IMAGINARY PROJECTIONS: COMPLEX VERSUS REAL COEFFICIENTS

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Abstract. Given a multivariate complex polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$, the imaginary projection $\mathcal{I}(p)$ of p is defined as the projection of the variety $\mathcal{V}(p)$ onto its imaginary part. We focus on studying the imaginary projection of complex polynomials and we state explicit results for certain families of them with arbitrarily large degree or dimension. Then, we restrict to complex conic sections and give a full characterization of their imaginary projections, which generalizes a classification for the case of real conics. That is, given a bivariate complex polynomial $p \in \mathbb{C}[z_1, z_2]$ of total degree two, we describe the number and the boundedness of the components in the complement of $\mathcal{I}(p)$ as well as their boundary curves and the spectrahedral structure of the components. We further show a realizability result for strictly convex complement components which is in sharp contrast to the case of real polynomials.

1. Introduction

Given a polynomial $p \in \mathbb{C}[\mathbf{z}] := \mathbb{C}[z_1, \ldots, z_n]$, the imaginary projection $\mathcal{I}(p)$ as introduced in [20] is the projection of the variety $\mathcal{V}(p) \subseteq \mathbb{C}^n$ onto its imaginary part, that is,

$$\mathcal{I}(p) = \{\mathbf{z}_{\mathrm{im}} = ((z_1)_{\mathrm{im}}, \ldots, (z_n)_{\mathrm{im}}) \ : \ \mathbf{z} \in \mathcal{V}(p)\} \ \subseteq \ \mathbb{R}^n, \tag{1}$$

where $(\cdot)_{\mathrm{im}}$ is the imaginary part of a complex number. Recently, there has been wide-spread research interest in mathematical branches which are directly connected to the imaginary projection of polynomials.

As a primary motivation, the imaginary projection provides a comprehensive geometric view for notions of *stability of polynomials* and generalizations thereof. A polynomial $p \in \mathbb{C}[\mathbf{z}]$ is called *stable*, if $p(\mathbf{z}) = 0$ implies $(z_j)_{\mathrm{im}} \leq 0$ for some $j \in [n]$. In terms of the imaginary projection $\mathcal{I}(p)$, we can express the stability of p as the condition $\mathcal{I}(p) \cap \mathbb{R}^n_{>0} = \emptyset$. Stable polynomials have applications in many branches of mathematics including combinatorics ([5] and see [8] for the connection of the imaginary projection to combinatorics), differential equations [3], optimization [34], probability theory [4], and applied algebraic geometry [37]. Further application areas include theoretical computer science [23, 24], statistical physics [2], and control theory [25], see also the surveys [29] and [38].

Recently, various generalizations and variations of the stability notion have been studied, such as stability with respect to a polyball [13, 14], conic stability [9, 18], Lorentzian polynomials [6], or positively hyperbolic varieties [31]. Exemplarily, regarding the conic stability, a polynomial $p \in \mathbb{C}[\mathbf{z}]$ is called *K-stable* for a proper cone $K \subset \mathbb{R}^n$ if $p(\mathbf{z}) \neq 0$, whenever $\mathbf{z}_{\mathrm{im}} \in \operatorname{int} K$, where int is the interior. In terms of the imaginary projection, this condition can be equivalently expressed as $\mathcal{I}(p) \cap \operatorname{int} K = \emptyset$.

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Another motivation comes from the close connection of the imaginary projection to hyperbolic polynomials and hyperbolicity cones [11]. As shown in [19], in case of a real *homogeneous* polynomial p, the components of the complement $\mathcal{I}(p)^{\mathsf{c}}$ coincide with the hyperbolicity cones of p. These concepts play a central role in hyperbolic programming, see [15, 26, 27, 32]. A prominent open question in this research direction is the generalized Lax conjecture, which claims that every hyperbolicity cone is spectrahedral, see [36]. Representing convex sets by spectrahedra is not only motivated by the general Lax conjecture, but also by the question of effective handling convex semialgebraic sets (see, for example, [1, 21]). Recently, the conjecture that every convex semialgebraic set would be the linear projection of a spectrahedron, the "Helton-Nie conjecture", has been disproven by Scheiderer [33].

Moreover, the imaginary projection closely relates to and complements the notions of *amoebas*, as introduced by Gel'fand, Kapranov and Zelevinsky [12], and *coamoebas*. The amoeba $\mathcal{A}(p)$ of a polynomial p is defined as $\mathcal{A}(p) := \{(\ln|z_1|, \ldots, \ln|z_n|) : \mathbf{z} \in \mathcal{V}(p) \cap (\mathbb{C}^*)^n\}$, so it considers the logarithm of the absolute value of a complex number rather than its imaginary part. The coamoeba of a polynomial deals with the phase of a complex number. Each of these three viewpoints of a complex variety gives a set in a real space with the characteristic property that the complement of the closure consists of finitely many *convex* connected components. See [10], [12] and [20] for the convexity properties of amoebas, coamoebas, and imaginary projections, respectively. Due to their convexity phenomenon, these structures provide natural classes in recent developments of convex algebraic geometry.

For amoebas, an exact upper bound on the number of components in the complement is known [12]. For the coamoeba of a polynomial p, it has been conjectured that there are at most $n! \operatorname{vol} \operatorname{New}(p)$ connected components in the complement, where vol denotes the volume and $\operatorname{New}(p)$ the Newton polytope of p, see [10] for more background as well as a proof for the special case $n = 2$. For imaginary projections, a tight upper bound is known in the homogeneous case [19], but for the non-homogeneous case there only exists a lower bound [20].

Currently, no efficient method is known to calculate the imaginary projection for a general real or complex polynomial. For some families of polynomials, the imaginary projection has been explicitly characterized, including complex linear polynomials and real quadratic polynomials, see [20] and [18, Proposition 3.2]. However, since imaginary projections for non-linear complex polynomials exhibit new structural phenomena compared to the real case, even the characterization of the imaginary projection of complex conics had remained elusive so far.

Our primary goal is to reveal fundamental and surprising differences between imaginary projections of *real polynomials* and *complex polynomials*. In fixed degree and dimension, for a polynomial p with non-real coefficients, the algebraic degree of the *boundary* of the imaginary projection $\partial\mathcal{I}(p) := \overline{\mathcal{I}(p)} \cap \overline{\mathcal{I}(p)^{\mathsf{c}}}$ can be higher than the case of real coefficients. Here $(.)^{\mathsf{c}}$ and $\overline{(.)}$ are the complement and Euclidean closure, respectively. These incidences already begin when the degree and dimension are both two. However, the contrast is not only concerning the boundary degrees, but also the arrangements and the strict convexity of the components in $\mathcal{I}(p)^{\mathsf{c}}$.

We start with structural results which serve to work out the differences between the case of real and complex coefficients. Our first result is a sufficient criterion on the roots

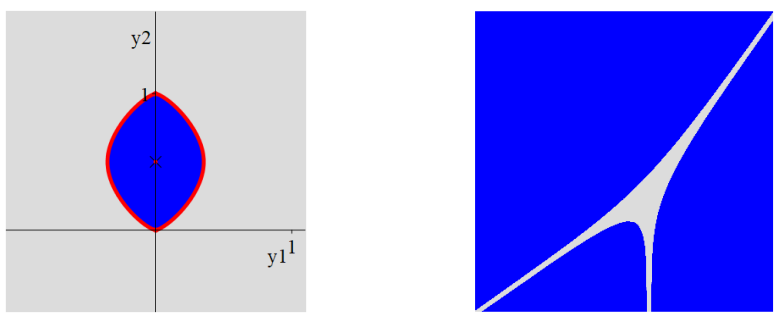
of the *initial form* of an arbitrarily large degree non-real bivariate complex polynomial to have the real plane as its imaginary projection, see Theorem 3.4 and Corollary 3.5.

Next, we characterize the imaginary projections of n-dimensional multivariate complex quadratics with hyperbolic initial form, see Theorem 4.5 and Corollary 4.6.

In the two-dimensional case, although by generalizing from real to complex conics, the bounds on the number of bounded and unbounded components in the complement of the imaginary projections remain unchanged, the possible arrangements of these components, strictness of their convexity, and the algebraic degrees of their boundaries strongly differ. See Corollaries 5.3 and 5.4. For conic sections with real coefficients, it was shown by Jörgens, Theobald, and de Wolff [20] that the boundary $\partial\mathcal{I}(p)$ consists of pieces which are algebraic curves of degree at most two. In sharp contrast to this, for complex polynomials, the boundary may not be algebraic and the degree of its irreducible pieces can go up to 8. For example, despite the simple expression of the polynomial $p = z_1^2 + \mathrm{i}z_2^2 + z_2$, an exact description of $\mathcal{I}(p)$ is

$$\begin{aligned}\mathcal{I}(p) \;=\; \{y \in \mathbb{R}^2 \,:\, & -64y_1^8 - 128y_1^4y_2^4 - 64y_2^8 + 256y_1^4y_2^3 + 256y_2^7 - 272y_1^4y_2^2 \\ & -400y_2^6 + 144y_1^4y_2 + 304y_2^5 - 27y_1^4 - 112y_2^4 + 16y_2^3 \leq 0\} \setminus \{(0, 1/2)\},\end{aligned} \tag{2}$$

and the describing polynomial in (2) is irreducible over $\mathbb{C}$. In this example, the set $\mathcal{I}(p)^{\mathsf{c}}$ consists of a single convex connected and bounded component. Any polynomial vanishing on the boundary will also vanish on the single point $(0, 1/2)$ which is not part of the boundary $\partial\mathcal{I}(p)$. Thus, $\partial\mathcal{I}(p)$ is not algebraic. See Figure 1 for an illustration and we return to this example in Section 3 and at the end of Section 6.



(A) (B)

FIGURE 1. (A) The gray area and its boundary form the imaginary projection $\mathcal{I}(p)$ of $p = z_1^2 + \mathrm{i}z_2^2 + z_2$. The polynomial in (2) vanishes on the red curve, which consists of a single point and another bounded component. The complement $\mathcal{I}(p)^{\mathsf{c}}$ contains the single point and it is bounded by the other component. (B) The amoeba of p is shown in gray.

Since the topology of the imaginary projection in $\mathbb{R}^n$ is invariant under the action of $G_n := \mathbb{C}^n \rtimes \mathrm{GL}_n(\mathbb{R})$, that is the semi-direct product of $\mathrm{GL}_n(\mathbb{R})$ and complex translations, the problem to understand the imaginary projections naturally leads to a polynomial classification problem.

As starting point, recall that under the action of the affine group $\mathrm{Aff}(\mathbb{C}^2)$, there are precisely five orbits for complex conics, with the following representatives:

$$z_1^2 \text{ (one line)}, \qquad z_1^2+1 \text{ (two parallel lines)}, \qquad z_1^2 - z_2 \text{ (parabola)},$$
$$z_1^2+z_2^2 \text{ (two crossing lines)}, \qquad z_1^2+z_2^2-1 \text{ (circle)}.$$

However, the arrangement of the components in $\mathcal{I}(p)^{\mathsf{c}}$ is not invariant under the action of $\mathrm{Aff}(\mathbb{C}^2)$, but only under its restriction to G_2. There are several other related classifications of complex conic sections. Newstead [28] has classified the set of projective complex conics under real linear transformations. However, out of a projective setting his method becomes ineffective as it is based on the arrangements of four intersection points between a conic and its conjugate. On the other hand, by considering the real part and the imaginary part of a complex conic p, under the action of G_2 the classification of conic sections has some relations to the problem of classifying pairs of real conics. Systematic classifications of this kind are mostly done in the projective setting and are well understood. See [7, 22, 30, 35]. However, those classifications rely on the invariance of the number and multiplicity of real intersection points between the two real conics. The drawback here is that under complex translations on p, these numbers are not invariant anymore, except at infinity.

