RADII OF SIMPLICES AND SOME APPLICATIONS TO GEOMETRIC INEQUALITIES

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Abstract. We review some recent results on inner and outer j-radii of simplices and general convex bodies. In particular, we discuss two lines of research whose present-day developments were strongly influenced by work of Bernulf Weißbach: radii of regular simplices and geometric inequalities among the radii.

1. Introduction

Let $\mathcal{L}_{j,n}$ be the set of all j-dimensional linear subspaces (hereafter *j-spaces*) in n-dimensional Euclidean space $\mathbb{E}^n$. The *inner j-radius* $r_j(C)$ of a convex body $C \subset \mathbb{E}^n$ is the radius of a largest *j-ball* (j-dimensional ball) contained in C, and the *outer j-radius* $R_j(C)$ is the radius of the smallest enclosing *j-ball* in an optimal orthogonal projection of C onto a j-space $J \in \mathcal{L}_{j,n}$, where the optimization is performed over $\mathcal{L}_{j,n}$.

Studying radii of polytopes is a fundamental topic in convex geometry (see [1, 2, 8, 10, 11, 14]. Applications in functional analysis, statistics, computer vision, robotics, and medical diagnosis (see [12] and the references therein) have initiated additional interest from the computational point of view. In the investigation of the inner and outer radii, the following two questions immediately arise.

a) What are the radii of special classes of convex bodies, such as (regular) simplices?
b) Is there a general order on the radii?

For the outer $(n-1)$-radius and its relation to the inner 1-radius (half diameter) these questions were studied by Weißbach [17, 18]. Section 3 contains an overview on Weißbach's results on the radii of regular simplices and extensions which were obtained recently.

Concerning relations among the radii, it is easy to see that for any convex body $C \subset \mathbb{E}^n$ we have $r_n(C) \leq \cdots \leq r_1(C)$, $R_1(C) \leq \cdots \leq R_n(C)$, and $R_1(C) \leq r_1(C)$. Moreover, in [13] it was shown $r_j(C) < R_{n+1-j}(C)$. The stated geometric inequalities are also displayed in Figure 1.

It is not difficult to see that for important classes of bodies (e.g., symmetric bodies) $R_{n-1}(C) \leq r_1(C)$ holds. Eggleston [9] first showed that for every $n \geq 3$ there are also bodies with $R_{n-1}(C) > r_1(C)$. A much simpler proof was provided by Weißbach [17].

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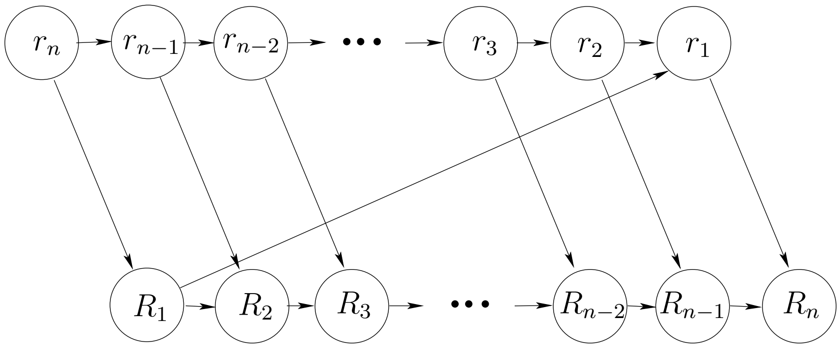

FIGURE 1. An arc between two radii represents a less than or equal relationship (from the origin to the sink), which holds for all n-dimensional bodies.

In Section 4 we will see that Figure 1 is complete in the sense that for any two radii which are not connected by a directed path there exist bodies C_1, C_2 such that the relationship between the two radii is 'less than' for C_1 and 'greater than' for C_2.

2. Preliminaries

Throughout the paper we work in Euclidean space $\mathbb{E}^n$, i.e., $\mathbb{R}^n$ with the usual scalar product $x \cdot y = \sum_{i=1}^n x_i y_i$ and norm $\|x\| = (x \cdot x)^{1/2}$. $\mathbb{B}^n$ and $\mathbb{S}^{n-1}$ denote the (closed) unit ball and unit sphere, respectively. For a set $A \subset \mathbb{E}^n$, the linear hull of A is denoted by $\operatorname{lin} A$ and the convex hull of A is denoted by $\operatorname{conv} A$.

