Nonnegativity of signomials with Newton simplex over convex sets

JONAS ELLWANGER, THORSTEN THEOBALD, AND TIMO DE WOLFF

Abstract. We study a class of signomials whose positive support is the set of vertices of a simplex and which may have multiple negative support points in the simplex. Various groups of authors have provided an exact characterization for the global nonnegativity of a signomial in this class in terms of circuit signomials and that characterization provides a tractable nonnegativity test. We generalize this characterization to the constrained nonnegativity over a convex set X. This provides a tractable X-nonnegativity test for the class in terms of relative entropy programming and in terms of the support function of X. Our proof methods rely on the convex cone of constrained SAGE signomials (sums of arithmetic-geometric exponentials) and the duality theory of this cone.

1. Introduction

Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a finite set. A *signomial* (or *exponential sum*) supported on $\mathcal{A}$ is a function of the form

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} c_{\boldsymbol{\alpha}} \exp\langle \boldsymbol{\alpha}, \boldsymbol{x} \rangle,$$

where $c_{\boldsymbol{\alpha}} \in \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ denotes the usual inner product. If all $\boldsymbol{\alpha}$ are nonnegative integer vectors, then the substitution $y_i = \exp(x_i)$ defines a polynomial function $p(\boldsymbol{y}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} c_{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\alpha}}$ on $\mathbb{R}^n_+$. To strengthen this connection, every monomial vector $\boldsymbol{\alpha} \in \mathcal{A}$ is associated with a "monomial" basis function $\mathrm{e}^{\boldsymbol{\alpha}}$ which takes values $\mathrm{e}^{\boldsymbol{\alpha}}(\boldsymbol{x}) = \exp\langle \boldsymbol{\alpha}, \boldsymbol{x} \rangle$. The corresponding signomial ring consists of all finite products and real-linear combinations of these basis functions $(\mathrm{e}^{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathcal{A}}$; see, e.g., [4]. From an optimization perspective, for a given signomial $f(\boldsymbol{x})$ one is interested in the problem

$$\min f(\boldsymbol{x}) \ \text{ such that } \ \boldsymbol{x} \in X \subseteq \mathbb{R}^n,$$

where X is some reasonably given feasibility region. In what follows, we assume that the constrained region X is convex. We refer the reader to [5] and its references for the multifaceted occurrences of signomials in mathematics.

In recent years investigating sparse settings became an intensely researched area in real algebraic geometry, and in polynomial and signomial optimization. Specifically, for given support $\mathcal{A}$ one considers the space $\mathbb{R}^{\mathcal{A}}$ of all real multivariate signomials supported on $\mathcal{A}$. If $\mathcal{A} \subseteq \mathbb{N}^n$, then $\mathbb{R}^{\mathcal{A}}$ can also denote the space of all polynomials supported on $\mathcal{A}$. An initiating moment for modern developments in sparse algebraic geometry were developments by Gelfand, Kapranov and Zelevinsky, which are summarized in [8], who coined the terminology "$\mathcal{A}$-philosophy", and who investigated the behavior of the space $\mathbb{R}^{\mathcal{A}}$ and structures within like $\mathcal{A}$-discriminants. More recent developments include fewnomial theory on the real algebraic geometry side, see e.g.,

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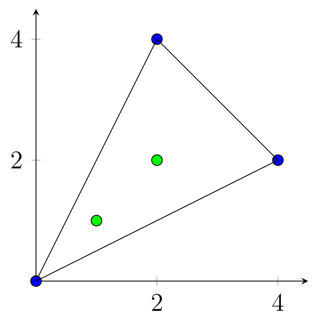

FIGURE 1. The support and the Newton polytope of f in Example 1.1

[19] for an overview, exploiting sparsity in a sums of squares context, see e.g. [25], and the development of other certificates which perform well in sparse settings; see e.g., [1]. This development is further motivated by the fact that in many real-world applications supports $\mathcal{A}$ of signomials and polynomials are sparse.

As solving signomial optimization problems are NP-hard even in the unconstrained case and notoriously hard to solve in practice, characterizing classes of sparse signomials for which nonnegativity on a convex set X can be decided efficiently and effectively is of ubiquitous interest. In the case of global nonnegativity of polynomials, a prominent classical result of this kind includes Hilbert's classification [10]. For homogeneous polynomials in two variables, homogeneous quadratic forms or homogeneous ternary quartics, this classification ensures the equality of nonnegative polynomials with sums of squares. From the viewpoint of convex optimization, the global nonnegativity problem can be formulated as a semidefinite program. These techniques do, however, neither extend to the case of signomials nor do they preserve sparsity of $\mathcal{A}$.

Let f be a signomial supported on $\mathcal{A}$, and let X be a convex set. In this article we consider the specific case that the Newton polytope of f (that is, the convex hull $\text{conv}(\mathcal{A})$ of its support vectors $\mathcal{A}$) is a simplex. We assume that terms corresponding to vertices of $\text{conv}(\mathcal{A})$ have positive coefficients, and all of the other coefficients are negative.

This class is quite broad, as there are no limitations on the number of negative terms, and no limitations on the set X beyond convexity, or on $\mathcal{A}$ beyond $\text{conv}(\mathcal{A})$ being a simplex. Consider the following toy example.

Example 1.1. Let

$$f(x,y) = 13 + \exp(4x+2y) + \exp(2x+4y) - 12\exp(x+y) - 3\exp(2x+2y) \quad (1)$$

be a signomial. The support points of f are shown in Figure 1. We aim to decide whether f is nonnegative over a convex set X, for example a polytope, or even more specific $X = [-1,0]^2$ being a box.

Results. In this paper we provide an exact characterization of nonnegativity of signomials and polynomials in the class described above. In the unconstrained situation $X = \mathbb{R}^n$ the question has been solved in various variants by Iliman and de Wolff [11], Murray, Chandrasekaran and Wierman [17], and by Wang [24]. All these papers build on methods for nonnegativity certificates based on the arithmetic-geometric inequality (AM/GM inequality), which define a full-dimensional convex subcone of the cone of nonnegative signomials $C(\mathcal{A})$, also known as SONC or SAGE cone; see [11, 3]. Here, we extend this result to the case of general convex sets X using a signed version of the constrained SAGE cone introduced by Murray, Chandrasekaran and Wierman [16]. Signed version means that we distinguish between the positive and the negative coefficients of the terms in the signomials, resulting in two separate support sets $\mathcal{A}$ and $\mathcal{B}$ and the signed constrained SAGE cone $C_X(\mathcal{A}, \mathcal{B})$, which we formally introduce in Section 2. This leads to the following main theorem.

Theorem 1.2. *Let $\mathcal{A} \subseteq \mathbb{R}^n$ be the vertices of a simplex, $\mathcal{B} \subseteq \mathrm{conv}(\mathcal{A}) \setminus \mathcal{A}$, and*

$$f = \sum_{\boldsymbol{\alpha}\in\mathcal{A}} c_{\boldsymbol{\alpha}} \mathrm{e}^{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\beta}\in\mathcal{B}} d_{\boldsymbol{\beta}} \mathrm{e}^{\boldsymbol{\beta}} \text{ with } c_{\boldsymbol{\alpha}} > 0 \text{ and } d_{\boldsymbol{\beta}} < 0. \tag{2}$$

Then f is nonnegative over X if and only if f is contained in the signed constrained SAGE cone $C_X(\mathcal{A}, \mathcal{B})$.