To capture the invariance under G_2, we develop a novel classification based on the initial forms of complex conics. This classification is adapted to the imaginary projection and it is rather fine but coarse enough to allow handling the inherent algebraic degree of 8 in the boundary description of the imaginary projection.

Finally, we show that non-real complex conics can significantly improve a realization result on the complement of the imaginary projections. In [19], for any given integer $k \geq 1$, they present a polynomial p of degree $d = 4\lceil \frac{k}{4} \rceil + 2$ as a product of real conics, such that $\mathcal{I}(p)^{\mathsf{c}}$ has at least k components that are strictly convex and bounded. Using non-real conics, we furnish a degree $d/2+1$ polynomial having exactly k components with these properties. See Theorem 7.1 and Question 7.2.

The paper is structured as follows. Section 2 provides our notation and the necessary background on the imaginary projection of polynomials and contains the classification of the imaginary projection for the case of real conics. Section 3 deals with complex plane curves and provides a highlighting example where the complex versus real coefficients make a remarkable difference in the complexity of the imaginary projection. Moreover, we determine a family of arbitrarily large degree non-real plane curves with a full-space imaginary projection, based on the arrangements of roots of the initial form. In Section 4, we set the degree to be two and let the dimension grow and we classify the imaginary projections of complex quadratics with hyperbolic initial form. In Sections 5 and 6, we restrict the degree and dimension both to be two and we provide a full classification of the imaginary projections for affine complex conics based on their initial forms. Moreover, we determine in which classes the components in the complement of the imaginary projection have a spectrahedral description and also state them explicitly.

Section 5 contains our main classification theorems and the corollaries differentiating the cases of complex and real coefficients. The part where the initial form is hyperbolic is already covered in 4. Each subsection of Section 6 treats one of the remaining classes and explains their spectrahedral structure. In particular, we show that the only class where the components in the complement are not necessarily spectrahedral is the case

where the initial form has two distinct non-real roots in $\mathbb{P}^1_{\mathbb{C}}$ such that they do not form a complex conjugate pair. In Section 7, we prove a realization result for strictly convex complement components, which highlights another contrast between the imaginary projections of complex and real polynomials. Section 8 gives some open questions.

2. Preliminaries and background

For a set $S \subseteq \mathbb{R}^n$, we denote by $\overline{S}$ the topological closure of S with respect to the Euclidean topology on $\mathbb{R}^n$ and by S^{c} the complement of S in $\mathbb{R}^n$. The *algebraic degree* of S is the degree of its closure with respect to the Zariski topology. The set of non-negative and the set of strictly positive real numbers are abbreviated by $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ throughout the text. Moreover, bold letters will denote n-dimensional vectors. By $\mathbb{P}^n$ and $\mathbb{P}^n_{\mathbb{R}}$, we denote the n-dimensional complex and real projective spaces, respectively.

For a polynomial $p \in \mathbb{C}[\mathbf{z}]$, the imaginary projection $\mathcal{I}(p)$ is defined in (1) and its boundary $\overline{\mathcal{I}(p)} \cap \overline{\mathcal{I}(p)^{\mathsf{c}}}$ is denote by $\partial\mathcal{I}(p)$.

Theorem 2.1. [20] *Let $p \in \mathbb{C}[\mathbf{z}]$ be a complex polynomial. The set $\overline{\mathcal{I}(p)}^{\mathsf{c}}$ consists of a finite number of convex connected components.*

We denote by a_{re} and a_{im} the real and the imaginary parts of a complex number $a \in \mathbb{C}$, i.e., a is written in the form $a_{\mathrm{re}} + \mathrm{i}a_{\mathrm{im}}$, such that $a_{\mathrm{re}}, a_{\mathrm{im}} \in \mathbb{R}$. Let $p \in \mathbb{C}[\mathbf{z}]$ be a complex polynomial. After substituting $z_j = x_j + \mathrm{i}y_j$ for all $1 \leq j \leq n$, the complex polynomial can be written in the form

$$p(\mathbf{z}) = p_{\mathrm{re}}(\mathbf{x}, \mathbf{y}) + \mathrm{i}p_{\mathrm{im}}(\mathbf{x}, \mathbf{y}),$$

such that $p_{\mathrm{re}}, p_{\mathrm{im}} \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$. We call the real polynomials p_{re} and p_{im}, the *real part* and the *imaginary part* of p, respectively. Thus, finding $\mathcal{I}(p)$ is equivalent to determining the values of $\mathbf{y}$ for which the real polynomial system

$$p_{\mathrm{re}}(\mathbf{x}, \mathbf{y}) = 0 \quad \text{and} \quad p_{\mathrm{im}}(\mathbf{x}, \mathbf{y}) = 0 \tag{3}$$

has real solutions for $\mathbf{x}$.

Definition 2.2. Let $p \in \mathbb{C}[z_1, z_2]$ be a quadratic polynomial, i.e., $p = az_1^2 + bz_1z_2 + cz_2^2 + dz_1 + ez_2 + f$ such that $a, b, c, d, e, f \in \mathbb{C}$. We say that p is the defining polynomial of a complex conic, or shortly, a *complex conic* if its total degree equals two, i.e., at least one of the coefficients a, b, or c is non-zero. A complex conic p is called a *real conic* if all coefficients of p are real.

The following lemma from [20] shows how real linear transformations and complex translations act on the imaginary projection. These are the key ingredients for computing the imaginary projection of every class of conic sections.

Lemma 2.3. *Let $p \in \mathbb{C}[\mathbf{z}]$ and $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then*

$$\mathcal{I}(p(A\mathbf{z})) = A^{-1}\mathcal{I}(p(\mathbf{z})).$$

Moreover, a real translation $\mathbf{z} \mapsto \mathbf{z} + \mathbf{a}$, $\mathbf{a} \in \mathbb{R}^n$ does not change the imaginary projection. An imaginary translation $\mathbf{z} \mapsto \mathbf{z} + \mathrm{i}\mathbf{a}$, $\mathbf{a} \in \mathbb{R}^n$ shifts the imaginary projection into the direction $-\mathbf{a}$.

By the previous lemma, to classify the imaginary projection of polynomials we consider their orbits under the action of the group $G_n := \mathbb{C}^n \rtimes \mathrm{GL}_n(\mathbb{R})$, given by real linear transformations and complex translations. Further let $\mathrm{Aff}(\mathbb{K}^n) := \mathbb{K}^n \rtimes \mathrm{GL}_n(\mathbb{K})$ be the general affine group for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The real dimensions of these groups are

$$\dim_{\mathbb{R}}(\mathrm{Aff}(\mathbb{C}^n)) = 2\dim_{\mathbb{R}}(\mathrm{Aff}(\mathbb{R}^n)) = 2(n^2+n), \qquad \dim_{\mathbb{R}}(G_n) = n^2+2n.$$

Up to the action of G_2, a real conic $p \in \mathbb{R}[z_1, z_2]$ is equivalent to a conic given by one of the following polynomials.

(i) $z_1^2 + z_2^2 - 1$ (ellipse),	(v) $z_1^2 - z_2^2$ (pair of crossing lines),
(ii) $z_1^2 - z_2^2 - 1$ (hyperbola),	(vi) $z_1^2 - 1$ (parallel lines/one line z_1^2),
(iii) $z_1^2 + z_2$ (parabola),	(vii) $z_1^2 + z_2^2$ (isolated point),
(iv) $z_1^2 + z_2^2 + 1$ (empty set),	$(viii)$ $z_1^2 + 1$ (empty set).

In [20], a full classification of the imaginary projection for real quadratics was shown. In particular, the following theorem is the classification for real conics. For illustrations of the cases, see Figure 2. The theorem that comes after provides the imaginary projection of some families of real quadratics. Furthermore, they state the subsequent question as an open problem.

Theorem 2.4. *Let $p \in \mathbb{R}[z_1, z_2]$ be a real conic. For the normal forms (i)–(viii) from above, the imaginary projections $\mathcal{I}(p) \subseteq \mathbb{R}^2$ are as follows.*

(i) $\mathcal{I}(p) = \mathbb{R}^2$,	(v) $\mathcal{I}(p) = \{\mathbf{y} \in \mathbb{R}^2 : y_1^2 = y_2^2\}$,
(ii) $\mathcal{I}(p) = \{-1 \le y_1^2 - y_2^2 < 0\} \cup \{\mathbf{0}\}$,	(vi) $\mathcal{I}(p) = \{\mathbf{y} \in \mathbb{R}^2 : y_1 = 0\}$,
(iii) $\mathcal{I}(p) = \mathbb{R}^2 \setminus \{(0, y_2) : y_2 \neq 0\}$,	(vii) $\mathcal{I}(p) = \mathbb{R}^2$,
(iv) $\mathcal{I}(p) = \{\mathbf{y} \in \mathbb{R}^2 : y_1^2 + y_2^2 - 1 \ge 0\}$,	$(viii)$ $\mathcal{I}(p) = \{\mathbf{y} \in \mathbb{R}^2 : y_1 = \pm 1\}$.

Theorem 2.5. *Let $p \in \mathbb{C}[z_1, \dots, z_n]$ be $p = \sum_{i=1}^{n-1} z_i^2 - z_n^2 + k$ for $k \in \{\pm 1\}$. Then*

$$\mathcal{I}(p) = \begin{cases} \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 < \sum_{i=1}^{n-1} y_i^2\right\} \cup \{\mathbf{0}\} & \text{if } k = 1, \\ \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 - \sum_{i=1}^{n-1} y_i^2 \le 1\right\} & \text{if } k = -1. \end{cases}$$

The following question, which is true for real quadratics $p \in \mathbb{C}[\mathbf{z}]$, was asked in [20, Open problem 3.4]. In Section 6.2, we show that it is not true in general even for complex conics.

Question 2.6. Let $p \in \mathbb{C}[\mathbf{z}]$ be a polynomial. Is $\mathcal{I}(p)$ open if and only if $\mathcal{I}(p) = \mathbb{R}^n$?

We use the *initial form* of p abbreviated by $\mathrm{in}(p)(\mathbf{z}) = p^h(\mathbf{z}, 0)$, where p^h is the homogenization of p. The initial form consists of the terms of p with the maximal total degree. Furthermore, a complex polynomial $p \in \mathbb{C}[\mathbf{z}]$ is called *hyperbolic* w.r.t. $\mathbf{e} \in \mathbb{R}^n$ if the univariate polynomial $t \mapsto p(\mathbf{x} + t\mathbf{e})$ is real-rooted. Note that any hyperbolic polynomial is a, possibly complex, multiple of a real polynomial.

Finally, a *spectrahedron* is a set of the form

$$\{\mathbf{x} \in \mathbb{R}^n \ : \ A_0 + \sum_{j=1}^{n} A_j x_j \succeq 0\},$$

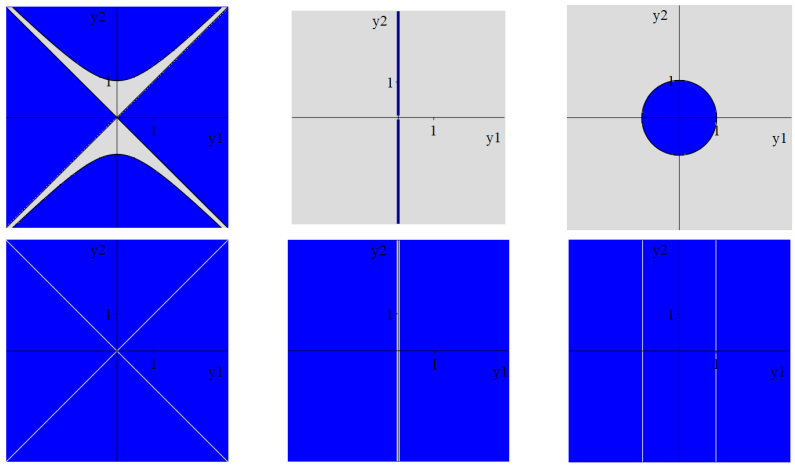


FIGURE 2. The imaginary projections of the real conic sections and their complements are colored in gray and blue, respectively. The cases (i) and (vii) are skipped, as their imaginary projection is the whole plane.

where $A_1, \ldots, A_n$ are real symmetric matrices of size d. Here, "$\succeq 0$" denotes the positive semidefiniteness of a matrix. We also speak of a spectrahedral set if the set is given by positive definite conditions, i.e., by strict conditions.