A set $C \subset \mathbb{E}^n$ is called a *body* if it is bounded, closed, convex and contains an interior point. Let $1 \leq j \leq n$. A *j-flat* is an affine subspace of dimension j, and a *j-cylinder* is a set of the form $J + \rho\mathbb{B}^n$ with an $(n-j)$-flat J and $\rho > 0$. For a body $C \subset \mathbb{E}^n$, the outer j-radius $R_j(C)$ of C (as defined in the introduction) is the radius ρ of a smallest enclosing j-cylinder of C. It follows from a standard compactness argument that this minimal radius is attained (see, e.g., [11]). Let $1 \leq j \leq k < n$. If $C' \subset \mathbb{E}^n$ is a compact, convex set whose affine hull F is a k-flat then $R_j(C')$ denotes the radius of a smallest enclosing j-cylinder $\mathcal{C}'$ relative to F, i.e., $\mathcal{C}' = J' + R_j(C')(\mathbb{B}^n \cap E)$ with a $(k-j)$-flat $J' \subset F$ and E the linear k-space parallel to F.

For a simplex $S = \operatorname{conv}\{v^{(1)}, \ldots, v^{(n+1)}\}$, let $S^{(i)}$ denote the facet of S which does not contain the vertex $v^{(i)}$, $i = 1, \ldots, n+1$. Whenever a statement is invariant under orthogonal transformations and translations we denote by T^n *the* regular simplex in $\mathbb{E}^n$ with edge length $\sqrt{2}$. The reason for the choice of $\sqrt{2}$ stems from the convenient embedding of T^n into $\mathbb{E}^{n+1}$. Let $\mathcal{H}^n_\alpha = \{x \in \mathbb{E}^{n+1} : \sum_{i=1}^{n+1} x_i = \alpha\}$. Then the *standard embedding* $\mathbf{T}^n$ of T^n is defined by

$$\mathbf{T}^n := \operatorname{conv}\left\{e^{(i)} \in \mathbb{E}^{n+1} : 1 \leq i \leq n+1\right\} \subset \mathcal{H}^n_1,$$

where $e^{(i)}$ denotes the i-th unit vector in $\mathbb{E}^{n+1}$. By $\mathcal{S}^{n-1} := \mathbb{S}^n \cap \mathcal{H}^n_0$ we denote the set of unit vectors parallel to the n-flat in which $\mathbf{T}^n$ is embedded.

A j-cylinder $\mathcal{C}$ containing some simplex S is called a *circumscribing* cylinder of S if all the vertices of S are contained in the boundary of $\mathcal{C}$.

3. Outer radii of regular simplices

One access to understand the geometry of functionals such as radii and volumes is to investigate their behavior on special subclasses of convex bodies. Here we mainly consider the class of regular simplices. In particular, for the analysis of geometric inequalities, they often attain the extreme values. In [17, Theorem 1] Weißbach showed that

$$R_{n-1}(T^n) \geq \sqrt{\frac{n-1}{n+1}}\ , \text{ with equality if and only if } n \text{ is odd.} \tag{3.1}$$

Instead of presenting a proof of (3.1) we will state several generalizations, which were obtained in a bit different form in [5].

Lemma 1. *For $1 \leq j \leq n$ it holds $R_j(T^n) \geq \sqrt{\frac{j}{n+1}}$, and in case of equality every minimal enclosing j-cylinder of T^n is a circumscribing j-cylinder of T^n.*

Proof. Let $s^{(1)}, \ldots, s^{(n)}$ be an orthonormal basis of $\mathcal{H}_0^n$, $p \in \mathcal{H}_1^n$ and $\rho > 0$ such that $\mathcal{C} = J + \rho(\mathbb{B}^{n+1} \cap \mathcal{H}_0^n)$ is an enclosing j-cylinder of $\mathbf{T}^n$ with $J = p + \mathrm{lin}\{s^{(j+1)}, \ldots, s^{(n)}\}$. Further let $x \mapsto Px$ be the orthogonal projection onto $\mathrm{lin}\{s^{(1)}, \ldots, s^{(j)}\}$, where $P = \sum_{k=1}^{j} s^{(k)}(s^{(k)})^T \in \mathbb{E}^{(n+1)\times(n+1)}$. Then

$$\|Pe^{(i)}\|^2 = \sum_{k=1}^{j} (s_i^{(k)})^2\ . \tag{3.2}$$

Assume there exists a point $x \in \mathcal{H}_0^n$ such that $\|x - Pe^{(i)}\| < \sqrt{\frac{j}{n+1}}$ for all $i = 1, \ldots, n+1$. Since $\sum_{i=1}^{n+1} s_i^{(k)} = 0$ and $\sum_{i=1}^{n+1} (s_i^{(k)})^2 = 1$, we obtain from summing over all i

$$\begin{aligned} j > \sum_{i=1}^{n+1} \|x - Pe^{(i)}\|^2 &= (n+1)\|x\|^2 - 2\sum_{i=1}^{n+1}\sum_{k=1}^{j} s_i^{(k)} x^T s^{(k)} + \sum_{i=1}^{n+1}\sum_{k=1}^{j} (s_i^{(k)})^2 \\ &= (n+1)\|x\|^2 + j \geq j \end{aligned} \tag{3.3}$$

which is a contradiction. This proves the first part of the theorem. For the second part, it is easy to see that if $R_j(T^n) = \sqrt{\frac{j}{n+1}}$ then (3.3) becomes an equality chain if and only if $x = 0$ and $\|x - Pe^{(i)}\|^2 = \frac{j}{n+1}$ for all $1 \leq i \leq n+1$. □

