorithms for problems which involve this algebraic core. Both of these investigations are also interesting from the point of view of enumerative real algebraic geometry [25] and of computational algebraic geometry.

3.1. **The problem of four lines.** Let us begin with the simplest case of our algebraic core problem, the classical and surprisingly non-linear problem of common transversals to four lines in space.

Let ℓ_1, ℓ_2, ℓ_3, and ℓ_4 be lines in general position in $\mathbb{R}^3$. The three mutually skew lines ℓ_1, ℓ_2, and ℓ_3 lie in one ruling of a doubly-ruled hyperboloid as shown in Figure 3. This is

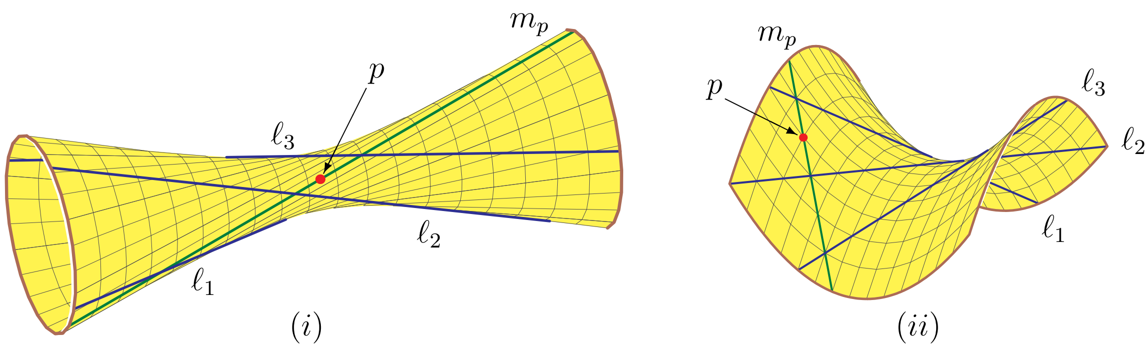
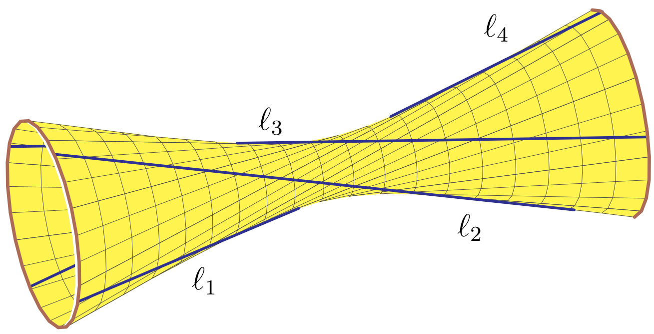


Figure 3. Hyperboloids through 3 lines.

either (i) a hyperboloid of one sheet, or (ii) a hyperbolic paraboloid. The line transversals to ℓ_1, ℓ_2, and ℓ_3 constitute the second ruling. Through every point p of the hyperboloid there is a unique line m_p in the second ruling which meets the lines ℓ_1, ℓ_2, and ℓ_3. To simplify this and other discussions, we will at times work in projective space where parallel lines meet. Thus for example, we include lines in the second ruling of (i) which are parallel to one of ℓ_1, ℓ_2, or ℓ_3.

The hyperboloid is defined by a quadratic polynomial and so the fourth line ℓ_4 will either meet the hyperboloid in two points or it will miss the hyperboloid. In the first case, there will be two transversals to ℓ_1, ℓ_2, ℓ_3, and ℓ_4, and in the second case there will be no transversals. But then ℓ_4 meets the hyperboloid in two conjugate complex points, giving two complex transversals.

If ℓ_4 is not in general position, it could be tangent to the hyperboloid, in which case there is a single common transversal occurring with a multiplicity of 2. More interestingly, it could lie in the ruling of the first three lines, and then there will be infinitely many common transversals.



Another possibility is for two of the first three lines to meet. If ℓ_1 and ℓ_2 meet in a point p, then they span a plane H, and the common transversals to ℓ_1 and ℓ_2 either lie in the plane H or meet the point p. In either case, further questions of transversals and tangents essentially become problems in a plane. Because of this reduction, we shall (almost always) assume that our lines are pairwise skew.

A different perspective on this problem of four lines is furnished by Plücker coordinates for lines. Recall from Section 2.1 that the set $\mathbb{G}_{1,3}$ of lines in $\mathbb{P}^3$ is a quadratic hypersurface in $\mathbb{P}^5$ defined by (1), and the condition for a line ℓ to meet a fixed line is a linear equation (2) in the Plücker coordinates of ℓ. Thus the Plücker coordinates of line transversals to ℓ_1, ℓ_2, ℓ_3, and ℓ_4 satisfy one quadratic and four linear equations. By Bézout's Theorem, we see again that there will be two line transversals to four given lines. Observe that this purely algebraic derivation of the number of solutions cannot distinguish between real and complex solutions. This is both a strength (finding complex solutions is a relaxation of the problem of finding real solutions) and a weakness of using algebra to study configurations of objects in $\mathbb{R}^3$.

3.2. **Line tangent to four spheres.** As in Section 2.1, a line is tangent to a sphere with equation $x^TQx = 0$ when its Plücker coordinates p satisfy the quadratic equation

$$p^T \wedge^2 Q\, p \;=\; 0\,. \tag{5}$$

It follows that the Plücker coordinates of common tangents to 4 spheres in $\mathbb{R}^3$ are defined by 5 quadratic equations. These are the Plücker equation (1) and 4 quadratic equations (5) expressing tangency to each sphere. Bézout's Theorem gives a bound of $2^5 = 32$ for the number of tangent lines. Unlike the problem of lines in Section 3.1, this is not a sharp upper bound, as not all solutions are isolated and geometrically meaningful.