3. Imaginary projections of complex plane curves

In this section, we determine the imaginary projection of some families of arbitrarily high degree complex plane curves. Our point of departure is the characterization of real conics in Theorem 2.4. In the following example, which is an affine version of case (B_+) in Newstead's classification [28], we show that by allowing non-real coefficients the imaginary projection of a complex conic can significantly change in terms of the algebraic degree of its boundary. See Corollary 5.3.

Remark 3.1. Recall that the discriminant of a univariate polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ is given by $\mathrm{Disc}(p) = (-1)^{\frac{1}{2}n(n-1)} \frac{1}{a_n} \mathrm{Res}(p, p')$, where Res denotes the resultant. For a quartic, having negative discriminant implies the existence of a real root. However, a positive discriminant can correspond to either four real roots or none. Let

$$P = 8a_2a_4 - 3a_3^2,\ R = a_3^3 + 8a_1a_4^2 - 4a_4a_3a_2,\ D = 64a_4^3a_0 - 16a_4^2a_2^2 + 16a_4a_3^2a_2 - 16a_4^2a_3a_1 - 3a_3^4.$$

If $\mathrm{Disc}(p) > 0$, then $p = 0$ has four real roots if $P < 0$ and $D < 0$, and no real roots otherwise. Finally, if the discriminant is zero, the only conditions under which there is no real solution is having $D = R = 0$ and $P > 0$ (see, e.g., [17, Theorem 9.13 (vii)]).

Example 3.2. Let $p = z_1^2 + \mathrm{i}z_2^2 + z_2$. For simplifying the calculations, we use the translation $z_2 \mapsto z_2 + \mathrm{i}/2$ to eliminate the linear term. This turns the equation $p = 0$ into $q := z_1^2 + \mathrm{i}z_2^2 + \mathrm{i}/4 = 0$. Building the real polynomial system as introduced in (3) implies

$$q_{\mathrm{re}} = x_1^2 - 2x_2y_2 - y_1^2 = 0 \ \text{ and } \ q_{\mathrm{im}} = 4x_2^2 + 8x_1y_1 - 4y_2^2 + 1 = 0.$$

First assume $y_1 \neq 0$. Substituting x_1 from $q_{\mathrm{im}} = 0$ into $q_{\mathrm{re}} = 0$ gives

$$16x_2^4 + (-32y_2^2 + 8)x_2^2 - 128y_1^2y_2x_2 - 64y_1^4 + 16y_2^4 - 8y_2^2 + 1 = 0.$$

We calculate the discriminant of the above equation with respect to x_2. By the previous remark, there is a real solution for x_2 if the discriminant is negative, i.e.,

$$-64y_1^8 - 128y_1^4y_2^4 - 64y_2^8 - 80y_1^4y_2^2 + 48y_2^6 + y_1^4 - 12y_2^4 + y_2^2 < 0.$$

Now we need to check the conditions where the discriminant is zero or positive. To show the positive discriminant implies no real solution for x_2, we rewrite the condition with the substitution $u = y_1^4$:

$$\triangle := -64u^2 + (-128y_2^4 - 80y_2^2 + 1)u - 64y_2^8 + 48y_2^6 - 12y_2^4 + y_2^2 > 0.$$

It is a quadratic polynomial in u with negative leading coefficient. It can only be positive between the two roots for u in $\triangle = 0$. Those are

$$-y_2^4 - \frac{5}{8}y_2^2 + \frac{1}{128} \pm \frac{\sqrt{32768y_2^6 + 3072y_2^4 + 96y_2^2 + 1}}{128}.$$

To obtain $\triangle > 0$, we need to have a solution $u > 0$, i.e., we need to have either $-y_2^4 - \frac{5}{8}y_2^2 + \frac{1}{128} \geq 0$ or otherwise

$$\left(-y_2^4 - \frac{5}{8}y_2^2 + \frac{1}{128}\right)^2 > \frac{32768y_2^6 + 3072y_2^4 + 96y_2^2 + 1}{128^2}.$$

The first inequality implies $y_2^2 \leq \frac{3\sqrt{3}-5}{16}$ and after simplifications the second inequality implies $y_2^2 < 1/4$. The polynomial P from the previous remark for the quartic polynomials evaluates to $4(1 - 4y_2^2)$, which is positive for $y_2^2 < 1/4$. Therefore, for $\triangle > 0$, there is no real solution for x_2. It remains now to consider the case $\triangle = 0$. Since $y_1 \neq 0$, to have $R = -262144y_2y_1^2 = 0$ we need $y_2 = 0$. Substituting $y_2 = 0$ in $D = 0$ implies $-4096y_1^4 - 960 = 0$, which is a contradiction. Therefore, if $y_1 \neq 0$, the imaginary projection of q consists of points $\mathbf{y} \in \mathbb{R}^2$ for which $\triangle \leq 0$.

Now assume $y_1 = 0$. From $q_{\text{im}} = 0$ we can observe that $\mathbf{0} \notin \mathcal{I}(q)$. Thus, assume $y_2 \neq 0$. Solving $q_{\text{re}} = 0$ for x_2 and substituting in $q_{\text{im}} = 0$ implies $x_1^4 - y_2^2(4y_2^2 - 1) = 0$. This equation has a real solution if and only if $-y_2^2(4y_2^2 - 1) \leq 0$. Substituting $y_1 = 0$ in $\triangle$ allows to write $\triangle$ in terms of y_2, which gives $\triangle_{y_2} = -y_2^2(4y_2^2 - 1)^3$. Therefore, the imaginary projection on the y_2-axis is $\{(0, y_2) \in \mathbb{R}^2 : \triangle_{y_2} \leq 0\} \setminus \{(0, 0)\}$. Thus,

$$\mathcal{I}(q) = \{\mathbf{y} \in \mathbb{R}^2 : -64y_1^8 - 128y_1^4y_2^4 - 64y_2^8 - 80y_1^4y_2^2 + 48y_2^6 + y_1^4 - 12y_2^4 + y_2^2 \leq 0\} \setminus \{\mathbf{0}\}.$$

The irreducibility of the polynomial above over $\mathbb{C}$ can be verified for example using Maple. For the original polynomial p, this gives the inequality description for $\mathcal{I}(p)$ stated in (2) in the Introduction.

Even in the case of real polynomials, extending the case of real conics by letting the degree or the number of variables be greater than two dramatically increases the difficulty of characterizing the imaginary projection. Let us see one such example of a cubic plane curve, i.e., where we have two unknowns and the total degree is three.

Example 3.3. Let $p \in \mathbb{R}[\mathbf{z}] = \mathbb{R}[z_1, z_2]$ be of the form $p = z_1^3 + z_2^3 - 1$. The similar attempt as before to calculate the imaginary projection $\mathcal{I}(p)$ is to separate the real and the imaginary parts of p according to (3),

$$p_{\text{re}} = x_1^3 - 3x_1y_1^2 + x_2^3 - 3x_2y_2^2 - 1 = 0 \text{ and } p_{\text{im}} = 3x_1^2y_1 + 3x_2^2y_2 - y_1^3 - y_2^3 = 0.$$

Despite the simplicity of the polynomial p, one cannot use the previous techniques to find the values of $\mathbf{y} \in \mathbb{R}^2$ such that the above system has real solutions for $\mathbf{x}$. The reason is that both x_1 and x_2 appear in higher degree than one in both equations. The resultant with respect to one of x_1 or x_2 is a univariate polynomial of degree six in the other, where we lack the exact tools to specify the reality of the roots.

In the following theorem, we show that the imaginary projection of a generic complex plane curve of odd degree is the whole plane.

Theorem 3.4. *Let $p \in \mathbb{C}[z_1, z_2]$ be a complex bivariate polynomial of total degree d such that its initial form has no real roots in $\mathbb{P}^1$. If d is odd then the imaginary projection $\mathcal{I}(p)$ is $\mathbb{R}^2$. As a consequence, the imaginary projection of a generic complex bivariate polynomial of odd total degree is $\mathbb{R}^2$.*

Proof. Since the initial form has no real roots, it can be written in the form

$$\mathrm{in}(p) = \prod_{j=1}^{d} (z_1 - \alpha_j z_2),$$

where $\alpha_j \notin \mathbb{R}$ for $1 \leq j \leq d$. Substitute $z_j = x_j + \mathrm{i}y_j$ for $j = 1, 2$ in p and form the polynomial system $p_{\mathrm{re}} = p_{\mathrm{im}} = 0$ as introduced in (3). For any fixed $\mathbf{y} \in \mathbb{R}^2$, both equations are of total degree d in x_1 and x_2. Denote by p^h_{re} and p^h_{im}, the homogenization of these two polynomials by a new variable x_3. Since both, p^h_{re} and p^h_{im}, have odd degree, the number of complex intersection points (counted with multiplicities) is odd while the number of non-real intersection points (counted with multiplicities) is even. Thus, there is a real intersection point in $\mathbb{P}^2_{\mathbb{R}}$. We claim that this intersection point lies in the affine piece where $x_3 = 1$. This implies that for any given $\mathbf{y} \in \mathbb{R}^2$, there exist $x_1, x_2 \in \mathbb{R}$ for which $p_{\mathrm{re}} = p_{\mathrm{im}} = 0$ and therefore completes the proof.

To prove our claim, we show that the two curves defined by $p^h_{\mathrm{re}} = 0$ and $p^h_{\mathrm{im}} = 0$ do not intersect at infinity, i.e., their intersection point has $x_3 \neq 0$. Let us assume that they intersect at infinity and set $x_3 = 0$ in p^h_{re} and p^h_{im}. This substitution turns the complex polynomial $p^h_{\mathrm{re}} + \mathrm{i}p^h_{\mathrm{im}}$ into

$$q := \prod_{j=1}^{d} (x_1 - \alpha_j x_2).$$

Thus, for the two projective curves to intersect at infinity we need to have $q = 0$. Since $\alpha_j \notin \mathbb{R}$ for $1 \leq j \leq d$, the only real solution for x_1 and x_2 is zero. This is a contradiction. □

Corollary 3.5. *Let $p \in \mathbb{C}[z_1, z_2]$ be a complex bivariate polynomial. The imaginary projection $\mathcal{I}(p)$ is $\mathbb{R}^2$ if p has a factor q such that the total degree of q is odd and its initial form has no real roots in $\mathbb{P}^1$.*

Proof. Since for $p_1, p_2 \in \mathbb{C}[\mathbf{z}]$, we have $\mathcal{I}(p_1 \cdot p_2) = \mathcal{I}(p_1) \cup \mathcal{I}(p_2)$, we claim that if there is a factor q in p whose imaginary projection is $\mathbb{R}^2$, then $\mathcal{I}(p) = \mathbb{R}^2$. The result now follows from the previous theorem. □

In the following section, instead of the dimension we set the degree to be two and characterize the imaginary projection for a certain family of quadratic hypersurfaces.

4. Complex quadratics with hyperbolic initial form

As we have seen in Example 3.2, the methods used to compute the imaginary projection of real quadratics is not always useful for complex ones. However, for a certain family, namely the quadratics with hyperbolic initial form, one can build up on the methods for the real case. To classify the imaginary projections of any family of polynomials, Lemma 2.3 suggests bringing them to their proper normal forms.