Also about the equality cases we can generalize Weißbach's result (3.1) for general j. If a sequence of orthogonal vectors $s^{(1)}, \ldots, s^{(j)} \in \mathcal{S}^{n-1}$ satisfies $\sum_{k=1}^{j} (s_i^{(k)})^2 = \frac{j}{n+1}$ for all $i = 1, \ldots, n+1$, we call it a *good subspace basis* (shortly, *gsb*). The proof of Lemma 1 implies that $R_j(T^n) = \sqrt{\frac{j}{n+1}}$ if and only if there exists a gsb for the pair (j, n). It is not hard to see that

$$\text{any orthonormal basis } s^{(1)}, \ldots, s^{(n)} \text{ of } \mathcal{H}_0^n \text{ is a gsb.} \tag{3.4}$$

It directly follows $R_n(T^n) = \sqrt{\frac{n}{n+1}}$, which is a long known fact, not only since the famous work of Jung [15].

As an easy consequence of (3.4), the basis completion theorem implies

$$R_j(T^n) = \sqrt{\frac{j}{n+1}} \text{ if and only if } R_{n-j}(T^n) = \sqrt{\frac{n-j}{n+1}}\,. \tag{3.5}$$

Property (3.5) immediately shows that Weißbach's 'if and only if n is odd' statement on equality in (3.1) corresponds directly to the property that the half width satisfies $R_1(T^n) = \sqrt{\frac{1}{n+1}}$ if and only if n is odd, also an old result, shown by Steinhagen [16]. In fact, Steinhagen first showed that all minimal enclosing 1-cylinders of a regular simplex are circumscribing. It was shown in [11] that minimal enclosing 1-cylinders are always circumscribing even for general simplices, and a similar statement for general j will be given in Proposition 5.

Weißbach showed that the axis direction $s^{(n)} \in \mathcal{S}^{n-1}$ of a minimal enclosing $(n-1)$-cylinder of $\mathbf{T}^n$ for odd n must be of the form $\sqrt{\frac{1}{n+1}}(1,\ldots,1,-1,\ldots,-1)^T$ where both the number of 1's and (-1)'s are $\frac{n+1}{2}$. However, for the analysis in which cases the lower bound $\sqrt{\frac{j}{n+1}}$ holds for general j, it seems to be more convenient to describe the space where $\mathbf{T}^n$ is projected on.

Definition 2. *Let $1 \le j < m$. A sequence $v^{(1)},\ldots,v^{(m)} \in \mathbb{S}^{j-1}$ is called* (j,m)-isotropic *if $\sum_{i=1}^m v^{(i)} = 0$ and $\sum_{i=1}^m v^{(i)}(v^{(i)})^T = mI$, where I denotes the j-dimensional identity matrix. A polytope $P \subset \mathbb{B}^j$ is called* (j,m)-isotropic *if there exists a (j,m)-isotropic sequence $v^{(1)},\ldots,v^{(m)}$ whose convex hull is P.*

The following Proposition is taken from [5]. The proof is quite easy but purely technical.

Proposition 3. *Let $1 \le j \le n$. There exists a gsb $s^{(1)},\ldots,s^{(j)} \in \mathcal{S}^{n-1}$ if and only if there exists a $(j,n+1)$-isotropic polytope $P \subset \mathbb{B}^j$. Moreover, for any gsb $s^{(1)},\ldots,s^{(j)} \in \mathcal{S}^{n-1}$ we can choose P such that it is the projection of $\mathbf{T}^n$ on $\operatorname{lin}\{s^{(1)},\ldots,s^{(j)}\}$ up to a linear transformation.*

Obviously, for every odd $n \ge 1$ the unique $(1,n+1)$-isotropic polytope is $[-1,1]$, and the underlying sequence consists of 1's and -1's, both $\frac{n+1}{2}$ times.

Also it is easy to see that every regular j-dimensional polytope with $n+1$ vertices, all on $\mathbb{S}^{n-1}$, is $(j,n+1)$-isotropic. Hence, at least for $j=2$ there exist $(2,n+1)$-isotropic polytopes for all $n \ge 2$. Moreover, for odd n we can choose a 3-dimensional prism with a regular $(\frac{n+1}{2})$-gon as the base. By appropriately choosing the height of the prism, it becomes $(3,n+1)$-isotropic. Finally, if $v^{(1)},\ldots,v^{(m_1)}$ is a (j,m_1)-isotropic sequence and $u^{(1)},\ldots,u^{(m_2)}$ a (j,m_2)-isotropic sequence, then $v^{(1)},\ldots,v^{(m_1)},u^{(1)},\ldots,u^{(m_2)}$ is a (j,m_1+m_2)-isotropic sequence. We call this the *additive rule*. Now we are ready to state the following theorem about the radii of regular simplices.