Indeed, a sphere with center $(c_1, c_2, c_3)^T \in \mathbb{R}^3$ and radius r is defined by the equation

$$(x_1 - c_1x_0)^2 + (x_2 - c_2x_0)^2 + (x_3 - c_3x_0)^2 \;=\; r^2x_0^2\,.$$

Every sphere in $\mathbb{R}^3$ contains the imaginary *spherical conic at infinity*,

$$C\ :\ x_0 = 0 \quad \text{and} \quad x_1^2 + x_2^2 + x_3^2 = 0\,. \tag{6}$$

In fact, spheres are exactly the quadrics that contain the spherical conic.

Each (necessarily imaginary) line at infinity that is tangent to C is tangent to every sphere, and there is a one-dimensional family of such lines. Thus the equations for lines tangent to four spheres not only define the tangents in $\mathbb{R}^3$, but also this one-dimensional excess component consisting of geometrically meaningless lines tangent to C.

We treat this excess component by defining it away. Represent a line ℓ in $\mathbb{R}^3$ by a point $p \in \ell$ and a direction vector $v \in \mathbb{RP}^2$. No such line can lie at infinity, so we are avoiding the excess component of lines at infinity tangent to C.

Lemma 3.1. *The set of direction vectors $v \in \mathbb{RP}^2$ of lines tangent to four spheres with affinely independent centers consists of the common solutions to a cubic and a quartic equation on $\mathbb{RP}^2$. Each direction vector gives one common tangent, and there are at most* 12 *common tangents.*

Proof. We follow the derivation of [30, Lemma 13], which generalizes the argument given in [15] to spheres of arbitrary radii. For vectors $x, y \in \mathbb{R}^3$, let $x \cdot y$ be their ordinary Euclidean dot product and write x^2 for $x \cdot x$, which is $\|x\|^2$.

Fix p to be the point of ℓ closest to the origin, so that

$$p \cdot v \;=\; 0\,. \tag{7}$$

The line ℓ is tangent to the sphere with radius r centered at $c \in \mathbb{R}^3$ if its distance to c is r,

$$\left\| (c-p) \times \frac{v}{\|v\|} \right\| \;=\; r\,.$$

Squaring and clearing the denominator v^2 gives

$$[(c-p)\times v]^2 \;=\; r^2 v^2\,. \tag{8}$$

This formulation requires that $v^2 \neq 0$. A line with $v^2 = 0$ has a complex direction vector and it meets the spherical conic at infinity.

We assume that one sphere is centered at the origin with radius r, while the other three have centers and radii (c_i, r_i) for $i = 1, 2, 3$. The condition for the line to be tangent to the sphere centered at the origin is

$$p^2 \;=\; r^2\,. \tag{9}$$

For the other spheres, we expand (8), use vector product identities, and the equations (7) and (9) to obtain the vector equation

$$2v^2 \begin{pmatrix} c_1^T \\ c_2^T \\ c_3^T \end{pmatrix} \cdot p \;=\; -\begin{pmatrix} (c_1\cdot v)^2 \\ (c_2\cdot v)^2 \\ (c_3\cdot v)^2 \end{pmatrix} \;+\; v^2 \begin{pmatrix} c_1^2 + r^2 - r_1^2 \\ c_2^2 + r^2 - r_2^2 \\ c_3^2 + r^2 - r_3^2 \end{pmatrix}\,. \tag{10}$$

Now suppose that the spheres have affinely independent centers. Then the matrix $(c_1, c_2, c_3)^T$ appearing in (10) is invertible. Assuming $v^2 \neq 0$, we may use (10) to write p as a quadratic function of v. Substituting this expression into equations (7) and (9), we obtain a cubic and a quartic equation for $v \in \mathbb{RP}^2$. 

Bézout's Theorem implies that there are at most $3 \cdot 4 = 12$ isolated solutions to these equations, and over $\mathbb{C}$ exactly 12 if they are generic. The equations are however far from generic as they involve only 13 parameters while the space of quartics has 14 parameters and the space of cubics has 9.

Example 3.2. Suppose that the spheres have equal radii, r, and have centers at the vertices of a regular tetrahedron with side length $2\sqrt{2}$,

$$(2,2,0)^T\,, \qquad (2,0,2)^T\,, \qquad (0,2,2)^T\,, \quad \text{and} \quad (0,0,0)^T\,.$$

In this symmetric case, the cubic factors into three linear factors. There are real common tangents only if $\sqrt{2} \leq r \leq 3/2$, and exactly 12 when the inequality is strict. If $r = \sqrt{2}$, then the spheres are pairwise tangent and there are three common tangents, one for each pair of non-intersecting edges of the tetrahedron. Each tangent has algebraic multiplicity 4. If $r = 3/2$, then there are six common tangents, each of multiplicity 2. The spheres meet pairwise in circles of radius $1/2$ lying in the plane equidistant from their centers. This plane also contains the centers of the other two spheres, as well as one common tangent which is parallel to the edge between those centers.