Lemma 4.1. *Under the action of G_n, any quadratic polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$ with hyperbolic initial form can be transformed to one of the following normal forms:*

(1) $z_1^2 + \alpha z_2 + r z_3 + \gamma$,

(2) $\sum_{i=1}^{j} z_i^2 - z_{j+1}^2 + \alpha z_{j+2} + r z_{j+3} + \gamma \qquad$ *for some* $j = 1, \dots, n-1$,

such that terms containing z_k do not appear for $k > n$, and $\alpha, r, \gamma \in \mathbb{C}$.

Proof. The initial form $\mathrm{in}(p)$ is a hyperbolic polynomial of degree two. That is, after a real linear transformation it can be either z_1^2 or of the form $\mathbf{z}'^T M \mathbf{z}'$ such that $\mathbf{z}' = (z_1, \dots, z_{j+1})$ for some $1 \le j \le n-1$ and M is a square matrix of size $j+1$ with signature $(j, 1)$. See [11]. This explains the initial forms in (1) and (2).

Any term λz_j for some $1 \le j \le n$, such that z_j appears in our transformed initial forms, cancels out by one of the translations $z_j \mapsto z_j \pm \frac{\lambda}{2}$ without changing the initial form. Finally, we show that the number of linear terms in the rest of the variables is at most two. Consider the complex linear form $\sum_{j=1}^{n} \lambda_j z_j$. For $1 \le j \le n$, let $\lambda_j = r_j + \mathrm{i} s_j$ such that $r_j, s_j \in \mathbb{R}$. We can now write the sum as $(\sum_{j=1}^{n} r_j z_j) + \mathrm{i}(\sum_{j=1}^{n} s_j z_j)$. If in the real part at least one of the r_j, say, r_1, is non-zero, then a sequence of linear transformations $z_1 \mapsto z_1 - \frac{r_j}{r_1} z_j$ for $j = 2, \dots, n$, cancels out $\sum_{j=2}^{n} r_j z_j$. Similarly, the complex part reduces to only one term. □

We first focus on the case where $n = 2$. In this case, we explicitly express the unbounded spectrahedral components forming $\mathcal{I}(p)^{\mathsf{c}}$. The following subsection covers part of the proof of Theorem 5.1.

4.1. Complex conics with hyperbolic initial form.

To match them with our classification of conics in Theorem 5.5, we do a real linear transformation in the case (2) and write them as

$$(1\text{a.1})\; p = z_1^2 + \gamma, \qquad (1\text{a.2})\; p = z_1^2 + \gamma z_2 \quad \gamma \neq 0, \qquad (1\text{b})\; p = z_1 z_2 + \gamma,$$

for some $\gamma \in \mathbb{C}$. To find $\mathcal{I}(p)$ for each normal form, we compute the resultant of the two real polynomials, as introduced in (3), with respect to x_i to have a univariate polynomial in x_j, where $i, j \in \{1, 2\}$, and $i \neq j$. Then we use the discriminantal conditions on the univariate polynomials to argue about the real roots.

First consider the normal form (1a.1). If $\gamma_{\mathrm{im}} = 0$, then we have the real conics of the cases (vi) and $(viii)$ in Theorem 2.4. The two real polynomials $p_{\mathrm{re}} = x_1^2 - y_1^2 + \gamma_{\mathrm{re}} = 0$ and $p_{\mathrm{im}} = 2x_1 y_1 + \gamma_{\mathrm{im}} = 0$ form the system (3) here. From $\gamma_{\mathrm{im}} \neq 0$, we need to have $y_1 \neq 0$. Now substituting $x_1 = \frac{-\gamma_{\mathrm{im}}}{2y_1}$ from $p_{\mathrm{im}} = 0$ into $p_{\mathrm{re}} = 0$ and solving for y_1^2 implies $y_1^2 = \frac{1}{2}\left(\gamma_{\mathrm{re}} + \sqrt{\gamma_{\mathrm{re}}^2 + \gamma_{\mathrm{im}}^2}\right)$. Therefore,

$$\mathcal{I}(p) = \begin{cases} \text{A unique line} & \text{if } \gamma \in \mathbb{R}_{\leq 0}, \\ \text{Two parallel lines} & \text{otherwise.} \end{cases} \tag{1a.1}$$

Clearly, the closures of the components in the complement are spectrahedra.

Now consider (1a.2) which is a generalization of the parabola case (*iii*) in Theorem 2.4, where $\gamma = 1$. Similarly to the previous case, we build the corresponding polynomial system as (3). The discriminantal condition after substituting x_2 from $p_{\text{im}} = 0$ into $p_{\text{re}} = 0$ implies that there exists a real x_1 if and only if $4|\gamma|^2(y_1^2 + \gamma_{\text{im}} y_2) \geq 0$. Hence, $\mathcal{I}(p)^{\mathsf{c}}$ consists of $\mathbf{y} \in \mathbb{R}^2$ such that $y_1^2 + \gamma_{\text{im}} y_2 < 0$. This inequality specifies the open subset of $\mathbb{R}^2$ bounded by the parabola $y_1^2 + \gamma_{\text{im}} y_2 = 0$ and containing its focus. Therefore,

$$\mathcal{I}(p) = \begin{cases} \mathbb{R}^2 \setminus \{(0, y_2) : y_2 \neq 0\} & \text{if } \gamma \in \mathbb{R}, \\ \{\mathbf{y} \in \mathbb{R}^2 : y_1^2 + \gamma_{\text{im}} y_2 \geq 0\} & \text{otherwise.} \end{cases} \tag{1a.2}$$

Notice that this incidence of $\mathcal{I}(p)^{\mathsf{c}}$ consisting of one unbounded component does not occur for real conics. See Corollary 5.4. Further, $\mathcal{I}(p)^{\mathsf{c}}$ for $\gamma \notin \mathbb{R}$ is given by the unbounded spectrahedral set defined by

$$\begin{pmatrix} 1 & y_1 \\ y_1 & -\gamma_{\text{im}} y_2 \end{pmatrix} \succ 0.$$

For the last case (1b) from the corresponding real polynomial system $p_{\text{re}} = p_{\text{im}} = 0$, one can simply check that $\gamma = 0$ implies $\mathcal{I}(p) = \{\mathbf{y} \in \mathbb{R}^2 : y_1 y_2 = 0\}$. Now let $\gamma \neq 0$ and first assume $y_1 y_2 \neq 0$. After the substitution of x_2 from $p_{\text{im}} = 0$ to $p_{\text{re}} = 0$, the discriminantal condition on the quadratic univariate polynomial to have a real x_1 implies

$$\gamma_{\text{re}} - |\gamma| \leq 2 y_1 y_2 \leq \gamma_{\text{re}} + |\gamma|.$$

If $\gamma \in \mathbb{R} \setminus \{0\}$, then $\mathbf{0}$ is the only point with $y_1 y_2 = 0$ that is included in $\mathcal{I}(p)$. If $\gamma \notin \mathbb{R}$, then the union of the two axes except the origin is included in $\mathcal{I}(p)$. Thus,

$$\mathcal{I}(p) = \begin{cases} \text{The union of the two axes } y_1 \text{ and } y_2 & \text{if } \gamma = 0, \\ \{\mathbf{y} \in \mathbb{R}^2 \ : \ 0 < y_1 y_2 \leq \gamma\} \cup \{\mathbf{0}\} & \text{if } \gamma \in \mathbb{R}_{>0}, \\ \{\mathbf{y} \in \mathbb{R}^2 \ : \ \gamma \leq y_1 y_2 < 0\} \cup \{\mathbf{0}\} & \text{if } \gamma \in \mathbb{R} \setminus \mathbb{R}_{\geq 0}, \\ \left\{\mathbf{y} \in \mathbb{R}^2 \ : \ \frac{1}{2}(\gamma_{\text{re}} - |\gamma|) \leq y_1 y_2 \leq \frac{1}{2}(\gamma_{\text{re}} + |\gamma|)\right\} \setminus \{\mathbf{0}\} & \text{if } \gamma \notin \mathbb{R}. \end{cases} \tag{1b}$$

Corollary 4.2. *Let $p \in \mathbb{C}[z_1, z_2]$ be a complex conic with hyperbolic initial form. The complement $\mathcal{I}(p)^{\mathsf{c}}$ of the imaginary projection consists of only unbounded spectrahedral components.*

Proof. We saw this already for the cases (1a.1) and (1a.2). Therefore, we only prove the statement for (1b). There are four unbounded components, namely in each quadrant one, and no bounded component in $\mathcal{I}(p)^{\mathsf{c}}$. The closures of the four unbounded components after setting

$$w = \sqrt{\frac{1}{2}(|\gamma| + \gamma_{\text{re}})} \quad \text{and} \quad u = \sqrt{\frac{1}{2}(|\gamma| - \gamma_{\text{re}})}$$

have the following representations as spectrahedra. In the quadrants $y_1y_2 \geq 0$, they are expressed by $y_1y_2 - \frac{1}{2}(\gamma_{\mathrm{re}} + |\gamma|) \geq 0$, or equivalently, $S_1(y_1, y_2) \succeq 0$ and $S_2(y_1, y_2) \succeq 0$, where

$$S_1(y_1, y_2) = \begin{pmatrix} y_1 & w \\ w & y_2 \end{pmatrix}, \quad S_2(y_1, y_2) = \begin{pmatrix} -y_1 & w \\ w & -y_2 \end{pmatrix}.$$

In the quadrants with $y_1y_2 \leq 0$, they are expressed by $y_1y_2 - \frac{1}{2}(\gamma_{\mathrm{re}} - |\gamma|) \leq 0$, or equivalently, $S_3(y_1, y_2) \succeq 0$ and $S_4(y_1, y_2) \succeq 0$, where

$$S_3(y_1, y_2) = \begin{pmatrix} y_1 & u \\ u & -y_2 \end{pmatrix}, \quad S_4(y_1, y_2) = \begin{pmatrix} -y_1 & u \\ u & y_2 \end{pmatrix}.$$

□

Given a conic q, an explicit description of the components of $\mathcal{I}(q)^{\mathsf{c}}$ can be derived by using those of its normal form p and applying on $\mathbf{y}$ the inverse operations turning q to p. We close this subsection by providing two examples for the cases (1a.2) and (1b) and their corresponding spectrahedral components.

Example 4.3. Let $q(z_1, z_2) = z_1^2 + 2z_1z_2 + z_2^2 + 2\mathrm{i}z_2 + 1$. By applying the transformation A and the translation $\mathbf{w}$ given by

$$A := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{w} := \begin{pmatrix} 0 \\ \mathrm{i}/2 \end{pmatrix},$$

the conic q is transformed to its normal form $p = z_1^2 + 2\mathrm{i}z_2$. Thus, we have

$$\mathcal{I}(p)^{\mathsf{c}} = \left\{ y \in \mathbb{R}^2 : \begin{pmatrix} 1 & y_1 \\ y_1 & -2y_2 \end{pmatrix} \succ 0 \right\} \text{ and } \mathcal{I}(q)^{\mathsf{c}} = \left\{ y \in \mathbb{R}^2 : \begin{pmatrix} 1 & y_1 + y_2 \\ y_1 + y_2 & -2y_2 + 1 \end{pmatrix} \succ 0 \right\},$$

such that $\mathcal{I}(q)^{\mathsf{c}}$ is obtained by the inverse transformations for $\mathbf{y}$ in $\mathcal{I}(p)^{\mathsf{c}}$. Figure 4 (1a) illustrates $\mathcal{I}(q)^{\mathsf{c}}$.