Theorem 4. *Let $1 \le j \le n$. If*
a) n is odd, or
b) j is even and $n \ne 2j$,
then $R_j(T^n) = \sqrt{\frac{j}{n+1}}$.

Proof. We will show by an inductive proof over j and n that whenever a) or b) holds there exists a $(j, n+1)$-isotropic polytope. It was shown before that a) or b) is true if $1 \leq j \leq 3$ or $n-2 \leq j \leq n$. So we can assume that $n \geq 7$, $4 \leq j \leq n-3$, and use the induction hypothesis that the theorem is true for all pairs (j', n') such that $j' < j$, $n' \leq n$ or $j' \leq j$, $n' < n$ which fulfill a) or b). In the inductive step we distinguish three cases: $n < 2j$, $n > 2j$, and $n = 2j$.

Suppose $n < 2j$. Then $j' := n - j < j$ and therefore the statement follows inductively by applying Property (3.5), as for odd n the precondition does not depend on j and for even n it holds that j' is even if j is even and $2j' = n$ would mean $2j = n$.

If $n > 2j$ we consider the two subcases $n-j \neq 2j$ and $n-j = 2j$. Let $n-j \neq 2j$. Define $n' = n-j-1$ and $n'' = j$. It is easy to see that each of the pairs (j, n') and (j, n'') fulfill a) or b). Hence, by the induction hypothesis there exists a $(j, n'+1)$-isotropic polytope and a $(j, n''+1)$-isotropic polytope, from which we obtain the existence of a $(j, n+1)$-isotropic polytope via the additive rule.

In the case that $n - j = 2j$ we set $n' = n - j - 3$ and $n'' = j + 2$. By the induction hypothesis the theorem is true for the pairs (j, n') and (j, n''). Applying the additive rule on a $(j, n'+1)$-isotropic polytope and a $(j, n''+1)$-isotropic polytope gives a $(j, n+1)$-isotropic polytope.

Finally, in the case $n = 2j$, n is even and therefore neither a) nor b) is satisfied. □

In [5] also a multiplicative rule to combine two isotropic polytopes was given, which allows to show that there exist $(j, n+1)$-isotropic polytopes even in some cases with even j and $n = 2j$. However, Steinhagen's and Weißbach's result for the cases $j = 1$ and $j = n - 1$ show that $(j, n+1)$-isotropic polytopes do not exist for all pairs (j, n) with $1 \leq j \leq n$.

There also exists a paper on the determination of $R_{n-1}(T^n)$ for even n, but unfortunately, that proof contained an error. At the end of 2002, Bernulf Weißbach suggested to us to work jointly on a new proof, but unfortunately he died in June 2003. In a letter, he stressed his opinion that a major step would be to show that every minimal enclosing $(n-1)$-cylinder of the regular simplex is always circumscribing. Recently, in [6] we were able to show the following statement:

Proposition 5. *Let $1 \leq j \leq n$ and S be a simplex in $\mathbb{E}^n$ with facets $S^{(1)}, \ldots, S^{(n+1)}$. Then one of the following is true.*

a) Every minimal enclosing j-cylinder of S is a circumscribing j-cylinder of S.
b) $R_j(S) = R_{j-1}(S^{(i)})$ for some $i \in \{1, \ldots, n+1\}$ and $j \geq 2$.
c) $R_j(S) = R_k(F)$, for some $k \in \{1, \ldots, j-1\}$, where F is a k-face of S.
If $j = 1$ or if $S = T^n$ then always case a) holds.

Rather than presenting the whole proof of Proposition 5 we will concentrate on the case $j = n - 1$ and $S = T^n$, as major ideas are already contained in this case.

Lemma 6. *The sequence $(R_{n-1}(T^n))_{n \geq 2}$ is strictly increasing.*

Proof. Let T^{n+1} be embedded in $\mathbb{E}^{n+1}$ such that $S^{(n+1)}$ (which is T^n) lies within $H := \{x \in \mathbb{E}^{n+1} : x_{n+1} = 0\}$, and let $\mathcal{C} = \ell + R_{n-1}(T^n)(\mathbb{B}^{n+1} \cap H)$ be any minimal enclosing cylinder of $S^{(n+1)}$ (taken in n-dimensional space) with ℓ a line in H. But, since

$$\mathrm{dist}(v^{(n+1)}, \ell) \geq \mathrm{dist}(v^{(n+1)}, H) = \sqrt{\frac{n+1}{n}} > 1 > R_n(T^{n+1})$$

(where $\mathrm{dist}(\cdot,\cdot)$ denotes the Euclidean distance), ℓ cannot be the axis of a minimal enclosing cylinder of T^{n+1}. Hence $R_{n-1}(T^n) < R_n(T^{n+1})$. $\square$

Theorem 7. *Let $n \geq 2$ and $\ell \subset \mathbb{E}^n$ be a line, such that $\mathcal{C} = \ell + R_{n-1}(T^n)\mathbb{B}^n$ is a minimal enclosing cylinder of T^n. Then $\mathcal{C}$ is also a circumscribing cylinder of T^n.*

Proof. The proof is split into three parts. In the first part we will exclude the special cases that ℓ is parallel or perpendicular to one of the facets $S^{(1)}, \ldots, S^{(n+1)}$ of T^n. The other two parts deal with the previously excluded cases. We can assume that T^n is embedded in $\mathbb{E}^n$ such that $S^{(n+1)} \subset H := \{x \in \mathbb{E}^n : x_n = 0\}$.