Figure 4 shows the cubic (which consists of three lines supporting the edges of an equilateral triangle) and the quartic, in an affine piece of the set $\mathbb{RP}^2$ of direction vectors. The vertices of the triangle are the direction vectors from the origin to the centers c_1, c_2, and c_3 of the other three spheres. The singular cases, (i) when $r = \sqrt{2}$ and (ii) when $r = 3/2$, are shown first, and then (iii) when $r = 1.425$. The 12 points of intersection in this third case are visible in the expanded view in (iii'). Each point of intersection gives

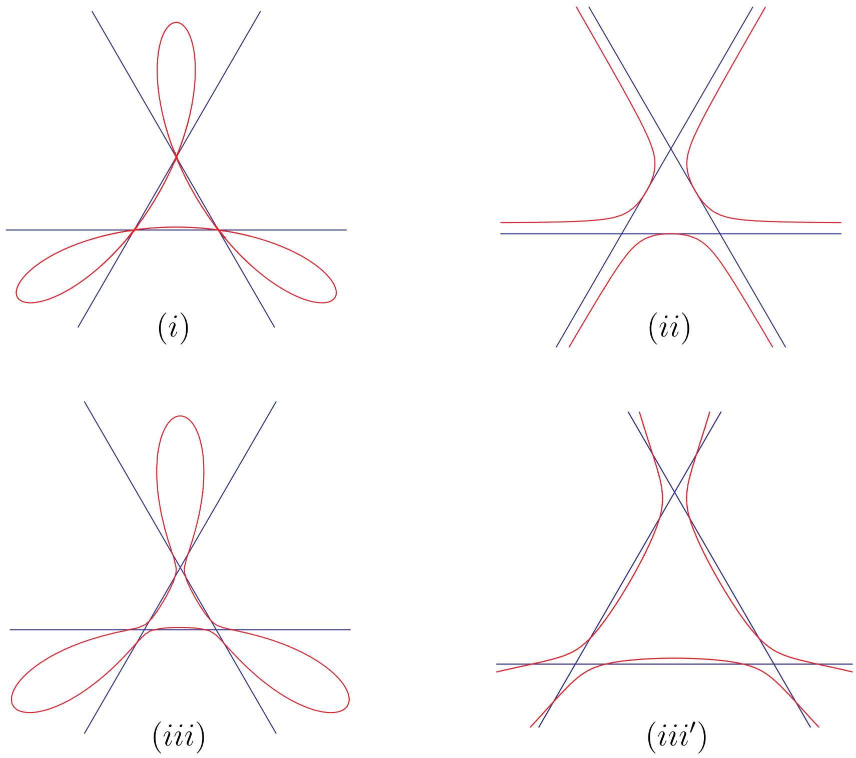


FIGURE 4. The cubic and quartic for symmetric configurations.

a real tangent, so there are 12 tangents to four spheres of equal radii 1.425 with centers at the vertices of the regular tetrahedron with edge length $2\sqrt{2}$.

One may also see this number 12 using group theory. The symmetry group of the tetrahedron, which is the group of permutations of the spheres, acts transitively on their common tangents and the isotropy group of any tangent has order 2. To see this, orient a common tangent and suppose that it meets the spheres a, b, c, d in order. Then the permutation $(a, d)(b, c)$ fixes that tangent but reverses its orientation, and the identity is the only other permutation fixing that tangent.

This example shows that the bound of 12 common tangents from Lemma 3.1 is in fact attained.

Theorem 3.3. *There are at most* 12 *common real tangent lines to four spheres whose centers are not coplanar, and there exist spheres with* 12 *common real tangents.*

Example 3.4. We give an example when the radii are distinct, namely 1.4, 1.42, 1.45, and 1.474. Figure 5 shows the quartic and cubic and the configuration of 4 spheres and their 12 common tangents.

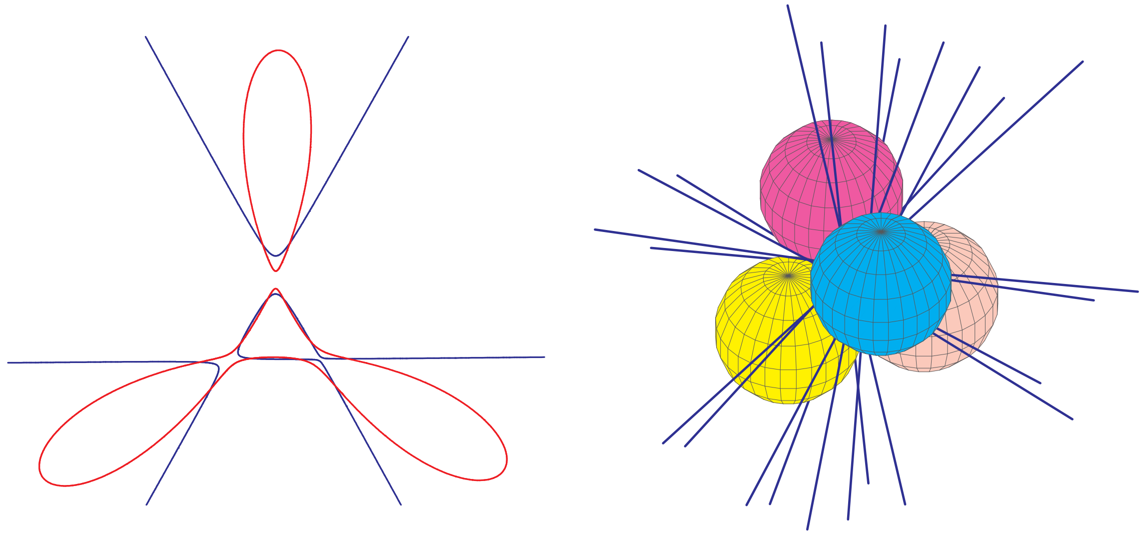


Figure 5. Spheres with 12 common tangents.