As a consequence, deciding whether a signomial in this class is nonnegative over X can be formulated as a convex optimization problem, specifically, a relative entropy program involving also the support function of X. Furthermore, the result reveals a hidden convexity structure in the nonnegativity question of our class of constrained signomials.

We complement our main result by showing that the condition $\mathcal{B} \subseteq \mathrm{conv}(\mathcal{A}) \setminus \mathcal{A}$ indeed is necessary via providing counterexamples to more general statements; see Lemma 3.8 and Example 3.9. Furthermore, we provide a full characterization of nonnegativity in the univariate case in terms of a separability result; see Theorem 3.6. A formulation of our main result in the language of polynomials is given in Corollary 3.10.

Our result strongly exploits the sign information of the support points. In the study of real polynomials, using sign information of the coefficients has a rich history dating back, in particular, to Descartes' rule of signs. Recently, Bihan and Dickenstein [2] and Feliu and Telek [6, 21] generalized Descartes' rule of signs.

2. Preliminaries

We review some basic concepts on SAGE certificates over a convex set.

2.1. **Sparse signomials.** From now on, $\mathcal{A} \subseteq \mathbb{R}^n$ is an affinely independent, finite set and $\mathcal{B} \subseteq \mathbb{R}^n$ is finite. Usually $\mathcal{A}$ and $\mathcal{B}$ are disjoint, but because we are adding constants to signomials in optimization settings we allow $\mathbf{0}$ to be contained in both sets. These sets act as the *positive* and *negative support* of our signomials. We denote by $\mathbb{R}^{\mathcal{A}}$ the $|\mathcal{A}|$-tuples of $\mathbb{R}$ indexed by $\mathcal{A}$. The set X is a fixed convex subset of $\mathbb{R}^n$.

Let f be a general signomial whose coefficients except at most one are positive,

$$f = \sum_{\boldsymbol{\alpha}\in\mathcal{A}} c_{\boldsymbol{\alpha}} \mathrm{e}^{\boldsymbol{\alpha}} + d\mathrm{e}^{\boldsymbol{\beta}} \text{ with } c_{\boldsymbol{\alpha}} > 0 \text{ and } d \in \mathbb{R}. \tag{3}$$

For $\boldsymbol{\beta} \in \mathcal{B}$, the set of all these functions which are nonnegative on X form a convex cone called the *constrained* $\boldsymbol{\beta}$-AGE *cone* with respect to $\mathcal{A}$. We write

$$C_X(\mathcal{A}, \boldsymbol{\beta}) = \left\{ f \mid f = \sum_{\boldsymbol{\alpha}\in\mathcal{A}} c_{\boldsymbol{\alpha}} \mathrm{e}^{\boldsymbol{\alpha}} + d\mathrm{e}^{\boldsymbol{\beta}} \geq 0 \text{ for all } \boldsymbol{x} \in X, c_{\boldsymbol{\alpha}} \geq 0 \right\}.$$

In our context, it is convenient to denote by

$$C_X(\mathcal{A},\mathcal{B}) = \sum_{\boldsymbol{\beta}\in\mathcal{B}} C_X(\mathcal{A},\boldsymbol{\beta}) \tag{4}$$

the *signed* X-SAGE *cone*, which allows negative coefficients only in the subset $\mathcal{B}$ of the support $\mathcal{A}\cup\mathcal{B}$. The signomials in $C_X(\mathcal{A},\mathcal{B})$ have positive support $\mathcal{A}$ and they may have multiple negative support points. If $f \in C_X(\mathcal{A},\mathcal{B})$ then we call f an X-SAGE-*signomial* with respect to $\mathcal{A}$ and $\mathcal{B}$. We omit to write $\mathcal{A}$ and $\mathcal{B}$ if they are clear from the context. The signed X-SAGE cone provides a common notation in optimization approaches to nonnegativity certificates based on the AM/GM inequality (see [12, 14, 17]). Note that our definitions are consistent with unsigned versions of the X-SAGE cone in the literature (e.g., [16, 23]), because by [16, Corollary 5], for any finite, disjoint sets $\mathcal{A}$ and $\mathcal{B}$ the signomial f from Theorem 1.2 is contained in the signed cone $C_X(\mathcal{A},\mathcal{B})$ if and only if f is contained in the corresponding unsigned X-SAGE cone. From an optimization perspective, a signomial f is contained in $C_X(\mathcal{A},\mathcal{B})$ if and only if the optimization problem

$$f_{\text{SAGE}} \coloneqq \sup\{\gamma \in \mathbb{R} \mid f-\gamma \in C_X(\mathcal{A},\mathcal{B}\cup\{\mathbf{0}\})\} \tag{5}$$

has a nonnegative optimal value.

Next we record how to decide membership to the X-SAGE cone. The relative entropy function of two vectors $\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^n_{>0}$ is defined as $D(\boldsymbol{u},\boldsymbol{v}) = \sum_{i=1}^n u_i \ln(u_i/v_i)$. Denote by $\sigma_X(\boldsymbol{y}) = \sup\{\langle \boldsymbol{y},\boldsymbol{x}\rangle \ : \ \boldsymbol{x}\in X\}$ the *support function* of X from classical convex geometry. The function σ_X is a convex function $\mathbb{R}^n \to \mathbb{R}_+\cup\{\infty\}$. If X is polyhedral, then σ_X is linear on every normal cone of X.

Theorem 2.1 ([3, 16])**.**

1. *The signomial f in* (3) *is nonnegative on $\mathbb{R}^n$ if and only if there exists $\boldsymbol{\nu}\in\mathbb{R}^{\mathcal{A}}_+$ with $\sum_{\boldsymbol{\alpha}\in\mathcal{A}}\nu_{\boldsymbol{\alpha}}\boldsymbol{\alpha} = (\sum_{\boldsymbol{\alpha}\in\mathcal{A}}\nu_{\boldsymbol{\alpha}})\boldsymbol{\beta}$ and $D(\boldsymbol{\nu},\mathrm{e}\boldsymbol{c})\le d$.*
2. *The signomial f in* (3) *is nonnegative on X if and only if there exists $\boldsymbol{\nu}\in\mathbb{R}^{\mathcal{A}}_+$ with $\sigma_X(\sum_{\boldsymbol{\alpha}\in\mathcal{A}}\nu_{\boldsymbol{\alpha}}(\boldsymbol{\beta}-\boldsymbol{\alpha})) + D(\boldsymbol{\nu},\mathrm{e}\boldsymbol{c})\le d$.*

This statement can be used to express the convex cones $C_X(\mathcal{A},\boldsymbol{\beta})$ and $C_X(\mathcal{A},\mathcal{B})$ in terms of the relative entropy function and in terms of the support function of X. As another consequence, if $(\mathcal{A}\cup\{\boldsymbol{\beta}\})^T X = \{(\langle\boldsymbol{\alpha},\boldsymbol{x}\rangle)_{\boldsymbol{\alpha}\in(\mathcal{A}\cup\{\boldsymbol{\beta}\})}\in\mathbb{R}^{\mathcal{A}\cup\{\boldsymbol{\beta}\}} \mid \boldsymbol{x}\in X\}$ is a rational polyhedron, then, by [18], the nonnegativity of f on X can be formulated as a second-order program.