Part 1: Suppose ℓ is neither perpendicular nor parallel to any of the facets of T^n. Now assume $v^{(n+1)} \notin \mathrm{bd}(\mathcal{C})$. Let $p, s \in \mathbb{E}^n$ such that $\ell = p + \mathrm{lin}\{s\}$. Since, by assumption, ℓ is not parallel to H, we can assume $p = 0 \in \ell \cap H$, and $s_n > 0$. For every $s'_n \in (0, s_n)$ and $s' := (s_1, \ldots, s_{n-1}, s'_n) \in \mathbb{E}^n$ let $\ell' = p + \mathrm{lin}\{s'\}$. Geometrically, ℓ' results from ℓ by rotating ℓ into the direction of the hyperplane H in such a way that the orthogonal projection of ℓ onto H remains invariant (see Figure 2). Since ℓ and H are not perpendicular we obtain

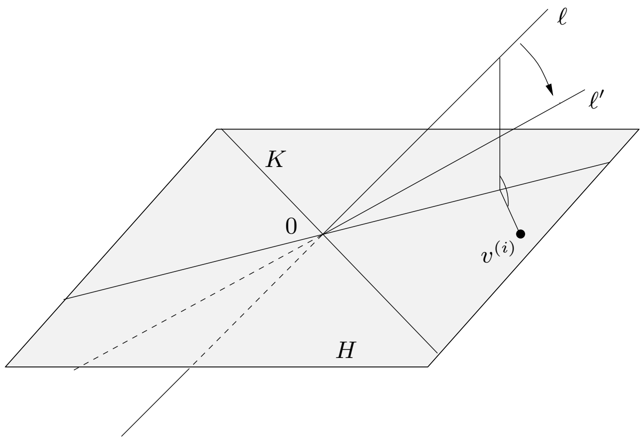

FIGURE 2. For $n = 3$ the figure shows how the underlying line ℓ of the cylinder $\mathcal{C}$ is rotated towards its orthogonal projection onto the plane H. The distances between the vertices $v^{(i)}$, $1 \leq i \leq n$, and the j-cylinder axis are not increased, and decreased if $v^{(i)} \notin K$.

$\ell \neq \ell'$. Further, since $v^{(1)}, \ldots, v^{(n)} \in H$, we have

$$\mathrm{dist}(v^{(i)}, \ell') \leq \mathrm{dist}(v^{(i)}, \ell)\,, \quad 1 \leq i \leq n\,, \tag{3.6}$$

where "$<$" holds whenever $v^{(i)} \notin K := \ell^{\perp} \cap H$. Obviously, $\dim(K) = n - 2$. If none of the $v^{(i)}$ lies in K, then, by choosing s'_n sufficiently close to s_n, all vertices of T^n lie in the interior of $\mathcal{C}' = \ell' + R_j(T^n)\mathbb{B}^n$, a contradiction to the minimality of $R_j(T^n)$. Hence, there must be some vertex of S in $K \cap \mathrm{bd}(\mathcal{C})$.

Let $k+1$ be the number of vertices in $K \cap \mathrm{bd}(\mathcal{C})$. By renumbering the vertices we can assume $v^{(1)}, \dots, v^{(k+1)} \in K \cap \mathrm{bd}(\mathcal{C})$, $0 \le k \le n-2$. Then $F := \mathrm{conv}\{v^{(1)}, \dots, v^{(k+1)}\}$ is a k-face of T^n. Since F and ℓ are perpendicular, F is congruent to T^k and $p = 0$ is the unique center of the circumball of F. Hence,

$$R_{n-1}(T^n) = R_k(F) = R_k(T^k) = \sqrt{\frac{k}{k+1}} = \sqrt{\frac{2k}{2k+2}} \, .$$

Now Theorem 4 and Lemma 6 imply that $n = 2k+1$. But in this case it follows already from Lemma 1 that $\mathcal{C}$ circumscribes T^n, contradicting $v^{(n+1)} \notin \mathrm{bd}(\mathcal{C})$.

Part 2: Now consider the case that ℓ is perpendicular to H and assume again $v^{(n+1)} \notin \mathrm{bd}(\mathcal{C})$. In this case any small perturbation of ℓ around $p := \ell \cap H$ keeps $v^{(n+1)}$ within the cylinder not increasing the distances of all the other vertices to the new axis. So the same argumentation as in the non-perpendicular case shows a contradiction to the assumptions.