Megyesi investigated configurations of spheres with coplanar centers [16]. A continuity argument shows that four general such spheres will have 12 complex common tangents (or infinitely many, but this possibility is precluded by the following example). Three spheres of radius $4/5$ centered at the vertices of an equilateral triangle with side length $\sqrt{3}$ and one of radius $1/3$ at the triangle's center have 12 common real tangents. We display this configuration in Figure 6. This configuration of spheres has symmetry group $\mathbb{Z}_2 \times D_3$,

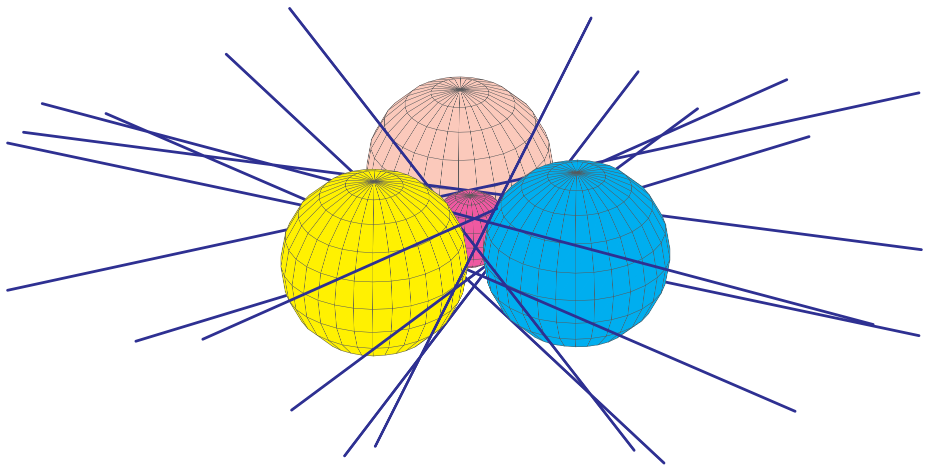


Figure 6. Spheres with coplanar centers and 12 common tangents.

which has order 12 and acts faithfully and transitively on the common tangents.

Note that these coplanar spheres, unlike the spheres of Example 3.2, have unequal radii. Megyesi showed that this is no coincidence.

Theorem 3.5. *If the centers of four unit spheres in $\mathbb{R}^3$ are coplanar but not colinear, then they have at most eight common real tangents, and this bound is sharp.*

The sharpness is shown by the configuration of four spheres of radius 9/10 with centers at the vertices of a rhombus in the plane $z = 0$, $(0,0,0)^T$, $(2,0,0)^T$, $(1\sqrt{3},0)^T$, and $(3,\sqrt{3},0)^T$, which have 8 common tangents. This construction is found in [29, Section 4]. Observe that the spheres here are disjoint.

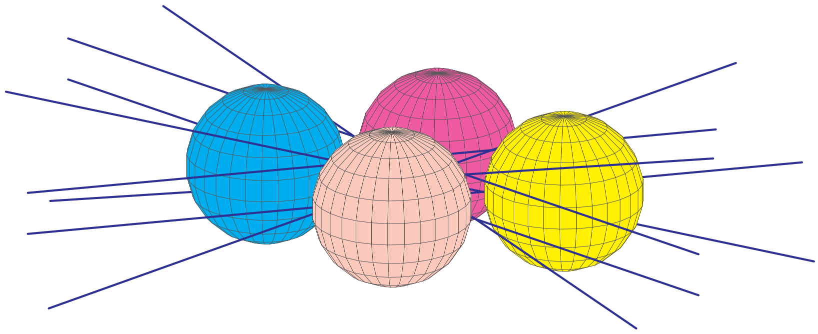


FIGURE 7. Four unit spheres with coplanar centers and 8 common tangents.

Each pair of spheres meet in the symmetric configuration of Example 3.2. However, it is not necessary for the spheres to meet pairwise, even when there are 12 common tangents. In fact, in both Figures 5 and 6 not all pairs of spheres meet. However, the union of the spheres is connected. This is not necessary. In the tetrahedral configuration, if one sphere has radius 1.38 and the other three have equal radii of 1.44, then the first sphere does not meet the others, but there are 12 tangents.

More interestingly, it is possible to have 12 common real tangents to four *disjoint* spheres. Figure 8 displays such a configuration. The three large spheres have radius 4/5

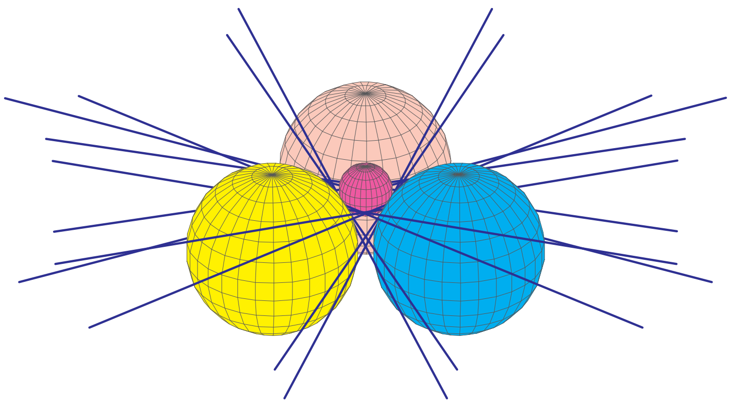


FIGURE 8. Four disjoint spheres with 12 common tangents.

and are centered at the vertices of an equilateral triangle of side length $\sqrt{3}$, while the smaller sphere has radius 1/4 and is centered on the axis of symmetry of the triangle, but at a distance of 35/100 from the plane of symmetry of the triangle. It remains an open question whether it is possible for four disjoint *unit* spheres to have 12 common tangents.

Macdonald, Pach, and Theobald [15] also addressed the question of degenerate configurations of spheres.

Theorem 3.6. *Four degenerate spheres of equal radii have colinear centers.*

The possible degenerate configurations of spheres having unequal radii remained open until recently when it was settled by Borcea, Goaoc, Lazard, and Petitjean [6].