Proposition 2.2 (Theorem 1 and Corollary 1 in [16])**.** *The constrained $\boldsymbol{\beta}$-AGE cone can be expressed as*

$$C_X(\mathcal{A},\boldsymbol{\beta}) = \Bigg\{ f = \sum_{\boldsymbol{\alpha}\in\mathcal{A}} c_{\boldsymbol{\alpha}}\mathrm{e}^{\boldsymbol{\alpha}} + d\mathrm{e}^{\boldsymbol{\beta}} \mid \textit{There exists } \boldsymbol{\nu}\in\mathbb{R}^{\mathcal{A}} \textit{ with}$$
$$\sigma_X\left(\sum_{\boldsymbol{\alpha}\in\mathcal{A}}\nu_{\boldsymbol{\alpha}}(\boldsymbol{\beta}-\boldsymbol{\alpha})\right) + D(\boldsymbol{\nu},\mathrm{e}\boldsymbol{c}) \le d \textit{ and } c_{\boldsymbol{\alpha}}\ge 0\Bigg\}.$$

The signed X-SAGE cone $C_X(\mathcal{A},\mathcal{B})$ can be expressed in terms of a Minkowski sum

$$C_X(\mathcal{A},\mathcal{B}) = \Bigg\{ f = \sum_{\boldsymbol{\alpha}\in\mathcal{A}} c_{\boldsymbol{\alpha}}\mathrm{e}^{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\beta}\in\mathcal{B}} d_{\boldsymbol{\beta}}\mathrm{e}^{\boldsymbol{\beta}} \textit{ with } c_{\boldsymbol{\alpha}}\ge 0,\ d_{\boldsymbol{\beta}}\le 0 \mid f\in\sum_{\boldsymbol{\beta}\in\mathcal{B}} C_X(\mathcal{A},\boldsymbol{\beta})\Bigg\}.$$

2.2. **The dual perspective.** Our proofs also employ the dual constrained AGE cones. To state these dual cones, we identify $C_X(\mathcal{A}, \boldsymbol{\beta})$ with the cone of its coefficient vectors. For $X = \mathbb{R}^n$, the dual cone $C^*_{\mathbb{R}^n}(\mathcal{A}, \boldsymbol{\beta})$ is given by

$$C^*_{\mathbb{R}^n}(\mathcal{A}, \boldsymbol{\beta}) = \operatorname{cl}\Bigg\{(\boldsymbol{v}, w_{\boldsymbol{\beta}}) \in \mathbb{R}^{\mathcal{A}}_{>0} \times \mathbb{R}_{>0} \mid \text{There exists } \boldsymbol{z} \in \mathbb{R}^n \text{ s.t. for all } \boldsymbol{\alpha} \in \mathcal{A} \quad w_{\boldsymbol{\beta}} \ln\left(\frac{w_{\boldsymbol{\beta}}}{v_{\boldsymbol{\alpha}}}\right) \leq \langle \boldsymbol{\beta} - \boldsymbol{\alpha}, \boldsymbol{z} \rangle \Bigg\},$$

where cl denotes the topological closure of a set in standard topology; see [3, 13]. In order to derive $C^*_X(\mathcal{A}, \boldsymbol{\beta})$ for a general convex set X, note that the epigraph of the support function $\operatorname{epi} \sigma_X = \{(\boldsymbol{y}, t) \in \mathbb{R}^n \times \mathbb{R} \mid \sigma_X(\boldsymbol{y}) \leq t\}$ is a convex cone with dual cone

$$(\operatorname{epi} \sigma_X)^* \;=\; \{\lambda(-\boldsymbol{x}, 1) \in \mathbb{R}^n \times \mathbb{R} \,:\, \boldsymbol{x} \in X,\, \lambda \geq 0\},$$

see [20, Sect. 1.7.1]. We obtain the following characterization.

Proposition 2.3 ([16])**.** *The dual cone to* $C_X(\mathcal{A}, \boldsymbol{\beta})$ *is*

$$C^*_X(\mathcal{A}, \boldsymbol{\beta}) = \operatorname{cl}\Bigg\{(\boldsymbol{v}, w_{\boldsymbol{\beta}}) \in \mathbb{R}^{\mathcal{A}}_{>0} \times \mathbb{R}_{>0} \mid \textit{There exists } \boldsymbol{z} \in \mathbb{R}^n \textit{ s.t. for all } \boldsymbol{\alpha} \in \mathcal{A} \quad w_{\boldsymbol{\beta}} \ln\left(\frac{w_{\boldsymbol{\beta}}}{v_{\boldsymbol{\alpha}}}\right) \leq \langle \boldsymbol{\beta} - \boldsymbol{\alpha}, \boldsymbol{z} \rangle \textit{ and } \frac{1}{w_{\boldsymbol{\beta}}}\boldsymbol{z} \in X \Bigg\},$$

which can be written more conveniently as

$$C^*_X(\mathcal{A}, \boldsymbol{\beta}) = \operatorname{cl}\Bigg\{(\boldsymbol{v}, w_{\boldsymbol{\beta}}) \in \mathbb{R}^{\mathcal{A}}_{>0} \times \mathbb{R}_{>0} \mid \textit{There exists } \boldsymbol{z} \in X \textit{ s.t. for all } \boldsymbol{\alpha} \in \mathcal{A} \quad \ln\left(\frac{w_{\boldsymbol{\beta}}}{v_{\boldsymbol{\alpha}}}\right) \leq \langle \boldsymbol{\beta} - \boldsymbol{\alpha}, \boldsymbol{z} \rangle \Bigg\}.$$

To express the dual signed X-SAGE cone $C^*_X(\mathcal{A}, \mathcal{B})$ formally, we embed each $C_X(\mathcal{A}, \boldsymbol{\beta})$ from $\mathbb{R}^{\mathcal{A}}_{>0} \times \mathbb{R}_{>0}$ into $\mathbb{R}^{\mathcal{A}}_{>0} \times \mathbb{R}^{\mathcal{B}}$ where every $w_{\boldsymbol{\beta}}$ is identified with one specific entry in the vector $\boldsymbol{w}$. Using this embedding, the intersection $\bigcap_{\boldsymbol{\beta} \in \mathcal{B}} C^*_X(\mathcal{A}, \boldsymbol{\beta})$ is well defined and

$$\begin{aligned} C^*_X(\mathcal{A}, \mathcal{B}) &= \bigcap_{\boldsymbol{\beta} \in \mathcal{B}} C^*_X(\mathcal{A}, \boldsymbol{\beta}) \\ &= \operatorname{cl}\Bigg\{(\boldsymbol{v}, \boldsymbol{w}) \in \mathbb{R}^{\mathcal{A}}_{>0} \times \mathbb{R}^{\mathcal{B}}_{>0} \mid \text{For all } \boldsymbol{\beta} \in \mathcal{B} \text{ there exists } \boldsymbol{z}_{\boldsymbol{\beta}} \in X \\ &\qquad \text{s.t. for all } \boldsymbol{\alpha} \in \mathcal{A} \ \ln\left(\frac{w_{\boldsymbol{\beta}}}{v_{\boldsymbol{\alpha}}}\right) \leq \langle \boldsymbol{\beta} - \boldsymbol{\alpha}, \boldsymbol{z}_{\boldsymbol{\beta}} \rangle \Bigg\}. \end{aligned}$$