Part 3: Finally, consider the case that ℓ is parallel to one of the facets of T^n. By Lemma 1 and Theorem 4, we only have to consider the case n even. Suppose ℓ is parallel to $S^{(n+1)}$ and that $v_n^{(n+1)} > 0$. Since $R_{n-1}(T^{n-1}) = \sqrt{(n-1)/n}$ we have $v_n^{(n+1)} = \sqrt{(n+1)/n}$. Let $p \in \ell$. Since $v_n^{(n+1)} > 0$ it holds $p_n \ge 0$ and obviously

$$R_{n-1}(T^n) \ge \sqrt{\frac{n+1}{n}} - p_n \, . \tag{3.7}$$

On the other hand, since ℓ is parallel to $S^{(n+1)}$, and since the cylinder radius of T^{n-1} is $\sqrt{(n-2)/n}$ (recall that $n-1$ is odd)

$$R_{n-1}(T^n)^2 = \frac{n-2}{n} + p_n^2. \tag{3.8}$$

Let

$$p_n^* = \frac{3}{2\sqrt{n(n+1)}} > 0$$

be the unique minimal solution for p_n to (3.7) and (3.8). As $\mathcal{C}$ is an optimal cylinder it must hold $p_n = p_n^*$. But because of Lemma 1 and the equalities in (3.7) and (3.8) for p_n^* this means that all vertices of T^n have the same distance to ℓ, which shows that $\mathcal{C}$ is circumscribing. □

As shown in [6], the optimal choice of ℓ for even n is the one parallel to a facet, but the proof is long and includes extensive study of the relevant polynomial equations, which we do not review here in detail. However, a crucial step towards a solution was the transformation of the original problem into an optimization problem over symmetric polynomials, an idea which also Weißbach had in mind for solving the problem. In [6] this transformation is presented for general outer j-radii of the regular simplex, but here we will keep our focus on the case $j = n-1$.

Let $\ell = p + \mathrm{lin}\{s\}$, where $s \in \mathcal{S}^{n-1}$ and $p \in \mathrm{lin}\{s\}$. Suppose $\mathcal{C} = \ell + R_{n-1}(T^n)(\mathbb{B}^{n+1} \cap \mathcal{H}_0^n)$ is a minimal enclosing (and circumscribing) cylinder of the regular simplex $\mathbf{T}^n$ in standard embedding. The orthogonal projection of a vector $x \in \mathcal{H}_1^n$ onto the orthogonal complement of $\mathrm{lin}\{s\}$ (relative to $\mathcal{H}_1^n$) can be written as $x \mapsto Px$ with $P = I - ss^T \in$

$\mathbb{E}^{(n+1)\times(n+1)}$. Hence, for a general polytope with vertices $v^{(1)}, \dots, v^{(m)}$ (embedded in $\mathcal{H}_1^n$) the computation of the square of R_{n-1} can be expressed by the following optimization problem. Here, we use the convention $x^2 := x \cdot x$.

$$
\begin{array}{llrcll}
 & & \min \rho^2 & & & \\
 & \text{(i)} & \text{s.t.} \quad (p - Pv^{(i)})^2 & \leq & \rho^2\,, & i = 1, \dots, m, \\
(3.9) & \text{(ii)} & p \cdot s & = & 0, & \\
 & \text{(iii)} & s & \in & \mathcal{S}^{n-1}, & \\
 & \text{(iv)} & p & \in & \mathcal{H}_1^n\,. &
\end{array}
$$

In the case of $\mathbf{T}^n$, (i) can be replaced by

$$
\text{(3.10)} \qquad \text{(i')} \qquad (p - e^{(i)} + s_i s)^2 \;=\; \rho^2\,, \quad i = 1, \dots, n+1\,,
$$

where the equality sign comes from the fact that $\mathcal{C}$ is circumscribing. By (ii) and $s \in \mathcal{S}^{n-1}$, (i') can be simplified to

$$
\text{(i'')} \qquad p^2 - \rho^2 \;=\; s_i^2 + 2p_i - 1\,, \quad i = 1, \dots, n+1\,.
$$

Summing over all i gives $(n+1)(p^2 - \rho^2) = 1 + 2 - (n+1)$, i.e., $p^2 - \rho^2 = \frac{2-n}{n+1}$. We substitute this value into (i'') and obtain $p_i = \frac{1}{2}\left(\frac{3}{n+1} - s_i^2\right)$. Hence, the p_i can be replaced in terms of the s_i,

$$
\begin{aligned}
\rho^2 &= \frac{4n-5}{4(n+1)} + \frac{1}{4}\sum_{i=1}^{n+1} s_i^4\,, \\
p \cdot s &= -\frac{1}{2}\sum_{i=1}^{n+1} s_i^3\,.
\end{aligned}
$$