Theorem 3.7. *Four degenerate spheres have colinear centers.*

Contrary to expectations gained in other work (see Section 3.4), their proof was refreshingly elementary. Beginning with the formulation of Lemma 3.1, they use algebraic manipulation (with $v^2 = 0$ playing an important role) to derive a contradiction to the assumption that the four spheres have infinitely many common tangents. Using a different formulation when the centers of the spheres span a plane, they again obtain a contradiction to their being degenerate. This leaves only the possibility that the four spheres have colinear centers, and they then classify degenerate configurations of spheres with colinear centers. These are displayed in Figure 9. The spheres (i) are tangent to each other at a

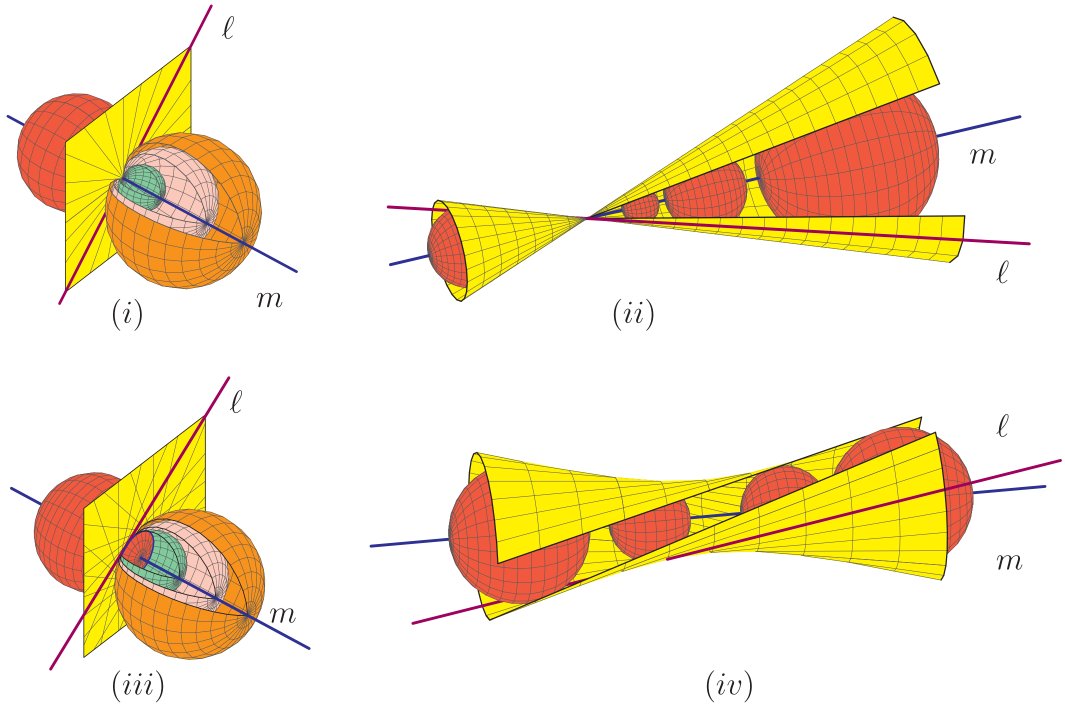


Figure 9. Configurations of four spheres with infinitely many common tangents.

point, or (ii) are inscribed in a circular cone or cylinder, or (iii) meet in a common circle, or (iv) are inscribed in a hyperboloid of revolution. The common tangents are generated from a single common tangent line ℓ by rotation about the line m containing their centers.

3.3. **Common tangents to k spheres which meet $4-k$ lines.** Theobald [30] studied the intermediate cases of our algebraic core problem, those involving *both* lines and spheres.

Theorem 3.8. *Bounds for the number N_k of common tangents to k spheres which also meet $4-k$ fixed lines for $1 \leq k \leq 3$ are as follows*

$$N_1 \ \leq \ 4\,, \qquad N_2 \ \leq \ 8\,, \quad \textit{and} \quad N_3 \ \leq \ 12\,,$$

and these are sharp. There exist unit spheres and lines having exactly these numbers of real common tangents.

Proof of bounds. In Plücker coordinates, the set of transversals and tangents is defined by $4-k$ linear equations and $k+1$ quadratic equations. Indeed, the condition for a line to meet a fixed line (2) is linear in the Plücker coordinates, the condition for a line to be tangent to a sphere (5) is quadratic in the Plücker coordinates, and the set of lines is defined by the quadratic Plücker equation (1). Bézout's Theorem gives the bounds

$$N_1 \ \leq \ 4\,, \qquad N_2 \ \leq \ 8\,, \quad \text{and} \quad N_3 \ \leq \ 16\,.$$

The Bézout bound of 16 for N_3 includes not only the geometrically meaningful tangents to the three spheres which meet a line, ℓ, but also a contribution from the lines tangent to the spherical conic (6) at infinity which also meet ℓ. Since there are two such lines, the contribution is $2m$, where m is the algebraic multiplicity of either line in the intersection. Theobald [30] computes $m = 2$, and so there are at most 12 lines tangent to three spheres which also meet a line.

We now give constructions which realize these bounds.

3.3.1. *One sphere and three lines.* Suppose that the three lines are in the (slightly) degenerate position where ℓ_1 and ℓ_2 are skew, but each meets ℓ_3 in points $p_1 := \ell_2 \cap \ell_3$ and $p_2 := \ell_1 \cap \ell_3$. Then the common transversals to the three lines form two pencils, the lines through p_1 which meet ℓ_1 and the lines through p_2 which meet ℓ_2. Let H_1 be the plane spanned by the first pencil and H_2 the plane spanned by the second pencil.

If S is a sphere which meets both H_1 and H_2 in circles C_1 and C_2 and contains neither p_1 nor p_2, then four of the transversals to the three lines will be tangent to S. Indeed, in each plane H_i there are two tangents t_{i1} and t_{i2} to the circle C_i through p_i, as p_i is exterior to C_i. See Figure 10 for a picture of such a configuration. There, the line ℓ_3 is the x-axis, ℓ_1 contains the point $p_2 := (2, 0, 0)$ and is parallel to the z-axis, ℓ_2 contains the point $p_1 := (-2, 0, 0)$ and is parallel to the y-axis, and the sphere is centered at the origin with radius 1.