Example 4.4. Let $q(z_1, z_2) = z_1^2 - z_2^2 + 2\mathrm{i}$. Applying $A = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$ transfers the conic q into $p = z_1z_2 + 2\mathrm{i} = 0$. The value of both u and w introduced in the proof of Corollary 4.2 is 1. By applying A^{-1} to $\mathbf{y}$, the matrices $S_1, \ldots, S_4$ transform to

$$T_1(y_1, y_2) = \begin{pmatrix} -y_1 - y_2 & 1 \\ 1 & -y_1 + y_2 \end{pmatrix}, \qquad T_2(y_1, y_2) = \begin{pmatrix} y_1 + y_2 & 1 \\ 1 & y_1 - y_2 \end{pmatrix},$$
$$T_3(y_1, y_2) = \begin{pmatrix} -y_1 - y_2 & 1 \\ 1 & y_1 - y_2 \end{pmatrix}, \qquad T_4(y_1, y_2) = \begin{pmatrix} y_1 + y_2 & 1 \\ 1 & -y_1 + y_2 \end{pmatrix}.$$

Thus, the complement of the imaginary projection as shown in Figure 3 is given by

$$\overline{\mathcal{I}(q)^{\mathsf{c}}} = \bigcup_{j=1}^{4} \left\{ \mathbf{y} \in \mathbb{R}^2 : T_j(y_1, y_2) \succeq 0 \right\}.$$

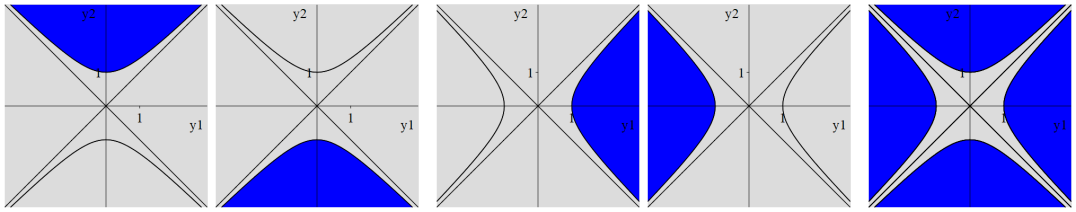


FIGURE 3. The first four pictures represent $T_j(y_1, y_2) \succeq 0$ for $1 \leq j \leq 4$, and the last one shows their union, which gives $\mathcal{I}(q)^{\mathsf{c}}$ for $q = z_1^2 - z_2^2 + 2\mathrm{i}$.

In the example above all four components are strictly convex, which can not occur in the case of real conics. This provides a key ingredient in the proof of Theorem 7.1.

4.2. **Higher dimensional complex quadratics.** We now let the dimension to be at least three and we use the normal forms provided in Lemma 4.1 to show the following classification of the imaginary projection. To avoid redundancy, for each quadratic polynomial we set n to be the largest index of z appearing in its normal form. Since we have already covered the case of conics, we need to consider $n \geq 3$.

Theorem 4.5. *Let $n \geq 3$ and $p \in \mathbb{C}[z_1, \ldots, z_n]$ be a quadratic polynomial with hyperbolic initial form. Up to the action of G_n, the imaginary projection $\mathcal{I}(p)$ is either $\mathbb{R}^n$, $\mathbb{R}^n \setminus \{(0, \ldots, 0, y_n) \in \mathbb{R}^n : y_n \neq 0\}$, or otherwise we can write p as $p = \sum_{i=1}^{n-1} z_i^2 - z_n^2 + \gamma$ for some $\gamma \in \mathbb{C}$ such that $|\gamma| = 1$ and we get*

$$\mathcal{I}(p) = \begin{cases} \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 < \sum_{i=1}^{n-1} y_i^2\right\} \cup \{\mathbf{0}\} & \text{if } \gamma = 1, \\ \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 - \sum_{i=1}^{n-1} y_i^2 \leq 1\right\} & \text{if } \gamma = -1, \\ \left\{\mathbf{y} \in \mathbb{R}^n \ : \ y_n^2 - \sum_{i=1}^{n-1} y_i^2 \leq \frac{1}{2}(1 - \gamma_{\mathrm{re}})\right\} \setminus \{\mathbf{0}\} & \text{if } \gamma \notin \mathbb{R}. \end{cases}$$

Proof. By real scaling and complex translations, any of the forms in Lemma 4.1 drops into one of the following cases:

$$\text{(a) } \alpha = r = \gamma = 0, \qquad \text{(b) } \alpha = 1, \text{ and } r = \gamma = 0, \qquad \text{(c) } \alpha \notin \mathbb{R}, \text{ and } r, \gamma = 0,$$
$$\text{(d) } \alpha \notin \mathbb{R},\ r = 1, \text{ and } \gamma = 0, \qquad \text{(e) } \alpha = r = 0, \text{ and } \gamma \neq 0.$$

For the normal form (1) all cases but (d) drop into the conic sections discussed previously. Case (d) is similar for both normal forms (1) and (2). Thus we focus on (2).

The imaginary projection for the cases (a) and (b) are known from the real classification and they are $\mathbb{R}^n$ and $\mathbb{R}^n \setminus \{(0, \ldots, 0, y_n) \in \mathbb{R}^n : y_n \neq 0\}$, respectively. See [20, Theorem 5.4].

In case (c) after building the system (3) and considering two cases, based on whether the real part of α is zero or not, one can then check that $\mathcal{I}(p) = \mathbb{R}^n$ as follows. We have

$$\begin{aligned} p_{\mathrm{re}} &= \textstyle\sum_{i=1}^{n-2} x_i^2 - x_{n-1}^2 - \sum_{i=1}^{n-2} y_i^2 + y_{n-1}^2 + \alpha_{\mathrm{re}} x_n - \alpha_{\mathrm{im}} y_n, \\ p_{\mathrm{im}} &= 2\textstyle\sum_{i=1}^{n-2} x_i y_i - 2x_{n-1}y_{n-1} + \alpha_{\mathrm{im}} x_n + \alpha_{\mathrm{re}} y_n. \end{aligned}$$

First assume $\alpha_{\mathrm{re}} = 0$. For any $\mathbf{y} \in \mathbb{R}^n$, the equation $p_{\mathrm{re}} = 0$ has solutions $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. By substituting any of those solutions in $p_{\mathrm{im}} = 0$ we can solve it for x_n and get a real solution. Now let $\alpha_{\mathrm{re}} \neq 0$. In this case, we substitute x_n from the second equation into the first. For any $\mathbf{y} \in \mathbb{R}^n$, we get $\sum_{i=1}^{n-2}(x_i - r_i)^2 - (x_{n-1} - r_{n-1})^2 = r_n$

for some $r_1, \dots, r_n \in \mathbb{R}$ and therefore, there always exists a real solution $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

Similarly, in the case (d), for any $\mathbf{y} \in \mathbb{R}^n$, there exists a real solution $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ for $p_{\mathrm{im}} = 0$ and for any $\mathbf{y} \in \mathbb{R}^n$ and any $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, there exists a real x_n for $p_{\mathrm{re}} = 0$. Thus $\mathcal{I}(p) = \mathbb{R}^n$ in this case, too.

Now we focus on case (e). Let $p = \sum_{i=1}^{n-1} z_i^2 - z_n^2 + \gamma$ for some $\gamma \in \mathbb{C} \setminus \{0\}$. Building the real system (3) for p yields

$$p_{\mathrm{re}} \;=\; \textstyle\sum_{i=1}^{n-1} x_i^2 - x_n^2 - \sum_{i=1}^{n-1} y_i^2 + y_n^2 + \gamma_{\mathrm{re}}, \qquad p_{\mathrm{im}} \;=\; 2\sum_{i=1}^{n-1} x_i y_i - 2x_n y_n + \gamma_{\mathrm{im}}.$$

We can assume $|\gamma| = 1$. Note that $\{\mathbf{0}\} \in \mathcal{I}(p)$ if and only if $\gamma \in \mathbb{R}$. We can thus exclude the origin in the following calculations. Moreover, Theorem 2.5 shows the cases where $\gamma = \pm 1$. Thus, we need to consider the case $\gamma \notin \mathbb{R}$.

Let T be an orthogonal transformation on $\mathbb{R}^{n-1}$. Invariance of the polynomials $\sum_{j=1}^{n-1} y_j^2$ and $\sum_{j=1}^{n-1} x_j y_j$ under the mapping $(x, y) \mapsto (T(x), T(y))$ implies

$$(y_1, y_2, \dots, y_n) \in \mathcal{I}(p) \qquad \text{if and only if} \qquad (y_1', \dots, y_{n-1}', y_n) \in \mathcal{I}(p),$$

where $(y_1', \dots, y_{n-1}') = T(y_1, \dots, y_{n-1})$. For a given $\mathbf{y} \in \mathcal{I}(p)$, let T be a transformation with the property $T(y_1, \dots, y_{n-1}) = (\sqrt{\sum_{i=1}^{n-1} y_i^2}, 0, \dots, 0)$ and set $(x_1', \dots, x_{n-1}') = T(x_1, \dots, x_{n-1})$. We can now rewrite the simplified polynomial system as

$$p_{\mathrm{re}} \;=\; \textstyle\sum_{i=1}^{n-1} {x_i'}^2 - x_n^2 - {y_1'}^2 + y_n^2 + \gamma_{\mathrm{re}}, \qquad p_{\mathrm{im}} \;=\; 2x_1' y_1' - 2x_n y_n + \gamma_{\mathrm{im}}.$$

First consider $y_1' = 0$. This implies $y_n \neq 0$. Solving $p_{\mathrm{im}} = 0$ for x_n and substituting in $p_{\mathrm{re}} = 0$ implies

$$4y_n^2\Big(\sum_{i=1}^{n-1} {x_i'}^2\Big) = \left(\gamma_{\mathrm{re}}^2 + \gamma_{\mathrm{im}}^2\right) - \left(2y_n^2 + \gamma_{\mathrm{re}}\right)^2 = 1 - \left(2y_n^2 + \gamma_{\mathrm{re}}\right)^2.$$

This has a real solution for $(x_1', \dots, x_{n-1}')$ if and only if $y_n^2 \leq \frac{1-\gamma_{\mathrm{re}}}{2}$. Now assume $y_1' \neq 0$. Observe that if ${y_1'}^2 = y_n^2$ then we always get a real solution. Thus assume $\frac{y_n^2}{{y_1'}^2} - 1 \neq 0$. Solving $p_{\mathrm{im}} = 0$ for x_1' and substituting in $p_{\mathrm{re}} = 0$ implies

$$\left(\frac{y_n^2}{{y_1'}^2} - 1\right)\left(x_n - \frac{\gamma_{\mathrm{im}} y_n}{2{y_1'}^2\left(\frac{y_n^2}{{y_1'}^2} - 1\right)}\right)^2 + \sum_{i=2}^{n-1} {x_i'}^2 + \frac{\left(y_n^2 - {y_1'}^2\right)^2 + \gamma_{\mathrm{re}}\left(y_n^2 - {y_1'}^2\right) - \left(\frac{\gamma_{\mathrm{im}}}{2}\right)^2}{y_n^2 - {y_1'}^2} = 0.$$

If ${y_1'}^2 > y_n^2$, there always is a real solution and otherwise, it has a real solution if and only if $\left(y_n^2 - {y_1'}^2\right)^2 + \gamma_{\mathrm{re}}\left(y_n^2 - {y_1'}^2\right) - \left(\frac{\gamma_{\mathrm{im}}}{2}\right)^2 \leq 0$. That is, $y_n^2 - {y_1'}^2 \leq \frac{1-\gamma_{\mathrm{re}}}{2}$. To get the imaginary projection of the original system, it is enough to do the inverse transformation T^{-1}. This completes the proof. □

Corollary 4.6. *Let $p \in \mathbb{C}[z_1, \dots, z_n]$ be a quadratic polynomial with hyperbolic initial form. Then*

(1) *the complement $\mathcal{I}(p)^{\mathsf{c}}$ is either empty or it consists of*
   - *one, two, three, or four unbounded components; or*
   - *two unbounded components and a single point.*

(2) *the complement of the closure $\overline{\mathcal{I}(p)}^{\mathsf{c}}$ is either empty or unbounded.*

(3) *the algebraic degrees of the irreducible components in* $\partial\mathcal{I}(p)$ *are at most two.*

5. The main classification of complex conics

In this section, we give a classification of the imaginary projection $\mathcal{I}(p)$ where $p \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, z_2]$ is a complex conic as in Definition 2.2. We state our topological classification in terms of the number and boundedness of the components in $\mathcal{I}(p)^{\mathsf{c}}$. In particular, this implies that the number of bounded and unbounded components do not exceed one and four, respectively. Furthermore, $\mathcal{I}(p)^{\mathsf{c}}$ cannot contain both bounded and unbounded components for some complex conic p.