Employing this dual signed SAGE cone, the optimization problem f_{SAGE} in (5) can be dualized to

$$f^*_{\text{SAGE}} = \inf\{\langle \boldsymbol{c}, \boldsymbol{v} \rangle + \langle \boldsymbol{d}, \boldsymbol{w} \rangle \mid v_{\boldsymbol{0}} = 1, (\boldsymbol{v}, \boldsymbol{w}) \in C^*_X(\mathcal{A}, \mathcal{B} \cup \{\boldsymbol{0}\})\}. \tag{6}$$

If we fix one vector of the positive support as the origin, we know that $1 \in C_X(\mathcal{A}, \mathcal{B})$. In this case, by [17], strong duality holds for the problems f_{SAGE} in (5) and f^*_{SAGE} in (6), because $C_X(\mathcal{A}, \mathcal{B})$ is a closed, convex and pointed cone.

Remark 2.4. The SAGE language, which we introduce here, has a counterpart called sums of nonnegative circuit functions, which is usually displayed in a polynomial setting. The main difference is that building blocks there are circuits, i.e., minimally affinely dependent support sets with one negative term, but the two resulting cones are identical up to technicalities due to Wang [24] and independently due to Murray, Chandrasekaran, and Wierman [17]. In the circuit setting, nonnegativity is decided by an invariant called circuit number, which is the counterpart of the entropy function in the SAGE language. Also in the circuit setting there exists a dual version, see, e.g., [9], and a generalized circuit concept, called sublinear circuits, was developed in [18].

3. Nonnegativity of signomials with Newton simplex

In this section we prove our main Theorem 1.2, and discuss further examples and extensions. In the proof, we reduce the nonnegativity statements to a finite number of auxiliary optimization problems. As a first step, we provide a solution to these problems via the following lemma.

Lemma 3.1. *Let $\mathcal{A} \subseteq \mathbb{R}^n$ be affinely independent and $\boldsymbol{\beta} \in \operatorname{conv}(\mathcal{A})$. Further let $X \subseteq \mathbb{R}^n$ be a convex set with $\mathbf{0} \in X$. Then the optimization problem*

$$\max_{\boldsymbol{x} \in X} \min_{\boldsymbol{\alpha} \in \mathcal{A}} \langle \boldsymbol{\beta} - \boldsymbol{\alpha}, \boldsymbol{x} \rangle$$

in the variable $\boldsymbol{x}$ has the optimal value 0 which is attained at $\boldsymbol{x} = \mathbf{0}$.

Proof. Since $\boldsymbol{\beta} \in \operatorname{conv}(\mathcal{A})$, we have that for every $\boldsymbol{x} \in X$ there exist an $\boldsymbol{\alpha}$ such that $\langle \boldsymbol{\alpha}, \boldsymbol{x} \rangle \geq \langle \boldsymbol{\beta}, \boldsymbol{x} \rangle$. Therefore the maximum is at most 0. Since $\mathbf{0}$ is contained in X, the maximum is exactly 0. □

Remark 3.2. Let f be the signomial in (2). Further, let $\boldsymbol{\gamma} \in \mathbb{R}^n$ be a vector. Then $f \geq 0$ for all $\boldsymbol{x} \in X$ if and only if $f \cdot \mathrm{e}^{\gamma} \geq 0$ for all $\boldsymbol{x} \in X$. Also, we have $f \in C_X(\mathcal{A}, \mathcal{B})$ if and only if $f \cdot \mathrm{e}^{\gamma} \in C_X(\mathcal{A} + \{\boldsymbol{\gamma}\}, \mathcal{B} + \{\boldsymbol{\gamma}\})$. Hence, for the question when the X-SAGE cone and the X-nonnegativity cone coincide we can fix one vector of $\mathcal{A}$ as the origin.

We can now prove Theorem 1.2.

Proof of Theorem 1.2. Using Remark 3.2, we can assume that $\mathbf{0}$ is a vertex of the Newton polytope of f. Let $f(\boldsymbol{x}) \geq 0$ for all $\boldsymbol{x} \in X$. Since f is contained in the X-SAGE cone if and only if $f_{\text{SAGE}} \geq 0$, we need to prove $f_{\text{SAGE}} \geq 0$, where f_{SAGE} is the supremum of the optimization problem defined in (5) associated to the SAGE decomposition. As pointed out in the Preliminaries, we know that strong duality holds in this case, so we can compute the dual problem (6) and obtain

$$f_{\text{SAGE}} = \inf\{\langle \boldsymbol{c}, \boldsymbol{v} \rangle + \langle \boldsymbol{d}, \boldsymbol{w} \rangle \mid v_{\mathbf{0}} = 1, (\boldsymbol{v}, \boldsymbol{w}) \in C_X^*(\mathcal{A}, \mathcal{B} \cup \{\mathbf{0}\})\}.$$

Using Proposition 2.3, we obtain

$$\begin{aligned} (7) \qquad f_{\text{SAGE}} = & \inf \langle \boldsymbol{c}, \boldsymbol{v} \rangle + \langle \boldsymbol{d}, \boldsymbol{w} \rangle, \\ & \text{s.t. } \boldsymbol{v} \in \mathbb{R}^{\mathcal{A}}_{>0}, \boldsymbol{w} \in \mathbb{R}^{\mathcal{B}}_{>0} \text{ with } v_{\mathbf{0}} = 1 \text{ and for all } \boldsymbol{\beta} \in \mathcal{B} \cup \{\mathbf{0}\} \text{ there} \\ & \text{exists } \boldsymbol{z}_{\boldsymbol{\beta}} \in X \text{ such that for all } \boldsymbol{\alpha} \in \mathcal{A} \ \ln\left(\frac{w_{\boldsymbol{\beta}}}{v_{\boldsymbol{\alpha}}}\right) \leq \langle \boldsymbol{\beta} - \boldsymbol{\alpha}, \boldsymbol{z}_{\boldsymbol{\beta}} \rangle. \end{aligned}$$

We simplify these conditions. We start by investigating the inequalities for $\boldsymbol{\beta} = \mathbf{0}$. The set $\mathcal{A} \setminus \{\mathbf{0}\}$ is linearly independent. Therefore, for every choice of $\boldsymbol{z_0}$, we can always choose the $v_{\boldsymbol{\alpha}}$ to satisfy the constraints

$$\ln\left(\frac{v_{\mathbf{0}}}{v_{\boldsymbol{\alpha}}}\right) \leq \langle \mathbf{0} - \boldsymbol{\alpha}, \boldsymbol{z_0} \rangle \text{ for } \boldsymbol{\alpha} \in \mathcal{A}$$

which we can rewrite as

$$\ln(v_{\boldsymbol{\alpha}}) \geq \langle \boldsymbol{\alpha}, \boldsymbol{z_0} \rangle$$

with equality. This is optimal with respect to (7), because the $v_{\boldsymbol{\alpha}}$ are multiplied with a positive value in the objective function so they should be as small as possible. Hence,

$$\ln(v_{\boldsymbol{\alpha}}) = \langle \boldsymbol{\alpha}, \boldsymbol{z_0} \rangle \text{ or } v_{\boldsymbol{\alpha}} = \exp\langle \boldsymbol{\alpha}, \boldsymbol{z_0} \rangle. \tag{8}$$