3.3.2. *Two spheres and two lines.* Suppose the lines ℓ_1 and ℓ_2 meet in a point p and span a plane H. Then there are two families of transversals to the two lines; lines that pass through p and lines contained in H. If H meets the spheres in disjoint circles, then the four common tangents to the circles in H are common tangents to the spheres which meet both ℓ_1 and ℓ_2. Dually, the point p and spheres can be located so that there are four common tangents to the spheres through p.

Supppose that S_1 and S_2 are spheres having radius $\sqrt{5}$ and centers $(\pm 2, 0, 0)$. If ℓ_1 and ℓ_2 have parametrization $(t, 2 \mp 2t, \pm 3t)$ for $t \in \mathbb{R}$, then they meet in the point $p = (0, 2, 0)$

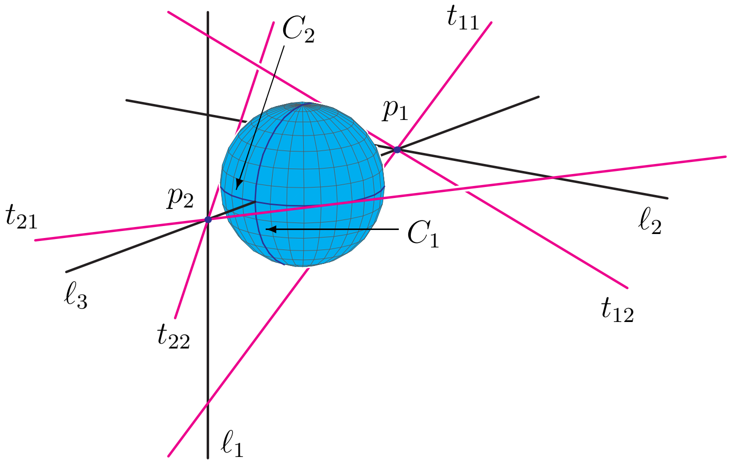


FIGURE 10. One sphere and three lines with four common tangents.

and span the plane H defined by $3y + 2z = 6$. We claim that there are eight common transversals to ℓ_1 and ℓ_2 which are tangent to S_1 and S_2.

Indeed, the plane H meets the spheres in disjoint circles, contributing four common tangents $t_1, \ldots, t_4$. There are four common tangents $t_5, \ldots, t_8$ through the point p; two in the plane $y = 2$ and two lying in the yz-plane. These have parametrizations

$$t_5, t_6 \;=\; (t, 2, \pm t/\sqrt{3}) \qquad \text{and} \qquad t_7, t_8 \;=\; (0, 2+t, \pm t/\sqrt{3})\,.$$

We display this configuration in Figure 11, showing the common tangents in the plane H on the left and those through the point p on the right. In the second picture, we draw the circles where the plane $y = 2$ meets the spheres.

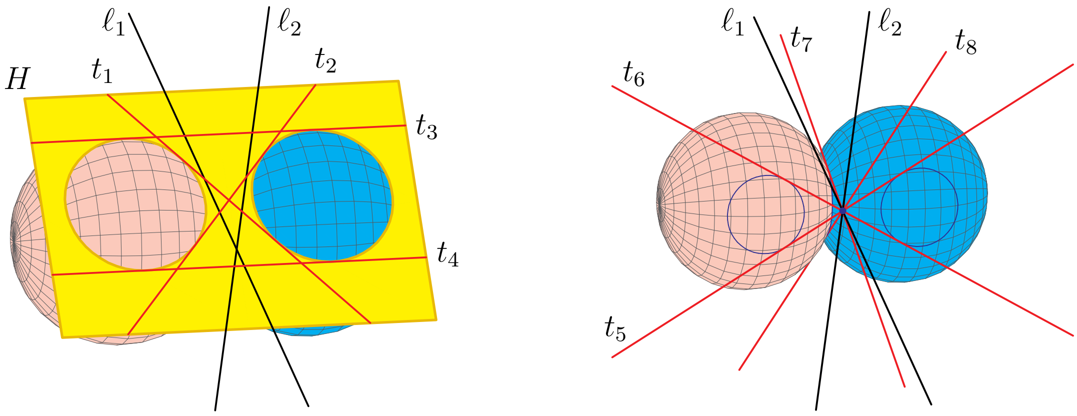


FIGURE 11. Eight tangents to two spheres that meet two lines.

3.3.3. *Three spheres and one line.* Suppose that S_1, S_2, and S_3 are unit spheres centered at the vertices of an equilateral triangle with edge length e, where $\sqrt{3} < e < 2$, and that

ℓ is the line perpendicular to the plane of the triangle passing through its center. Then there are 12 common real tangents to the sphere which meet ℓ.

To see this, note that any common tangent lies in a vertical plane through ℓ which meets all three spheres. Such a plane meets each sphere in a circle, and one of the three circles necessarily contains another, unless the plane contains a vertex of the triangle. A plane through ℓ and through a vertex meets the spheres in two disjoint circles—one is the intersection of two spheres. The four common tangents to these circles in this plane are common tangents to the spheres which meet ℓ. As there are three such planes, we obtain 12 common tangents. Figure 12 shows this configuration when $e = 1.9$.

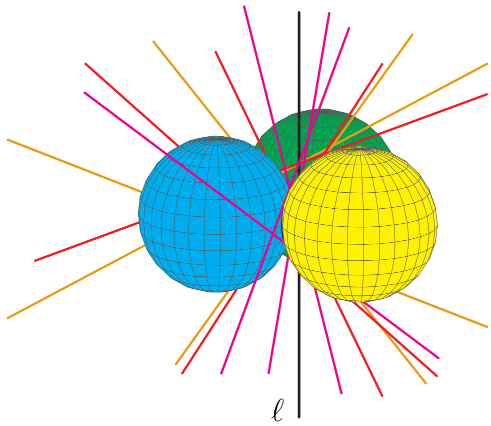


FIGURE 12. Three spheres and one line with 12 common tangents.