A main achievement of this section is to establish a suitable classification and normal forms of complex conics under the action of the group G_2. There are infinitely many orbits on the set of complex conics under this action, since the real dimension of G_2 is 8 and the set of complex conics has real dimension 10. Each of our normal forms corresponds to infinitely many orbits that share their topology of imaginary projection by Lemma 2.3.

As a consequence of the obstructions in the existing classifications of conics that we discussed in the Introduction, we developed our own classification of conic sections. It is based on the five distinct arrangement possibilities for the roots of the initial form in $\mathbb{P}^1$ that are grouped in two main cases, depending on whether the initial form of the complex conic is hyperbolic or not:

Hyperbolic initial form	Non-hyperbolic initial form
(1a) A double real root	(2a) A double non-real root
(1b) Two distinct real roots	(2b) One real and one non-real root
	(2c) Two distinct non-real roots

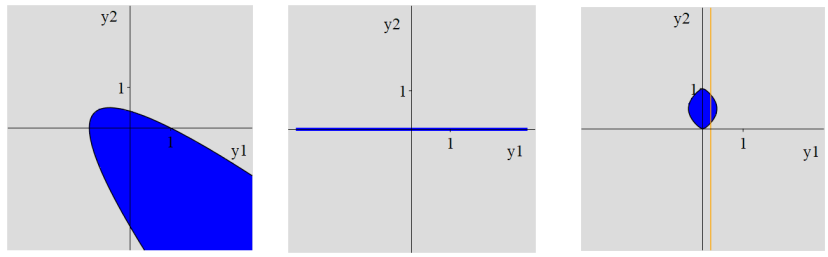
Theorem 5.1 (**Topological Classification**). *Let* $p \in \mathbb{C}[z_1, z_2]$ *be a complex conic. For the above five cases, the set* $\mathcal{I}(p)^{\mathsf{c}}$ *is*

(1a) *the union of one, two, or three unbounded components.*	(2a) *empty.*
(1b) *the union of four unbounded components.*	(2b) *empty, a single point, or a line segment.*
	(2c) *empty or one bounded component, possibly open.*

In particular, the components of $\mathcal{I}(p)^{\mathsf{c}}$ *are spectrahedral in all the first four classes. This is not true in general for the last class* (2c).

The following corollary relates the boundedness of the components in $\mathcal{I}(p)^{\mathsf{c}}$ to the hyperbolicity of the initial form $\mathrm{in}(p)$.

Corollary 5.2. *Let* $p \in \mathbb{C}[z_1, z_2]$ *be a complex conic. Then* $\mathcal{I}(p)^{\mathsf{c}}$ *consists of unbounded components if and only if the initial form of* p *is hyperbolic. Otherwise,* $\mathcal{I}(p)^{\mathsf{c}}$ *is empty or consists of one bounded component. Moreover, if there is a bounded component with non-empty interior, then* $\mathrm{in}(p)$ *has two distinct non-real roots.*



(1a) (2b) (2c)

FIGURE 4. The complements of the imaginary projections are colored in blue. The pictures show cases in the classification of the imaginary projection for complex conics which do not appear for real conics. The orange line in the right figure represents a generic line intersecting the boundary in two points, which is used to prove the non-spectrahedrality of this example in Section 6.

Figure 4 represents the types that do not appear for real coefficients. For instance, the middle picture, labeled as (2b), shows the case where $\mathcal{I}(p)^{\mathsf{c}}$ consists of a bounded component with empty interior. This can not occur if p has only real coefficients. The other two pictures are discussed in the next two corollaries. The following corollary compares the algebraic degrees of the irreducible components in the boundary $\partial\mathcal{I}(p)$. Its proof comes at the end of the next section.

Corollary 5.3. *Let $p \in \mathbb{C}[z_1, z_2]$ be a complex conic.*

(1) *The boundary $\partial\mathcal{I}(p)$ may not be algebraic. The algebraic degree of any irreducible component in its Zariski closure is at most 8. The bound is tight. If $\mathcal{I}(p)^{\mathsf{c}}$ has no bounded components, then $\partial\mathcal{I}(p)$ is algebraic and it consists of irreducible pieces of degree at most two.*

(2) *If all coefficients are real, then $\partial\mathcal{I}(p)$ is algebraic and it consists of irreducible pieces of degree at most two.*

Example 3.2, that is shown in Figure 4 (2c), illustrates an instance where the above contrast appears. The next corollary compares the number and strict convexity of the unbounded components that occur in $\mathcal{I}(p)^{\mathsf{c}}$ when p is a complex or a real conic.

Corollary 5.4. *Let $p \in \mathbb{C}[z_1, z_2]$ be a complex conic.*

(1) *The number of unbounded components in $\mathcal{I}(p)^{\mathsf{c}}$ can be any integer $0 \leq k \leq 4$ and up to 4 of them can be strictly convex.*

(2) *If all coefficients are real, the number of unbounded components in $\mathcal{I}(p)^{\mathsf{c}}$ can be any integer $0 \leq k \leq 4$ except for $k = 1$ and up to 2 of them can be strictly convex.*

The proof follows from Theorems 2.4 and 5.1, together with Example 4.4. The highlighting difference in the previous corollary, i.e., when $\mathcal{I}(p)^{\mathsf{c}}$ has one unbounded component, appears in the first class (1a) where the initial form has a double real root. Example 4.3 provides such an instance and is shown in Figure 4 (1a).

Theorem 5.1 is only proven by the end of Section 6. In the previous section, we discussed the case where p has hyperbolic initial form in details. It remains to consider the case where $\mathrm{in}(p)$ is not hyperbolic. As in Subsection 4.1, we first need to compute proper normal forms and then by Lemma 2.3, it suffices to compute the imaginary projections of those forms for each case.

Theorem 5.5 (**Normal Form Classification**). *With respect to the group G_2, there are infinitely many orbits for the complex conic sections with the following representatives.*

(1a) (1a.1) $p = z_1^2 + \gamma$
(1a.2) $p = z_1^2 + \gamma z_2$

(1b) $p = z_1 z_2 + \gamma$

(2a) (2a.1) $p = (z_1 - \mathrm{i} z_2)^2 + \gamma$
(2a.2) $p = (z_1 - \mathrm{i} z_2)^2 + \gamma z_2$

(2b) $p = z_2(z_1 - \alpha z_2) + \gamma$

(2c) (2c.1) $p = z_1^2 + z_2^2 + \gamma$
(2c.2) $p = (z_1 - \mathrm{i} z_2)(z_1 - \alpha z_2) + \gamma$

for some $\gamma, \alpha \in \mathbb{C}$ such that, to avoid overlapping, we assume $\gamma \neq 0$ in (1a.2) *and* (2a.2)*, $\alpha \notin \mathbb{R}$ in* (2b) *and* (2c.2)*, and finally $\alpha \neq \pm\mathrm{i}$ in* (2c.2).

Proof. By applying a real linear transformation we first map the roots of $\mathrm{in}(p)$ to $(0:1)$ in (1a), to $(1:0)$ and $(0:1)$ in (1b), to $(\mathrm{i}:1)$ in (2a), to $(1:0)$ and $(\alpha, 1)$ such that $\alpha \notin \mathbb{R}$ in (2b), to $(\pm\mathrm{i}:1)$ in (2c.1), and to $(\mathrm{i}:1)$ and $(\alpha:1)$ such that $\alpha \notin \mathbb{R}$ and $\alpha \neq \pm\mathrm{i}$ in (2c.2). Then, similar to the proof of Lemma 4.1, by eliminating some linear terms or the constant by complex translations we arrive at the given normal forms for each case. Since the arrangements of the two roots in $\mathbb{P}^1$ is invariant under the action of G_2, the given five cases lie in different orbits. Note that the orbits of the subcases in each case do not overlap. For the subcases of (1a), in (1a.2), z_1 and z_2 may be transformed to $az_1 + bz_2 + e$ and $cz_1 + dz_2 + f$ with $a, b, c, d \in \mathbb{R}$ and $e, f \in \mathbb{C}$. This leads to $(az_1 + bz_2 + e)^2 + \gamma$. Since z_2^2 does not appear in the normal form of case (1a.2), we get $b = 0$ and thus z_2 can not appear. Further $z_1^2 + \gamma_1$ and $z_1^2 + \gamma_2$ with $\gamma_1 \neq \gamma_2$ belong to different orbits since the previous argument enforces $a = 1, b = 0, e = 0$. The other cases are similar. Thus, for any of the eight normal forms, there are infinitely many orbits corresponding to each $\gamma \in \mathbb{C}$ (and $\alpha \in \mathbb{C}$ in some cases). □

6. Complex conics with non-hyperbolic initial form

We complete the proof of the Topological Classification Theorem 5.1 by treating the case where the complex conic $p \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, z_2]$ does not have a hyperbolic initial form. In particular, we see that, as previously stated in Corollary 5.2, if the initial form of p is not hyperbolic, then $\mathcal{I}(p)^{\mathsf{c}}$ is empty or consists of one bounded component whose interior is non-empty only if $\mathrm{in}(p)$ has two distinct non-real roots in $\mathbb{P}^1$.

The overall steps in computing the imaginary projection of the cases with non-hyperbolic initial form are as follows. After building up the real polynomial system for the classes (2b) and (2c.1) of Theorem 5.5 as in (3), we use the same techniques as in Subsection 4.1. However, in the case (2a), by the nature of the polynomial system, we directly argue that the imaginary projection is $\mathbb{R}^2$. In the last case (2c.2), we do not explicitly represent the components of $\mathcal{I}(p)^{\mathsf{c}}$. Instead, in Theorem 6.1 we prove that it does not contain any unbounded components and the number of bounded components does not exceed one.

6.1. **A double non-real root (2a).** We show that in this case we have a full space imaginary projection. First consider the normal form (2a.1). We have

$$\begin{aligned} p_{\mathrm{re}} &= x_1^2 - x_2^2 + 2y_2x_1 + 2y_1x_2 + \gamma_{\mathrm{re}}x_2 - y_1^2 + y_2^2 - \gamma_{\mathrm{im}}y_2 &= 0, \\ p_{\mathrm{im}} &= -2x_1x_2 + 2y_1x_1 - 2y_2x_2 + \gamma_{\mathrm{im}}x_2 + 2y_1y_2 + \gamma_{\mathrm{re}}y_2 &= 0. \end{aligned}$$

We prove $\mathcal{I}(p) = \mathbb{R}^2$ by showing that for every given $\mathbf{y} \in \mathbb{R}^2$, these two real conics in $\mathbf{x} = (x_1, x_2)$ have a real intersection point. For any fixed $\mathbf{y} \in \mathbb{R}^2$, the bivariate polynomial p_{re} in $\mathbf{x}$ has the quadratic part $x_1^2 - x_2^2$, and hence, the equation $p_{\mathrm{re}} = 0$ defines a real hyperbola in $\mathbf{x}$ with asymptotes $x_1 = x_2 + c_1$ and $x_1 = -x_2 + c_2$ for some constants $c_1, c_2 \in \mathbb{R}$; possibly the hyperbola degenerates to a union of these two lines. The degree two part of the polynomial p_{im} is given by $-2x_1x_2$ and hence, the equation $p_{\mathrm{im}} = 0$ defines a real hyperbola in $\mathbf{x}$ with asymptotes $x_1 = d_1$ and $x_2 = d_2$ for some constants $d_1, d_2 \in \mathbb{R}$; possibly the hyperbola may degenerate to a union of these two lines. Since the two hyperbolas have a real intersection point, the claim follows. The case (2a.2) is similar.