Applying (8) to the remaining inequalities, we obtain that for every $\boldsymbol{\beta} \in \mathcal{B}$ we need to find a $z_{\boldsymbol{\beta}}$ such that

$$\ln(w_{\boldsymbol{\beta}}) \leq \langle \boldsymbol{\beta}, \boldsymbol{z_\beta} \rangle + \langle \boldsymbol{\alpha}, \boldsymbol{z_0} - \boldsymbol{z_\beta} \rangle \text{ for all } \boldsymbol{\alpha} \in \mathcal{A}. \tag{9}$$

In the objective function we have to minimize $\langle \boldsymbol{w}, \boldsymbol{d} \rangle$ and all the components of $\boldsymbol{d}$ are negative. Thus, we need to maximize every entry of $\boldsymbol{w}$. To determine the maximal possible value for $w_{\boldsymbol{\beta}}$ we look at the following optimization problem for every $\boldsymbol{\beta}$:

$$\max_{\boldsymbol{z_\beta} \in X} \min_{\boldsymbol{\alpha} \in \mathcal{A}} \langle \boldsymbol{\beta}, \boldsymbol{z_\beta} \rangle + \langle \boldsymbol{\alpha}, \boldsymbol{z_0} - \boldsymbol{z_\beta} \rangle \tag{10}$$

in the variable $\boldsymbol{z_\beta} \in X$. With $\boldsymbol{y} = \boldsymbol{z_\beta} - \boldsymbol{z_0}$ and $X' = \{\boldsymbol{x} - \boldsymbol{z_0} \mid \boldsymbol{x} \in X\}$, we can restate the problem as

$$\begin{aligned} &\max_{\boldsymbol{y} \in X'} \min_{\boldsymbol{\alpha} \in \mathcal{A}} \langle \boldsymbol{\beta}, \boldsymbol{y} + \boldsymbol{z_0} \rangle - \langle \boldsymbol{\alpha}, \boldsymbol{y} \rangle \\ = &\langle \boldsymbol{\beta}, \boldsymbol{z_0} \rangle + \max_{\boldsymbol{y} \in X'} \min_{\boldsymbol{\alpha} \in \mathcal{A}} \langle \boldsymbol{\beta} - \boldsymbol{\alpha}, \boldsymbol{y} \rangle. \end{aligned}$$

This is the problem solved in Lemma 3.1 since $\boldsymbol{z_0} \in X$, and therefore $\mathbf{0} \in X'$. So, the optimal solution for (10) is given by $\langle \boldsymbol{\beta}, \boldsymbol{z_0} \rangle$. Substituting into (9), we have determined that

$$\ln(w_{\boldsymbol{\beta}}) = \langle \boldsymbol{\beta}, \boldsymbol{z_0} \rangle \text{ or } w_{\boldsymbol{\beta}} = \exp\langle \boldsymbol{\beta}, \boldsymbol{z_0} \rangle \text{ for all } \boldsymbol{\beta} \in \mathcal{B} \tag{11}$$

holds. Taking the optimal solutions for $v_{\boldsymbol{\alpha}}$ from (8) and for $w_{\boldsymbol{\beta}}$ from (11), we obtain

$$f_{\text{SAGE}} = \inf_{\boldsymbol{z_0} \in X} \sum_{\boldsymbol{\alpha} \in \mathcal{A}} c_{\boldsymbol{\alpha}} \exp\langle \boldsymbol{\alpha}, \boldsymbol{z_0} \rangle + \sum_{\boldsymbol{\beta} \in \mathcal{B}} d_{\boldsymbol{\beta}} \exp\langle \boldsymbol{\beta}, \boldsymbol{z_0} \rangle = \inf_{\boldsymbol{z_0} \in X} f(\boldsymbol{z_0})$$

and because f is nonnegative over X it follows $f_{\text{SAGE}} \geq 0$. □

Example 3.3. With the main theorem proven, we answer the question from Example 1.1 whether the function

$$f(x, y) = 13 + \exp(4x + 2y) + \exp(2x + 4y) - 12\exp(x + y) - 3\exp(2x + 2y)$$

is nonnegative over $X = [-1, 0]^2$. All the negative support points of f are contained in the convex hull of the positive support points. Therefore, it suffices to calculate f_{SAGE} as seen in Theorem 1.2. For any $\gamma \in \mathbb{R}$ such that $f + \gamma$ has a SAGE

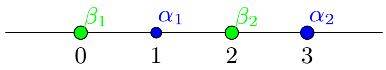

FIGURE 2. The support points of f in Example 3.4

decomposition $f + \gamma = f_1 + f_2$ with

$$f_1 = c_0^{(1)} + c_1^{(1)} \exp(4x + 2y) + c_2^{(1)} \exp(2x + 4y) - 12 \exp(x + y) \text{ and}$$
$$f_2 = c_0^{(2)} + c_1^{(2)} \exp(4x + 2y) + c_2^{(2)} \exp(2x + 4y) - 3 \exp(2x + 2y)$$

and $c_j^{(1)} + c_j^{(2)} = c_j$ for $j \in \{1, 2, 3\}$, we can choose f_1 and f_2 to have the same global minimizer. The smallest value of γ for which $f + \gamma$ is globally nonnegative is $\gamma = 7$. The resulting f_1 and f_2 are given by

$$f_1(x, y) = 16 + 0.5 \exp(4x + 2y) + 0.5 \exp(2x + 4y) - 12 \exp(x + y) \text{ and}$$
$$f_2(x, y) = 4 + 0.5 \exp(4x + 2y) + 0.5 \exp(2x + 4y) - 3 \exp(2x + 2y).$$

These signomials share the same global minimizer $(\ln(\sqrt{2}), \ln(\sqrt{2}))$. By calculating the derivative of f_1 in x,

$$\partial_x f_1(x, y) = 2 \exp(4x + 2y) + \exp(2x + 4y) - 12 \exp(x + y),$$

we can check that for every $x, y \in [-1, 0]$ this is a negative value. This is also the case for the derivative in y since f is symmetric in (x, y). Therefore, the minimum of f_1 over X is attained at $(1, 1)$. By the same argument, it follows that f_2 and therefore also f attains its minimum over X at $(1, 1)$. This means $\min_{(x,y)\in X} f(x, y) = 0$ and thus f is nonnegative on X. We can also see this if we calculate f_{SAGE} as in (7) by setting $\boldsymbol{z_0} = (1, 1)$, $v_{\boldsymbol{\alpha}} = \exp\langle \boldsymbol{\alpha}, \boldsymbol{z_0} \rangle$ and $w_{\boldsymbol{\beta}} = \exp\langle \boldsymbol{\beta}, \boldsymbol{z_0} \rangle$.