We arrive at the following characterization of the minimal enclosing cylinders:

Theorem 8. *Let $n \geq 2$. A vector $s \in \mathcal{S}^{n-1}$ represents the axis direction of a minimal enclosing cylinder of $\mathbf{T}^n \subset \mathcal{H}_1^n$ if and only if it is an optimal solution of the problem*

$$
\text{(3.11)} \qquad
\begin{array}{lrcl}
 & \min \; \sum_{i=1}^{n+1} s_i^4 & & \\
s.t. & \sum_{i=1}^{n+1} s_i^3 & = & 0\,, \\
 & \sum_{i=1}^{n+1} s_i^2 & = & 1\,, \\
 & \sum_{i=1}^{n+1} s_i & = & 0\,.
\end{array}
$$

Solving this system for even n in connection with Weißbach's results for odd n yields the following proposition.

Proposition 9. *Let $n \in \mathbb{N}$ and T^n be a regular simplex in $\mathbb{E}^n$ with edge length $\sqrt{2}$. Then*

$$R_{n-1}(T^n) = \begin{cases} \sqrt{\frac{n-1}{n+1}} & \text{if } n \text{ is odd,} \\ \frac{2n-1}{2\sqrt{n(n+1)}} & \text{if } n \text{ is even.} \end{cases}$$

4. Geometric inequalities

One reason why Weißbach considered the outer $(n-1)$-radius of the regular simplex was an older result of Eggleston [9], showing that for $n \geq 3$ there exists a body $C \subset \mathbb{E}^n$ with $R_{n-1}(C) > r_1(C)$. In [17] Weißbach gives a much simpler proof. By Lemma 1,

$$R_{n-1}(T^n) \geq \sqrt{\frac{n-1}{n+1}} > \sqrt{\frac{1}{2}} = r_1(T^n) \quad \text{for } n \geq 4 \,. \tag{4.1}$$

However, for $n = 3$ it holds $R_{n-1}(T^n) = r_1(T^n)$. By an elegant construction, Weißbach also provides a proof of the remaining case.

Proposition 10. *The Weißbach polytope*

$$W = \operatorname{conv}\left(\left\{\begin{pmatrix} 0 \\ -\sqrt{\frac{1}{2}} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{\frac{1}{2}} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{1}{2}} \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\sqrt{\frac{1}{2}} \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{1-\sqrt{6}}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}-2\sqrt{3}}{4} \\ \frac{-\sqrt{2}+2\sqrt{3}}{4} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{-\sqrt{2}+2\sqrt{3}}{4} \\ \frac{-\sqrt{2}+2\sqrt{3}}{4} \\ 0 \end{pmatrix}\right\}\right)$$

fulfills $R_2(W) > r_1(W)$.

Note that the first four vertices of W are the vertices of a regular simplex with edge length $\sqrt{2}$. Hence, it suffices to show that $r_1(W) = r_1(T^3)$ and in each of the three optimal projection directions of T^3 the projection of W does not fit into the same circumball. This situation is visualized in Figure 3.[1]

It turns out that the boundary of a minimal enclosing cylinder of W contains five vertices of W, and that this condition can be used to compute the numerical value of $R_2(W)$. E.g., one of the minimal enclosing cylinders of W contains the vertices 1, 3, 4, 6, and 7. For any five points in general position, there are six (complex) cylinders whose surface passes through the given points (see [3]). Thus, solving the corresponding systems of polynomial equations yields six cylinders (which are all real for our configuration), among which the one with radius $\rho \approx \sqrt{2} \cdot 0.50095$ is the smallest one containing W. Hence, $R_2(W)/r_1(W) \approx 1.0019$.

In fact, $R_2 > r_1$ would already be obtained by adding only two of the three new vertices to the regular simplex.

[1]A colored, dynamic 3D model of the Weißbach polytope is available from the homepages of the authors.

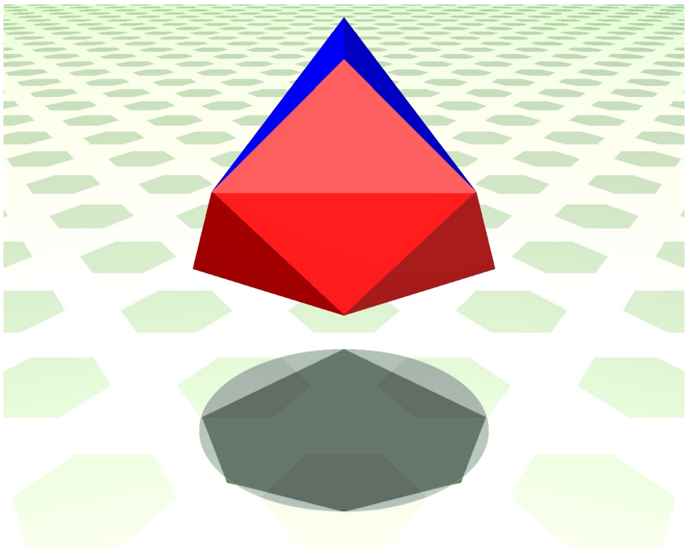

FIGURE 3. In the picture, all seven vertices of the Weißbach polytope W are visible. The upper two and the left most and right most belong to the original regular simplex T^3. In all three optimal projection directions of T^3 two of the additional vertices of W are projected outside the circumball of the projection of T^3. The picture shows one of these three projections as well as the circumball of the projection of T^3.