3.4. **Degenerate configurations.** We characterize configurations of k spheres and $4-k$ lines which are degenerate in that they have infinitely many common tangents.

3.4.1. *One sphere and three lines.* We leave the case when some of the lines meet as an exercise to the reader, and suppose that the three lines are pairwise skew so that their common transversals are one of the families of lines on a hyperboloid H which contains them as in Figure 3. Each point where the sphere is tangent to H gives a common transversal which is tangent to the sphere. Thus the configuration of lines and the sphere is degenerate if and only if the sphere is tangent to H along a curve, C.

The intersection of the sphere with the hyperboloid H will have multiplicity at least 2 along C. Since C cannot be a line, and Bézout's Theorem implies that the product of this multiplicity and the degree of C is at most 4, we conclude that the multiplicity is 2 and that C has degree 2. Thus C is a plane conic lying on the sphere, and is thus a circle. This will imply that the hyperboloid is a right circular hyperboloid inscribing the sphere. We do not give a picture as it is similar to Figures 9(iv) and 14(iv).

3.4.2. *Two spheres and two lines.* The most interesting geometry occurs when we study two spheres and two lines. Megyesi, Sottile, and Theobald [18] considered the more general situation where we replace the spheres by quadrics. Fix two skew lines ℓ_1 and ℓ_2 and a quadric Q in general position. Then the set τ of tangents to Q which meet both ℓ_1 and ℓ_2

is an irreducible algebraic curve (in fact it has genus 1). We consider the set $\mathcal{Q}$ of quadrics Q which are tangent to every line in τ. This set $\mathcal{Q}$ consists of quadrics Q' such that Q', Q, ℓ_1, and ℓ_2 are degenerate.

Since a quadric is defined as the zero set of a quadric form, quadrics correspond to symmetric 4×4 matrices, modulo scalars. This identifies $\mathbb{P}^9$ as the space of quadrics. The following theorem describes the set $\mathcal{Q}$ as a complex algebraic variety.

Theorem 3.9. *The set $\mathcal{Q}$ of quadrics having degenerate position with respect to skew lines ℓ_1 and ℓ_2 and a general quadric Q in $\mathbb{P}^3$ is an algebraic curve in $\mathbb{P}^9$ of degree* 24 *having* 12 *irreducible components, each a plane conic.*

The proof of this statement uses some geometric reductions, and then a very challenging symbolic computation using the computer algebra system Singular [11], which uses Gröbner bases to manipulate polynomials.

That $\mathcal{Q}$ is composed of algebraic curves is not too hard to see and it involves some beautiful geometry. It turns out that the set of non-zero real numbers $\mathbb{R}^\times$ (or $\mathbb{C}^\times$ if we are working over the complex numbers) acts on $\mathbb{P}^3$ with fixed points ℓ_1 and ℓ_2 so that the orbits are the transversals to ℓ_1 and ℓ_2. One way to see this is to choose coordinates so that ℓ_1 is the x-axis and ℓ_2 is the line at infinity in the yz-plane. Then the common transversals to ℓ_1 and ℓ_2 are the lines perpendicular to the x-axis, and $\mathbb{R}^\times$ acts by simultaneously scaling the y-and z-coordinates. It follows that if Q' is tangent to every line of τ, then so is any translate of Q'.

Not all of the 12 families of quadrics in $\mathcal{Q}$ will contain real smooth quadrics. From the calculations in [18], which are archived on a webpage[1], if Q is a sphere or ellipsoid then four of the families will be real, and there is a choice of Q for which all 12 families are real. Furter analysis reveals that if ℓ_1 and ℓ_2 are in $\mathbb{R}^3$ and if Q is a sphere, then there are no other spheres in $\mathcal{Q}$.

Theorem 3.9 underlies the following classification of degenerate configurations from [18], which is illustrated in Figure 13.

Theorem 3.10. *Let $S_1 \neq S_2$ be spheres in $\mathbb{R}^3$, and let $\ell_1 \neq \ell_2$ be two skew lines in $\mathbb{R}^3$. There are infinitely transversals to ℓ_1 and ℓ_2 which are tangent to S_1 and S_2 in exactly the following cases.*

- (*i*) *S_1 and S_2 are tangent to each other at a point p which lies on one line, and the second line lies in the common tangent plane to the spheres at the point p.*
- (*ii*) *The lines ℓ_1 and ℓ_2 are each tangent to both S_1 and S_2, and they are images of each other under a rotation about the line connecting the centers of S_1 and S_2.*

3.4.3. *Three spheres and one line.* Degenerate positions of three spheres and one line were classified by Megyesi and Sottile [17]. This was more complicated than the classifications with fewer than three spheres and did not contain as much interesting geometry as we just saw. Here is the classification, which is illustrated in Figure 14.

[1] `http://www.math.tamu.edu/~sottile/pages/2l2s/index.html`

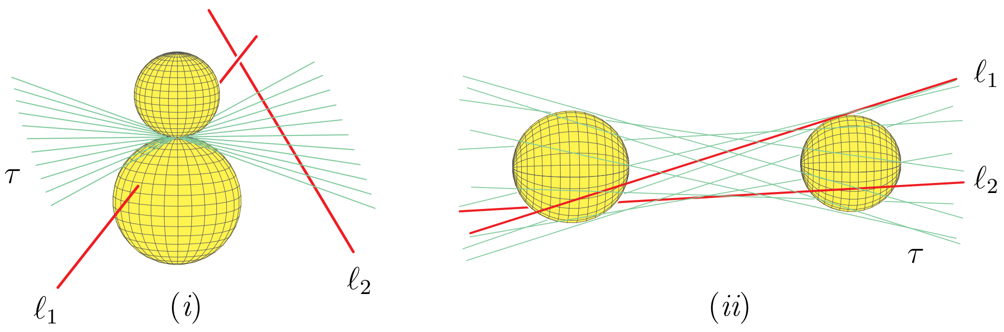


FIGURE 13. Examples from Theorem 3.10.