Example 3.4. If we only consider signomials with at most one negative term, the X-nonnegativity cone clearly coincides with the signed X-SAGE cone by definition. Thus, in this example, we examine the case with two negative terms to show that, in general, these cones do not coincide when there is at least one $\boldsymbol{\beta} \notin \text{conv}(\mathcal{A})$.

Let $f = \sum_{i=0}^{3} c_i e^i$ with $c_0, c_2 \leq 0$ and $c_1, c_3 \geq 0$ on the set $X = \mathbb{R}_+$. The support points of f are shown in Figure 2. We can assume $c_3 = 1$. Consider a family of nonnegative signomials which have a double root, say, at $\ln(2)$. This family is a one-dimensional family, and parametrizing it in terms of c_2 gives $f = \exp(3x) + c_2 \exp(2x) + (-12 - 4c_2) \exp(x) + (16 + 4c_2)$. For $c_2 \in (-5, -4)$, the signs of the coefficients are as desired and $f(0) > 0$. Hence, for $c_2 \in (-5, -4)$, the signomial f describes a family of nonnegative signomials satisfying the sign constraints.

However, none of the signomials in this family has an X-SAGE certificate. If one of these signomials had an X-SAGE certificate, then there were a decomposition $f = f_1 + f_2$ with $\operatorname{supp} f_1 = \{1, 2, 3\}$ and negative coefficient for the exponent 2, as well as $\operatorname{supp} f_2 = \{0, 1, 3\}$ and negative coefficient for the exponent 0. However, by Descartes' rule, a signomial $f_2 \neq 0$ with a double root at $\ln(2)$ cannot exist. Hence, for $c_2 \in (-5, -4)$, the signomial f with the double root at $\ln(2)$ is not an X-SAGE signomial.

Even though in general the signed X-SAGE cone does not coincide with the signed X-nonnegativity cone when $\boldsymbol{\beta} \notin \text{conv}(\mathcal{A})$, there are cases in which those two cones coincide. In the one-dimensional case we can exactly characterize these cases. To this end, we start with the following definition.

Definition 3.5. Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}$ be two sets. We say $\mathcal{A}$ separates $\mathcal{B}$ if there exists $\alpha \in \mathcal{A}$ and $\beta_1, \beta_2 \in \mathcal{B}$ with $\beta_1 < \alpha < \beta_2$.

Theorem 3.6. *For arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$ the X-SAGE cone $C_X(\{\alpha_1, \alpha_2\}, \mathcal{B})$ coincides with the nonnegativity cone if and only if $\{\alpha_1, \alpha_2\}$ does not separate $\mathcal{B}$.*

We prove the theorem through the following two lemmas. The first one shows that the condition on $\mathcal{B}$ is sufficient for a nonnegative signomial to be X-SAGE. The second lemma states that if $\mathcal{A}$ separates $\mathcal{B}$ then there are nonnegative signomials which are not X-SAGE.

Lemma 3.7. *Let $f = c_1 e^{\alpha_1} + c_2 e^{\alpha_2} + \sum_{\beta\in\mathcal{B}} d_\beta e^{\beta}$ be a signomial in one variable with $c_i \geq 0$ for $i \in 1, 2$ and $d_\beta < 0$ for $\beta \in \mathcal{B}$ such that $\{\alpha_1, \alpha_2\}$ does not separate $\mathcal{B}$. Then f is nonnegative on X if and only if f is X-SAGE.*

This lemma does not only cover the univariate case of Theorem 1.2, but also the case where the negative exponents are not contained in the convex hull of the positive ones.

Proof. The case $\alpha_1 < \beta < \alpha_2$ for every $\beta \in \mathcal{B}$ is proven in Theorem 1.2. We only need to consider the case, where $\beta < \alpha_1 < \alpha_2$ for every $\beta \in \mathcal{B}$. The case $\alpha_1 < \alpha_2 < \beta$ is analogous by considering $f(-x)$ instead of $f(x)$. For f to be nonnegative on X the set X needs to be bounded in the direction of $-\infty$. Let $a := \inf(X)$ (recall here that X is a subset of $\mathbb{R}$). By Remark 3.2, we can choose without loss of generality $\alpha_1 = 0$. To calculate the minimum of f on X we use the derivative

$$f' = c_2\alpha_2 e^{\alpha_2} + \sum_{\beta\in\mathcal{B}} \underbrace{d_\beta \beta}_{>0} e^{\beta} > 0.$$

Hence, the infimum is attained at a and the signomial is nonnegative on X if and only if

$$f(a) = c_1 + c_2 \exp(\alpha_2 a) + \sum_{\beta\in\mathcal{B}} d_\beta \exp(\beta a) \geq 0.$$

Now we show that this is also the condition for f to be X-SAGE. Calculating f_{SAGE} as in the proof of Theorem 1.2 gives

$$\begin{aligned} f_{\mathrm{SAGE}} = &\inf \langle \boldsymbol{c}, \boldsymbol{v}\rangle + \langle \boldsymbol{d}, \boldsymbol{w}\rangle, \\ &\text{s.t. } \boldsymbol{v} \in \mathbb{R}^{\{\alpha_1,\alpha_2\}}_{>0}, \boldsymbol{w} \in \mathbb{R}^{\mathcal{B}}_{>0} \text{ with } v_0 = 1 \text{ and for all } \beta \in \mathcal{B}\cup\{0\} \text{ there} \\ &\text{exists } z_\beta \in X \text{ such that for all } \alpha \in \{\alpha_1, \alpha_2\} \ \ln\left(\frac{w_\beta}{v_\alpha}\right) \leq (\beta - \alpha) z_\beta. \end{aligned}$$

As in the proof of Theorem 1.2, we choose $\ln(v_\alpha) = \alpha z_0$ to reach the optimal value for f_{SAGE}. For an optimal choice of w_β, following the proof of Theorem 1.2, we consider the optimization problem

$$\max_{z_\beta \in X} \min_{\alpha\in\{\alpha_1,\alpha_2\}} \alpha z_0 + z_\beta(\beta - \alpha). \tag{12}$$

Since $\beta - \alpha$ is negative for both α, the maximum is always attained at $z_\beta = a$. We can restate the problem (12) as

$$\min_{\alpha \in \{\alpha_1, \alpha_2\}} \alpha z_0 + a(\beta - \alpha) = a\beta + \min_{\alpha \in \{\alpha_1, \alpha_2\}} \alpha \underbrace{(z_0 - a)}_{>0}.$$

Since α_2 and $z_0 - a$ are positive numbers, the solution of (12) is given by $a\beta$. This means that the variables w_β are given by $\exp(a\beta)$. We obtain

$$f_{\text{SAGE}} = \inf_{z_0 \in X} c_1 + c_2 \exp(\alpha_2 z_0) + \sum_{\beta \in \mathcal{B}} d_\beta \exp(\beta a).$$

This infimum is attained at $z_0 = a$ because the function is monotone in z_0. Hence, $f_{\text{SAGE}} = f(a)$. This is the desired result because $f \geq 0$ on X if and only if f is X-SAGE. $\square$

Let $\text{rec}(X) = \{y \in \mathbb{R} \mid x + \lambda y \in X \text{ for all } x \in X \text{ and for all } \lambda \geq 0\}$ denote the *recession cone* of X and $\text{rec}(X)^*$ its dual cone. Then the converse direction in Theorem 3.6 follows from the following lemma.