6.2. **One real and one non-real root (2b).** This case gives the system of equations

$$\begin{aligned} p_{\mathrm{re}} &= -\alpha_{\mathrm{re}}x_2^2 + x_1x_2 + 2\alpha_{\mathrm{im}}y_2x_2 + \alpha_{\mathrm{re}}y_2^2 - y_1y_2 + \gamma_{\mathrm{re}} &= 0, \\ p_{\mathrm{im}} &= -\alpha_{\mathrm{im}}x_2^2 + y_2x_1 + y_1x_2 - 2\alpha_{\mathrm{re}}y_2x_2 + \alpha_{\mathrm{im}}y_2^2 + \gamma_{\mathrm{im}} &= 0. \end{aligned}$$

First assume $y_2 \neq 0$. By solving the second equation for x_1, substituting the solution into the first equation and clearing the denominator, we get a univariate cubic polynomial in x_2 with non-zero leading coefficient. Since real cubic polynomials always have a real root, this shows that for $\mathbf{y} \in \mathbb{R}^2$ with $y_2 \neq 0$, there is a solution $\mathbf{x} \in \mathbb{R}^2$.

It remains to consider $y_2 = 0$. In this case, the second equation has a real solution in x_2 whenever the corresponding discriminant $y_1^2 + 4\alpha_{\mathrm{im}}\gamma_{\mathrm{im}}$ is non-negative, and if one of these solutions is non-zero, the first equation then gives a real solution for x_1. The special case that in the second equation both solutions for x_2 are zero, can only occur for $y_1 = 0$ and $\gamma_{\mathrm{im}} = 0$. Then the first equation has a real solution for x_1 if and only if $\gamma_{\mathrm{re}} = 0$. Altogether, we obtain

$$\mathcal{I}(p) = \begin{cases} \mathbb{R}^2 & \text{if } \gamma = 0 \ \text{ or } \ \alpha_{\mathrm{im}}\gamma_{\mathrm{im}} > 0, \\ \mathbb{R}^2 \setminus \{\mathbf{0}\} & \text{if } \gamma \in \mathbb{R} \setminus \{0\}, \\ \mathbb{R}^2 \setminus \{(y_1, 0) \ : \ y_1^2 < -4\alpha_{\mathrm{im}}\gamma_{\mathrm{im}}\} & \text{if } \alpha_{\mathrm{im}}\gamma_{\mathrm{im}} < 0. \end{cases} \tag{2b}$$

Note that when $\gamma \in \mathbb{R} \setminus \{0\}$ then $\mathcal{I}(p)$ is open but not $\mathbb{R}^2$. This answers Question 2.6. See Figure 4 (2b) for the imaginary projection of $p = z_2(z_1 - \mathrm{i}z_2) - \mathrm{i}$ from this class.

6.3. **Two distinct non-real roots (2c).** First we show that in (2c.1), i.e., where the roots of the initial form are complex conjugate, the imaginary projection is one open bounded component. After forming the polynomial system (3), the same methods as those in Subsection 4.1, i.e., taking the resultant of the two polynomials p_{re} and p_{im} with respect to x_2 and checking the discriminantal conditions to have a real x_1, lead to the imaginary projection

$$\mathcal{I}(p) = \Big\{\mathbf{y} \in \mathbb{R}^2 : y_1^2 + y_2^2 \geq \frac{1}{2}(\gamma_{\mathrm{re}} + \sqrt{\gamma_{\mathrm{re}}^2 + \gamma_{\mathrm{im}}^2})\Big\}. \tag{2c.1}$$

In particular, we have $\mathcal{I}(p) = \mathbb{R}^2$ if and only if $\gamma_{\text{im}} = 0$ and $\gamma_{\text{re}} \leq 0$. Hence, in the case of two non-real conjugate roots, $\mathcal{I}(p)^{\mathsf{c}}$ consists of either one or zero bounded component and it is a spectrahedral set.

The subsequent lemma shows that for the case (2c) in general $\mathcal{I}(p)^{\mathsf{c}}$ is either empty or consists of one bounded component.

Lemma 6.1. *Let $p = (z_1 - \alpha z_2)(z_1 - \beta z_2) + dz_1 + ez_2 + f$ with $\alpha, \beta \notin \mathbb{R}$ and $d, e, f \in \mathbb{C}$. Then*

(1) *$\mathcal{I}(p)^{\mathsf{c}}$ has at most one bounded component.*
(2) *$\mathcal{I}(p)^{\mathsf{c}}$ does not have unbounded components.*

Proof. (1) Assume that there are at least two bounded components in $\mathcal{I}(p)^{\mathsf{c}}$. By Lemma 2.3, we can assume without loss of generality that the y_1-axis intersects both components. Solving $p = 0$ for z_1 gives

$$z_1 = \frac{\alpha+\beta}{2} z_2 - \frac{d}{2} + \sqrt[\mathbb{C}]{\left(\frac{\alpha-\beta}{2}\right)^2 z_2^2 - ez_2 - f}\,. \tag{4}$$

By letting $z_2 \in \mathbb{R}$ we obtain two continuous branches $y_1^{(1)}(z_2)$ and $y_1^{(2)}(z_2)$ satisfying (4). Therefore, the set $\mathcal{I}(p) \cap \{\mathbf{y} \in \mathbb{R}^2 : y_2 = 0\}$ has at most two connected components. This is a contradiction to our assumption that the y_1-axis intersects the two bounded components in $\mathcal{I}(p)^{\mathsf{c}}$.

For (2), assume that there exists an unbounded component in the complement of $\mathcal{I}(p)$. The convexity implies that it must contain a ray. By Lemma 2.3, we can assume without loss of generality that the ray is the non-negative part of the y_1-axis. Similarly to the proof of (1), we set $y_2 = 0$ and check the imaginary projection on y_1-axis, using the two complex solutions in (4). Since $\alpha \neq \beta$, we have $D := \left(\frac{\alpha-\beta}{2}\right)^2 \neq 0$, where D is the discriminant of $\text{in}(p)$ with z_2 substituted to 1. We consider two cases: $D \notin \mathbb{R}_{>0}$ and $D \in \mathbb{R}_{>0}$. In both cases we get into a contradiction to the assumption that the unbounded component contains the non-negative part of the y_1-axis.

First assume $D \notin \mathbb{R}_{>0}$. For $z_2 \to \pm\infty$, the imaginary part of the radicand is dominated by the imaginary part of the square root of D. Since $D \notin \mathbb{R}_{>0}$ at least one of the two expressions

$$\left(\frac{\alpha+\beta}{2}\right)_{\text{im}} \pm \sqrt{\frac{-D_{\text{re}} + \sqrt{D_{\text{re}}^2 + D_{\text{im}}^2}}{2}}$$

is non-zero. Thus, letting $z_2 \mapsto \pm\infty$, implies $y_1 \mapsto +\infty$ in at least one of the branches.

Now assume $D \in \mathbb{R}_{>0}$. This implies $(\alpha - \beta)/2 \in \mathbb{R}$. Thus $(\alpha + \beta)/2 \notin \mathbb{R}$, since otherwise it contradicts with $\alpha, \beta \notin \mathbb{R}$. In this case, by letting z_2 grow to infinity, the dominating expression for y_1 is $\frac{1}{2}(\alpha + \beta)_{\text{im}} z_2$. Therefore, y_1 converges to $+\infty$ in one of the two branches. In both cases, for some $s > 0$, the ray $\{(y_1, 0) \in \mathbb{R}^2 : y_1 \geq s\}$ lies in the imaginary projection. This completes the proof. □

Before, in Example 3.2 we have shown that the defining polynomial of the imaginary projection can be irreducible of degree 8. The previous lemma enables us to show that $\mathcal{I}(q)^{\mathsf{c}}$ has exactly one bounded component. Note that $\mathbf{0} \in \mathcal{I}(q)^{\mathsf{c}}$. Let B_ϵ be an open ball with center at the origin and radius ϵ. By letting y_1 and y_2 converge to zero, the dominating part of $\triangle$ is $y_1^4 + y_2^2$. Thus, for sufficiently small ϵ, any non-zero point in B_ϵ

has $\triangle > 0$. Therefore, $\mathcal{I}(q)^{\mathsf{c}}$ contains an open ball around the origin. Now the claim follows from Theorems 6.1.

In this example, the imaginary projection is Euclidean closed, i.e., $\overline{\mathcal{I}(q)} = \mathcal{I}(q)$, however, its boundary is not Zariski closed. We claim that the set $\mathcal{I}(q)^{\mathsf{c}}$ is not a spectrahedron. By the characterization of Helton and Vinnikov [16], it suffices to show that $\overline{\mathcal{I}(q)}$ is not rigidly convex. That is, if h is a defining polynomial of minimal degree for the component $\mathcal{I}(q)^{\mathsf{c}}$, then we have to show that a generic line ℓ through the interior of $\mathcal{I}(q)^{\mathsf{c}}$ does not meet the variety $V := \{\mathbf{x} \in \mathbb{R}^2 : h(\mathbf{x}) = 0\}$ in exactly $\deg(h)$ many real points, counting multiplicities. However, this can be checked immediately. For example, the line $y_1 = 1/3$ intersects the variety V in exactly two real points, and any sufficiently small perturbation of the line preserves the number of real intersection points. See Figure 4 (2c).

This completes the proof of Theorem 5.1. We now prove Corollary 5.3 by showing that 8 is an upper bound.

Proof of Corollary 5.3. For the first four classes we have precisely computed the boundaries $\partial\mathcal{I}(p)$ and they are algebraic with irreducible components of degree at most two. It remains to consider the case (2c), more precisely (2c.2), where $p = (z_1 - \mathrm{i}z_2)(z_1 - \alpha z_2) + \gamma$ for some $\alpha, \gamma \in \mathbb{C}$, $\alpha \notin \mathbb{R}$, and $\alpha \neq \pm\mathrm{i}$. Using Remark 3.1, we show that the degrees of the irreducible components in the Zariski closure of $\partial\mathcal{I}(p)$ do not exceed 8. This, together with Example 3.2, completes the proof of (1). We separate the real and the imaginary parts as before.

$$p_{\mathrm{re}} = x_1^2 + ((\alpha_{\mathrm{im}}+1)y_2) - \alpha_{\mathrm{re}}x_2)x_1 - \alpha_{\mathrm{im}}x_2^2 + ((\alpha_{\mathrm{im}}+1)y_1 - 2\alpha_{\mathrm{re}}y_2)x_2 + \alpha_{\mathrm{re}}y_2y_1 + \alpha_{\mathrm{im}}y_2^2 - y_1^2 + \gamma_{\mathrm{re}} = 0,$$

$$p_{\mathrm{im}} = ((\alpha_{\mathrm{im}}+1)x_2 + \alpha_{\mathrm{re}}y_2 - 2y_1)x_1 - \alpha_{\mathrm{re}}x_2^2 + (\alpha_{\mathrm{re}}y_1 + 2\alpha_{\mathrm{im}}y_2)x_2 + \alpha_{\mathrm{re}}y_2^2 - (\alpha_{\mathrm{im}}+1)y_1y_2 - \gamma_{\mathrm{im}} = 0.$$