Theorem 3.11. *Let ℓ be a line and S_1, S_2, and S_3 be spheres in $\mathbb{R}^3$. Then there are infinitely many lines that meet ℓ and are tangent to each sphere in precisely the following cases.*

- (*i*) *The spheres are tangent to each other at the same point and either* (*a*) *ℓ meets that point, or* (*b*) *it lies in the common tangent plane, or both. The common tangent lines are the lines in the tangent plane meeting the point of tangency.*
- (*ii*) *The spheres are tangent to a cone whose apex lies on ℓ. The common tangent lines are the ruling of the cone.*
- (*iii*) *The spheres meet in a common circle and the line ℓ lies in the plane of that circle. The common tangents are the lines in that plane tangent to the circle.*
- (*iv*) *The centers of the spheres lie on a line m and ℓ is tangent to all three spheres. The common tangent lines are one ruling on the hyperboloid of revolution obtained by rotating ℓ about m.*

In cases (*ib*), (*iii*) *and* (*iv*) *lines parallel to ℓ are excluded.*

For this, they begin with two spheres and one line, and consider the common tangents to the spheres which also meet the line. These tangents form a curve τ of degree 8 in Plücker space. When the configuration of the two spheres and the fixed line becomes degenerate, this curve becomes reducible. It is impossible for any component σ of τ to have degree 3 or 5 or 7, and if a component σ has degree 4 or 6 or 8, then it determines the two spheres. Only in the cases when σ has degree 1 or 2 can there be more than 2 spheres; these are described in Theorem 3.11.

4. Open problems

As we have seen, the tangent problems lead to a variety of problems combining discrete and algebraic geometry. Some open questions have already been mentioned, in particular determining the maximum number of real common tangent lines to four disjoint unit balls in $\mathbb{R}^3$. An open problem related to this is to determine how many different geometric permutations are to four unit balls in 3-space. It is known that 2 is a lower bound (Figure 7 is one such configuration) and 3 is an upper bound (see [8]). If the balls have different radii, then Figure 8 gives four balls with six geometric permutations.

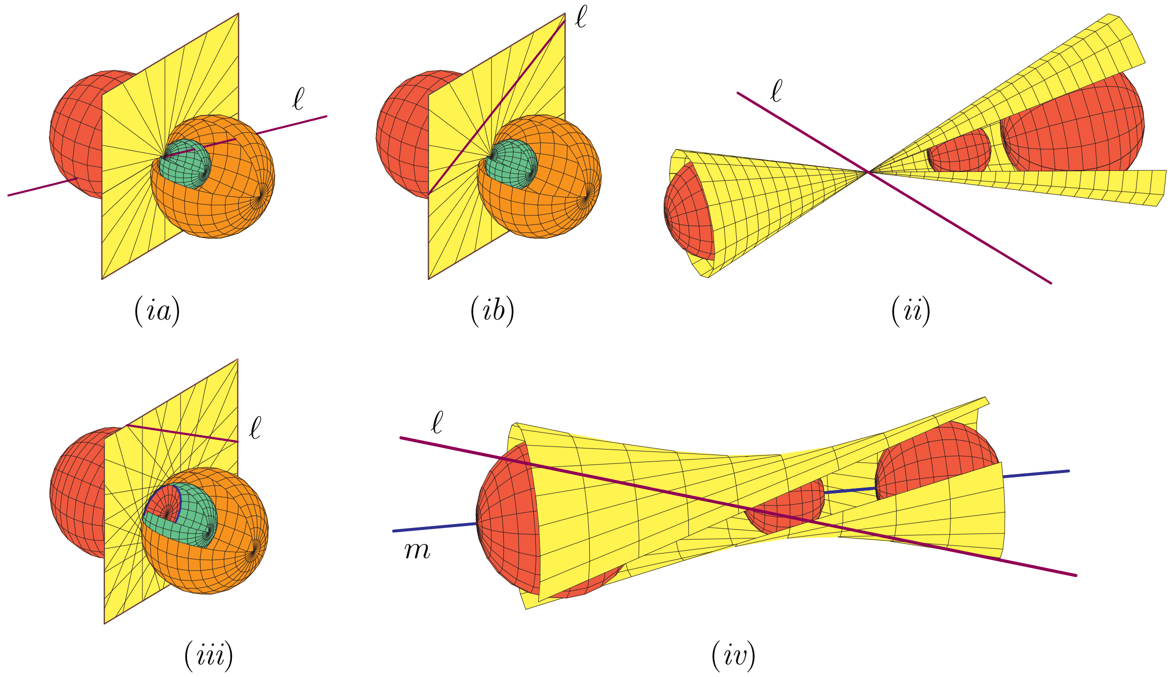


FIGURE 14. The geometry of Theorem 3.11.

We already noted that if we generalize the problems from spheres to quadric surfaces, the number of solutions can increase. In particular, four general quadric surfaces in $\mathbb{R}^3$ have 32 complex common tangent lines, and all these tangent lines can be real [27]. However, a complete characterization of the degenerate configurations is not known [5].

From the viewpoint of the inherent algebraic degree, the problem becomes more difficult for tangents to spheres in $\mathbb{R}^d$. If $d \geq 3$, then there are $3 \cdot 2^{d-1}$ complex common tangent lines to $2d-2$ general spheres in $\mathbb{R}^d$ and all of the tangent lines can be real [26]. A characterization of the degenerate configurations seems currently to be out of reach.

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