Lemma 3.8. *Let $f = c_1 e^{\alpha_1} + c_2 e^{\alpha_2} + \sum_{\beta \in \mathcal{B}} d_\beta e^\beta$ be a signomial in one variable with $\beta \in \text{conv}\{\alpha_1, \alpha_2\} - \text{rec}(X)^*$ for every $\beta \in \mathcal{B}$. If $\{\alpha_1, \alpha_2\}$ separates $\mathcal{B}$ there exist coefficients $c_1, c_2 > 0$ and d_β, such that $f \geq 0$ on X but $f \notin C_X(\{\alpha_1, \alpha_2\}, \mathcal{B})$.*

The minus sign in $\beta \in \text{conv}\{\alpha_1, \alpha_2\} - \text{rec}(X)^*$ denotes the Minkowski difference. Note that this precondition on every β is a necessary condition for f to be nonnegative on X [22].

Proof. Without loss of generality (using Remark 3.2) we choose $\alpha_1 = 0$ and $\alpha_2 > 0$. The SAGE bound f_{SAGE} from the proof of Theorem 1.2 gives

$$\begin{aligned} f_{\text{SAGE}} = &\inf \langle \boldsymbol{c}, \boldsymbol{v} \rangle + \langle \boldsymbol{d}, \boldsymbol{w} \rangle, \\ &\text{s.t. } \boldsymbol{v} \in \mathbb{R}_{>0}^{\{\alpha_1, \alpha_2\}}, \boldsymbol{w} \in \mathbb{R}_{>0}^{\mathcal{B}} \text{ with } v_0 = 1 \text{ and for all } \beta \in \mathcal{B} \cup \{0\} \text{ there} \\ &\text{exists } z_\beta \in X \text{ such that for all } \alpha \in \{\alpha_1, \alpha_2\} \ \ln\left(\frac{w_\beta}{v_\alpha}\right) \leq (\beta - \alpha) z_\beta. \end{aligned}$$

As in that proof, we choose $\ln(v_\alpha) = \alpha z_0$ to reach the optimal value for f_{SAGE}. For an optimal choice of w_β, we consider the optimization problem

$$\max_{z_\beta \in X} \min_{\alpha \in \{\alpha_1, \alpha_2\}} \alpha z_0 + z_\beta(\beta - \alpha). \tag{13}$$

Since $\{\alpha_1, \alpha_2\}$ separates $\mathcal{B}$, there is at least one $\beta_1 \in \mathcal{B}$ which is not contained in $\text{conv}\{\alpha_1, \alpha_2\}$. We consider the case where $\beta_1 < \alpha_1$. The other possibility, $\beta_1 > \alpha_2$, can be treated similarly. We know from the proof of Lemma 3.7 that $\inf(X) =: a$ exists and $w_{\beta_1} = \exp(a\beta_1)$ is optimal regarding (13). Since $\{\alpha_1, \alpha_2\}$ separates $\mathcal{B}$, there is at least one $\beta_2 > \alpha_1$. We now discuss two cases. If $\beta_2 < \alpha_2$ the optimal (regarding (13)) w_{β_2} is given by $\exp(\beta_2 z_0)$ as seen in the proof of Theorem 1.2. Setting $c_\beta = 0$ for all $\beta \in \mathcal{B} \setminus \{\beta_1, \beta_2\}$ gives

$$\begin{aligned} f_{\text{SAGE}} &= \inf_{z_0 \in X} c_1 + c_2 \exp(\alpha z_0) + d_{\beta_2} \exp(\beta_2 z_0) + d_{\beta_1} \exp(\beta_1 a) \\ &\leq \inf_{z_0 \in X} c_1 + c_2 \exp(\alpha z_0) + d_{\beta_2} \exp(\beta_2 z_0) + d_{\beta_1} \exp(\beta_1 z_0) = \inf_{z_0 \in X} f(z_0). \end{aligned}$$

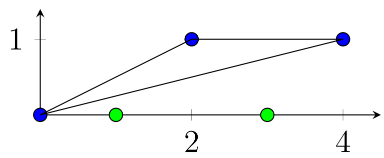

FIGURE 3. The support points of f in Example 3.9

We can choose the coefficients c_1, c_2, d_{β_2} such that the global minimum of $g(x) := c_1 + c_2 \exp(\alpha z_0) + d_{\beta_2} \exp(\beta_2 z_0)$ is $-d_{\beta_1} \exp(\beta_1 a)$ and it is attained in $b \in X \setminus \{a\}$. Therefore, $f_{\mathrm{SAGE}} = 0$. However,

$$\inf_{z_0 \in X} f(z_0) = \inf_{z_0 \in X} \underbrace{(g(z_0) + d_{\beta_1} \exp(\beta_1 a))}_{\geq 0} + \underbrace{(d_{\beta_1} \exp(\beta_1 z_0) - d_{\beta_1} \exp(\beta_1 a))}_{\geq 0} > 0,$$

since both functions are nonnegative on X and attain their minimum at different points. If we now set $c_1 = c_1 - \varepsilon$ with $0 < \varepsilon < \inf_{x \in X} f(x)$, we obtain coefficients with the desired properties.

In the case $\beta_2 > \alpha_2$, the supremum $\sup(X) =: b$ exists and the optimal (regarding (13)) w_{β_2} is given by $w_{\beta_2} = \exp(b(\beta_2 - \alpha_2) + \alpha_2 z_0)$. This is equal to $\exp(\beta_2 z_0)$ if and only if $z_0 = b$ and larger than $\exp(\beta_2 z_0)$ in any other case. Therefore, setting $c_\beta = 0$ for all $\beta \in \mathcal{B} \setminus \{\beta_1, \beta_2\}$ yields

$$\begin{aligned} f_{\mathrm{SAGE}} &= \inf_{z_0 \in X} c_1 + c_2 \exp(\alpha z_0) + d_{\beta_2} \exp(w_{\beta_2}) + d_{\beta_1} \exp(\beta_1 a) \\ &\leq \inf_{z_0 \in X} c_1 + c_2 \exp(\alpha z_0) + d_{\beta_2} \exp(\beta_2 z_0) + d_{\beta_1} \exp(\beta_1 a). \end{aligned}$$

Now we can proceed as in the first case. □

An open question is whether and how Theorem 3.6 can be generalized to the multivariate case. The following counterexample shows that in general it does not suffice that the negative support points are contained in the same polyhedral cell of the natural polyhedral subdivision of $\mathbb{R}^2$, which would be a possible way to generalize the statement.

Example 3.9. Let

$$f \;=\; \exp(4x + y) - 10\exp(3x) + 37\exp(2x + y) - 60\exp(x) + 36$$

and $X = \mathbb{R}^2_+$. The support of f is depicted in Figure 3. For $y = 0$ we obtain

$$f_{y=0} = (\exp(x) - 2)^2(\exp(x) - 3)^2)\,,$$

so that $f_{y=0}$ has double roots at $\ln(2)$ and at $\ln(3)$. For fixed x, the function f is strictly monotonically increasing in y, so the minimum of f on X must be located on the nonnegative x-axis. Hence, the two zeroes of f are the global minimizers. Since f has two isolated roots, it cannot be a SAGE signomial.