First we assume $(\alpha_{\mathrm{im}} + 1)x_2 + \alpha_{\mathrm{re}}y_2 - 2y_1 \neq 0$. Solving $p_{\mathrm{im}} = 0$ for x_1 and substituting in $p_{\mathrm{re}} = 0$ returns

$$\Big(\alpha_{\mathrm{im}}(\alpha_{\mathrm{re}}^2 + (\alpha_{\mathrm{im}}+1)^2)\Big)x_2^4 - \Big((\alpha_1^2+\alpha_2^2+6\alpha_2+1)(-\alpha_1y_2 + y_1(\alpha_2+1))\Big)x_2^3 + \Big((\alpha_1^2+5\alpha_2^2+14\alpha_2+5)y_1^2$$
$$-y_1\alpha_1(\alpha_1^2+\alpha_2^2+14\alpha_2+9)y_2 + (4\alpha_1^2 + \alpha_2(\alpha_1^2+(\alpha_2-1)^2))y_2^2 + (k_2\alpha_1 - 2k_1 - k_1\alpha_2)\alpha_2 - k_2\alpha_1 - k_1\Big)x_2^2$$
$$+\Big(8(-\alpha_2-1)y_1^3 + 8\alpha_1(\alpha_2+2)y_1^2y_2 - (\alpha_2(\alpha_1^2+\alpha_2^2-\alpha_2-1) + 9\alpha_1^2+1)y_1y_2^2 + \alpha_1(\alpha_1^2+(\alpha_2-1)^2)y_2^3$$
$$+4k_1(\alpha_2+1)y_1 + ((\alpha_1^2-(\alpha_2-1)^2)k_2 - 2k_1\alpha_1(\alpha_2+1))y_2\Big)x_2 + 4y_1^4 - 8\alpha_1y_1^3y_2 + (5\alpha_1^2+(\alpha_2-1)^2)y_1^2y_2^2$$
$$-\alpha_1(\alpha_1^2 + (\alpha_2-1)^2)y_1y_2^3 - 4k_1y_1^2 + 4\alpha_1k_1y_1y_2 - \alpha_1(k_1\alpha_1 + \alpha_2k_2 - k_2)y_2^2 - k_2^2.$$

Since $\alpha \notin \mathbb{R}$, the leading coefficient is non-zero. Therefore, we have a quartic univariate polynomial in x_2. The relevant polynomials for the decision of whether this polynomial has a real root for x_2 are P, D and the discriminant Disc from Remark 3.1. By computing these polynomials, we observe that Disc decomposes as $Q_1^2 \cdot q$, where Q_1 is a quadratic polynomial and q is of degree 8 in $\mathbf{y}$. The total degrees of P and D are 2 and 4, respectively.

Now let us assume $(\alpha_{\mathrm{im}} + 1)x_2 + \alpha_{\mathrm{re}}y_2 - 2y_1 = 0$. If $\alpha_{\mathrm{im}} \neq -1$, then substituting $x_2 = \frac{-\alpha_{\mathrm{re}}y_2 + 2y_1}{\alpha_{\mathrm{im}}+1}$ into $p_{\mathrm{im}} = 0$ is the quadratic Q_1. Otherwise, the substitution $\alpha_{\mathrm{im}} = -1$

and $y_1 = \frac{\alpha_{\mathrm{re}} y_2}{2}$ in p_{re} and p_{im}, and setting $s = 2p_{\mathrm{im}} - \alpha_{\mathrm{re}} p_{\mathrm{re}}$ simplifies the original system to

$$\begin{aligned} p_{\mathrm{re}} &= \alpha_{\mathrm{re}}^2 y_2^2 - 4\alpha_{\mathrm{re}} x_1 x_2 - 8\alpha_{\mathrm{re}} x_2 y_2 + 4x_1^2 + 4x_2^2 - 4y_2^2 + 4\gamma_{\mathrm{re}} &&= 0, \\ s &= 2(2\alpha_{\mathrm{re}}^2 x_1 + 3\alpha_{\mathrm{re}}^2 y_2 + 4y_2)x_2 - (\alpha_{\mathrm{re}}^3 y_2^2 + 4\alpha_{\mathrm{re}} x_1^2 + 4\gamma_{\mathrm{re}}\alpha_1 - 4\gamma_{\mathrm{im}}) &&= 0. \end{aligned}$$

If the coefficient of x_2 in s is non-zero, then solving $s = 0$ for x_2 and substituting in $p_{\mathrm{re}} = 0$ results in a quartic polynomial in x_1 with non-zero leading coefficient. In this case, the polynomials Disc, P, and D from Remark 3.1 are all univariate in y_2. The decomposition of the discriminant in this case consists of the polynomial q after the substitution $y_1 = \frac{\alpha_{\mathrm{re}} y_2}{2}$ and the square of a quadratic polynomial Q_2. The total degrees of P and D are 2 and 4, respectively.

Otherwise, solving $2\alpha_{\mathrm{re}}^2 x_1 + 3\alpha_{\mathrm{re}}^2 y_2 + 4y_2 = 0$ for x_1 and substituting in $s = 0$, results in Q_2. In all the cases that we have discussed above, the degree of none of the irreducible factors appearing in the polynomials that could possibly form the $\partial\mathcal{I}(p)$ exceeds 8. Example 3.2 shows an example where this bound is reached. This completes the proof of (1). (2) follows from Theorem 2.4. □

We have precisely verified the imaginary projections for all the normal forms in Theorem 5.5 except for (2c.2) . In particular, we have shown that if p is not of the class (2c.2), then $\mathcal{I}(p) = \mathbb{R}^2$ if and only if there exist some $\gamma, \alpha \in \mathbb{C}$, and $\alpha \notin \mathbb{R}$ such that p can be transformed to one of the following normal forms.

$$\begin{cases} (2a) : (z_1 - \mathrm{i}z_2)^2 + \gamma z_2 \quad \text{or} \quad (z_1 - \mathrm{i}z_2)^2 + \gamma & \\ (2b) : z_2(z_1 - \alpha z_2) + \gamma & \text{for} \quad \gamma = 0 \ \text{ or } \ \alpha_{\mathrm{im}}\gamma_{\mathrm{im}} < 0, \\ (2c.1) : z_1^2 + z_2^2 + \gamma & \text{for} \quad \gamma_{\mathrm{im}} = 0 \ \text{ and } \ \gamma_{\mathrm{re}} \leq 0. \end{cases} \tag{5}$$

An example for a complex conic of class (2c.2) where the imaginary projection is $\mathbb{R}^2$ is $p = z_1^2 - 3\mathrm{i}z_1z_2 - 2z_2^2$. The reason is that for any given $(y_1, y_2) \in \mathbb{R}^2$, the polynomial p vanishes on the point $(-y_2 + \mathrm{i}y_1, y_1 + \mathrm{i}y_2)$. Answering the following question completes the verification of complex conics with a full-space imaginary projection.

Question 6.2. Let $p \in \mathbb{C}[z_1, z_2]$ be a complex conic of the form $p = (z_1 - \mathrm{i}z_2)(z_1 - \alpha z_2) + \gamma$ such that $\alpha \notin \mathbb{R}$ and $\alpha \neq \pm\mathrm{i}$. Under which conditions on the coefficients $\gamma, \alpha \in \mathbb{C}$ does $\mathcal{I}(p)$ coincide with $\mathbb{R}^2$?

7. CONVEXITY RESULTS

For the case of complex plane conics, we have shown in Theorem 6.1 that there can be at most one bounded component in the complement of its imaginary projection. An example of such a conic is $z_1^2 + z_2^2 + 1 = 0$, where the unique bounded component is the unit disc, which in particular is strictly convex. In the following theorem, we show that for any $k > 0$, there exists a complex plane curve whose complement of the imaginary projection has exactly k strictly convex bounded components. For the case of real coefficients, only the lower bound of k and no exactness result is known (see [19, Theorem 1.3]).

Allowing non-real coefficients lets us break the symmetry of the imaginary projection with respect to the origin and this enables us to fix the number of components exactly instead of giving a lower bound. Furthermore, using a non-real conic which has four

strictly convex unbounded components, illustrated in Figure 3, notably drops the degree of the corresponding polynomial.

Theorem 7.1. *For any $k > 0$ there exists a polynomial $p \in \mathbb{C}[z_1, z_2]$ of degree $2\lceil \frac{k}{4} \rceil + 2$ such that $\mathcal{I}(p)^{\mathsf{c}}$ consists of exactly k strictly convex bounded components.*

Proof. Let R^{φ} be the rotation map and $g : \mathbb{C}^2 \to \mathbb{C}^2$ be defined as

$$g(z_1, z_2) = z_1 z_2 + 2\mathrm{i}.$$

Note that the equation

$$\prod_{j=0}^{m-1} (g \circ R^{\pi j/2m})(z_1, z_2) = 0 \tag{6}$$

where $m = \lceil \frac{k}{4} \rceil$ as before, has $4m$ unbounded components in the complement of its imaginary projection. We need to find a circle that intersects with k of them and does not intersect with the rest $4m - k$ components. By symmetry of the construction of the equation above, the smallest distance between the origin O and each component is the same for all the components. The following picture shows the case $m = 2$.

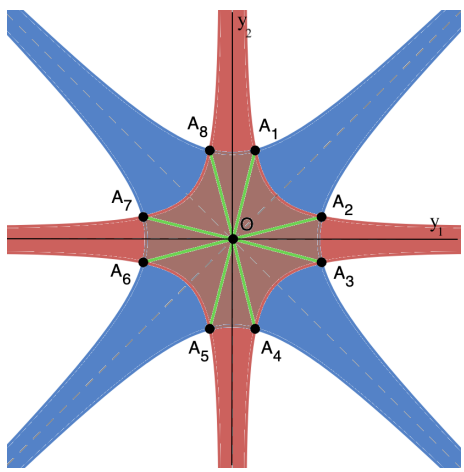


FIGURE 5. The imaginary projection of (6) for $m = 2$ is the union of the imaginary projections for polynomials corresponding to $j = 0$ and $j = 1$.

Let C be the boundary of the imaginary projection of $z_1^2 + z_2^2 + r^2$ where $r = |OA_1|$. The center of C is the origin and it passes through all $4m$ points $A_1, \ldots, A_{4m}$ that minimize the distance from the origin to each component. A sufficiently small perturbation of the center and the diameter can result in a circle C' with center (a, b) and radius s that only intersects the interiors of the first k unbounded components. Now define

$$q := (z_1 - \mathrm{i}a)^2 + (z_2 - \mathrm{i}b)^2 + s^2.$$

By Lemma 2.3 and the fact that the imaginary projection of the multiplication of two polynomials is the union of their imaginary projections, the polynomial

$$p := q \cdot \prod_{j=0}^{m-1} (g \circ R^{\pi j/2m})(z_1, z_2),$$

has exactly k strictly convex bounded components in $\mathcal{I}(p)^{\mathsf{c}}$. $\square$

Although, by generalizing from real to complex coefficients, we improved the degree of the desired polynomial from $d = 4\lceil \frac{k}{4} \rceil + 2$ to $d/2 + 1$, it is not the optimal degree. For instance if $k = 1$, the polynomial $z_1^2 + z_2^2 + 1$ has the desired imaginary projection, while the degree is $2 < 4$. Thus, we can ask the following question.

Question 7.2. For $k > 0$, what is the smallest integer $d > 0$ for which there exists a polynomial $p \in \mathbb{C}[z_1, z_2]$ of degree d such that $\mathcal{I}(p)^{\mathsf{c}}$ consists of exactly k strictly convex bounded components.

8. Conclusion and open questions

We have classified the imaginary projections of complex conics and revealed some phenomena for polynomials with complex coefficients in higher degrees and dimensions. It seems widely open to come up with a classification of the imaginary projections of bivariate cubic polynomials, even in the case of real coefficients. In particular, the maximum number of components in the complement of the imaginary projection for both complex and real polynomials of degree d where $d \geq 3$ is currently unknown. We have shown that in degree two they coincide for real and complex conics, however, this may not be the case for cubic polynomials.

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