Since $R_{n-1}(C) < r_1(C)$ for many symmetric bodies, (4.1) and Proposition 10 show that R_{n-1} and r_1 are *incomparable*, i.e., no arc between the nodes for R_{n-1} and r_1 can be added in Figure 1. Now the question arises which other pairs of radii (which are not connected by a directed path in that figure) are provably incomparable. For certain pairs, already the regular simplex in addition with the class of ellipsoids shows incomparability, but for example it holds $R_2(T^n) \leq r_1(T^n)$ for $n \geq 3$ and $R_1(T^n) \leq r_2(T^n)$ for $n \geq 5$ like for all symmetric bodies. On the other hand it follows from the monotonicity of the outer and inner radii chains, that bodies C, C' with $R_2(C) > r_1(C)$ and $R_1(C') > r_2(C')$ imply that the diagram in Figure 1 is already complete (since in both cases the other direction of the inequalities are fulfilled by ellipsoids).

Bodies C which satisfy the inequalities $R_2(C) > r_1(C)$ and $R_1(C) > r_2(C)$ simultaneously were considered in [7]. There, a body of constant breadth C with $R_2(C) > r_1(C)$ is called a *totally non-spherical body* as every projection of C onto arbitrary subspaces of dimension at least 2 is different from the ball.

Proposition 11. *For all $n \geq 3$ there exists a totally non-spherical body $C \subset \mathbb{E}^n$, i.e.,*

$$r_n(C) \leq \cdots \leq r_2(C) < r_1(C) = R_1(C) < R_2(C) \leq \cdots \leq R_n(C)\,.$$

Finally, we want to describe a small application of the existence of bodies (such as the Weißbach polytope) which satisfy radii relations in the unusual direction (say, $R_2(C) > r_1(C)$).

The following Proposition was shown in [11, (1.11)] by use of Helly's theorem.

Proposition 12. *If C is a body in $\mathbb{E}^n$ then*

$$R_n(C) = \sup_{\substack{S \subset C \\ S \text{ Simplex}}} R_n(S)\,.$$

Proposition 12 provides an algorithmic reduction of the problem to compute $R_n(P)$ of a polytope P to the computation of the outer n-radius of simplices defined by vertices of P. Using the concept of non-spherical bodies, the following results imply that a similar result does *not* hold for general outer j-radii, not even for the outer 2-radius in $\mathbb{E}^3$ (the radius of the smallest enclosing cylinder). We make use of the following result, shown in [4, Theorem 3.17].

Proposition 13. *Let $1 \leq j \leq n$, such that $n - j + 1$ divides $n + 1$, and let S be an n-dimensional simplex. Then*

$$\frac{R_j(S)}{r_1(S)} \leq \sqrt{\frac{2j}{n+1}}\,,$$

with equality if $S = T^n$.

Proposition 10 and the case $(j, n) = (2, 3)$ in Proposition 13 imply that the Weißbach polytope W satisfies

$$R_2(W) > r_1(W) \geq r_1(S) \geq R_2(S) \text{ for all simplices } S \subset W.$$

More generally, we can state the following theorem.

Theorem 14. *If $1 \leq j \leq \frac{n+1}{2}$ then every totally non-spherical body $C \subset \mathbb{E}^n$ satisfies*

$$R_j(C) > \sup_{\substack{S \subset C \\ S \text{ Simplex}}} R_j(S)\,.$$

Proof. In case $j = 1$, note that the constant breadth of C implies $r_1(C) = R_1(C)$. However, since no simplex can be of constant breadth, $R_1(S) < r_1(S) \leq r_1(C)$ for any simplex S contained in C.

Now suppose $j \geq 2$ and let $S \subset C$ be a simplex. Since $r_1(S) \leq r_1(C) < R_j(C)$, it remains to show $R_j(S) \leq r_1(S)$. If $n + 1$ is even, this follows from the monotonicity of the outer radii and Proposition 13. If n is even, let $\bar{S}$ be an $(n+1)$-simplex, such that S is congruent to one of the facets of $\bar{S}$ and $r_1(S) = r_1(\bar{S})$. Since $j + 1 \leq \frac{n+2}{2}$, we have $R_j(S) \leq R_{j+1}(\bar{S}) \leq r_1(\bar{S}) = r_1(S)$. □

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