Finally, we provide a formulation of Theorem 1.2 in terms of polynomials. Let

$$p = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\beta} \in \mathcal{B}} d_{\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}} \tag{14}$$

where $\mathcal{A} \subseteq \mathbb{N}^n$ is an affinely independent finite set and $\mathcal{B} \subseteq \mathbb{N}^n$ is finite and disjoint from $\mathcal{A}$. Furthermore, let $c_{\boldsymbol{\alpha}} > 0$, $d_{\boldsymbol{\beta}} < 0$ and X be a logarithmically convex subset of $\mathbb{R}^n_{>0}$. We can formulate the corresponding optimization problem

$$
\begin{aligned}
p_{\text{SAGE}}^{\text{Poly}} = &\inf \langle \boldsymbol{c}, \boldsymbol{v} \rangle + \langle \boldsymbol{d}, \boldsymbol{w} \rangle \\
&\text{s.t. } \boldsymbol{v} \in \mathbb{R}^{\mathcal{A}}_{>0}, \boldsymbol{w} \in \mathbb{R}^{\mathcal{B}}_{>0} \text{ with } v_{\boldsymbol{0}} = 1 \text{ and for all } \boldsymbol{\beta} \in \mathcal{B} \cup \{\boldsymbol{0}\} \text{ there} \\
&\text{exists } z_{\boldsymbol{\beta}} \in \ln(X) \text{ such that for all } \boldsymbol{\alpha} \in \mathcal{A} \ \ln\left(\frac{w_{\boldsymbol{\beta}}}{v_{\boldsymbol{\alpha}}}\right) \leq \langle \boldsymbol{\beta} - \boldsymbol{\alpha}, z_{\boldsymbol{\beta}} \rangle.
\end{aligned}
$$

These constraints coming from the relative entropy function are convex constraints. Also note that if X is, for example, a box in $\mathbb{R}^n_{>0}$ then $\ln(X)$ is still a box. In the language of polynomials, Theorem 1.2 can be stated as follows.

Corollary 3.10. *Let $\mathcal{A} \subseteq \mathbb{N}^n$ be the vertices of a simplex and $\mathcal{B} \subseteq \operatorname{conv}(\mathcal{A}) \setminus \mathcal{A}$. The polynomial p in* (14) *is nonnegative on X if and only if $p_{\text{SAGE}}^{\text{Poly}} \geq 0$.*

4. Outlook

In this work, we have provided an exact nonnegativity characterization for a class of signomials whose Newton polytope is a simplex. It remains a general open question to extend our and other existing classes of signomials regarding nonnegativity results. We mention that the papers [7, 17] and, in the symmetric setting [15], study other classes of polynomials or signomials with respect to exactness in the unconstrained case. One specific question is whether those results also extend to constrained settings.

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References

[1] G. Averkov, B. Peters, and S. Sager. Sparse convex relaxations in polynomial optimization. Preprint, arXiv:2403.03560, 2024.
[2] F. Bihan and A. Dickenstein. Descartes' rule of signs for polynomial systems supported on circuits. *Internat. Math. Res. Notices*, 2017(22):6867–6893, 2017.
[3] V. Chandrasekaran and P. Shah. Relative entropy relaxations for signomial optimization. *SIAM J. Optim.*, 26(2):1147–1173, 2016.
[4] M. Dressler and R. Murray. Algebraic perspectives on signomial optimization. *SIAM J. Appl. Algebra Geom.*, 6(4):650–684, 2022.
[5] A. A. Ergür, G. Paouris, and J. M. Rojas. Tropical varieties for exponential sums. *Math. Annalen*, 377:863–882, 2020.
[6] E. Feliu and M. L. Telek. On generalizing Descartes' rule of signs to hypersurfaces. *Adv. Math.*, 408:108582, 2022.
[7] J. Forsgård and T. de Wolff. The algebraic boundary of the SONC cone. *SIAM J. Appl. Algebra Geom.*, 6(3):468–502, 2022.
[8] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, Resultants and Multidimensional Determinants*. Birkhäuser, Boston, 1994.
[9] J. Heuer and T. de Wolff. The duality of SONC: Advances in circuit-based certificates. Preprint, arXiv:2204.03918, 2025.
[10] D. Hilbert. Über die Darstellung definiter Formen als Summe von Formenquadraten. *Math. Annalen*, 32(3):342–350, 1888.
[11] S. Iliman and T. de Wolff. Amoebas, nonnegative polynomials and sums of squares supported on circuits. *Res. Math. Sci.*, 3(paper no. 9), 2016.
[12] S. Iliman and T. de Wolff. Lower bounds for polynomials with simplex Newton polytopes based on geometric programming. *SIAM J. Optim.*, 26(2):1128–1146, 2016.

[13] L. Katthän, H. Naumann, and T. Theobald. A unified framework of SAGE and SONC polynomials and its duality theory. *Math. Comput.*, 90:1297–1322, 2021.

[14] P. Moustrou, H. Naumann, C. Riener, T. Theobald, and H. Verdure. Symmetry reduction in AM/GM-based optimization. *SIAM J. Optim.*, 32(2):765–785, 2022.

[15] P. Moustrou, C. Riener, T. Theobald, and H. Verdure. Symmetric SAGE and SONC forms, exactness and quantitative gaps. *J. Symb. Comp.*, 127:102374, 2025.

[16] R. Murray, V. Chandrasekaran, and A. Wierman. Signomial and polynomial optimization via relative entropy and partial dualization. *Math. Program. Computation*, 13(2):257–295, 2020.

[17] R. Murray, V. Chandrasekaran, and A. Wierman. Newton polytopes and relative entropy optimization. *Found. Comp. Math.*, 21(6):1707–1737, 2021.

[18] R. Murray, H. Naumann, and T. Theobald. Sublinear circuits and the constrained signomial nonnegativity problem. *Math. Program.*, 198:471–505, 2023.

[19] M. Rojas. An introduction to algorithmic fewnomial theory over $\mathbb{R}$. To appear in Notices of the AMS, 2025.

[20] R. Schneider. *Convex Bodies: the Brunn-Minkowski Theory*. Cambridge University Press, 2014.

[21] M. L. Telek. Geometry of the signed support of a multivariate polynomial and Descartes' rule of signs. *SIAM J. Appl. Algebra Geom.*, 8(4):968–1000, 2024.

[22] T. Theobald. Relative entropy methods in constrained polynomial and signomial optimzation. In M. Kočvara, B. Mourrain, C. Riener, editor, *Polynomial Optimization, Moments and Applications*. Springer, 2023.

[23] T. Theobald. *Real Algebraic Geometry and Optimization*. Graduate Studies in Mathematics, vol. 241. Amer. Math. Soc., Providence, RI, 2024.

[24] J. Wang. Nonnegative polynomials and circuit polynomials. *SIAM J. Appl. Algebra Geom.*, 6(2):111–133, 2022.

[25] J. Wang, V. Magron, and J.-B. Lasserre. TSSOS: A moment-SOS hierarchy that exploits term sparsity. *SIAM J. Optim.*, 31(1):30–58, 2021.

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