

Nonnegativity of signomials with Newton simplex over \mathcal{A} -convex sets

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ABSTRACT. We study a class of signomials whose positive support is the set of vertices of a simplex and which may have several negative support points in the simplex. Various groups of authors have provided an exact characterization for the global nonnegativity of a signomial in this class in terms of circuit signomials and that characterization provides a tractable nonnegativity test. We generalize this characterization to the constrained nonnegativity over a set X under an additional convexity precondition in the exponential moment space. This provides a tractable nonnegativity test over X for the class in terms of a power cone program. Our proof methods rely on a variant of the convex cone of constrained SAGE signomials (sums of arithmetic-geometric exponentials) and the duality theory.

1. INTRODUCTION

Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a finite set. A *signomial* (or *exponential sum*) supported on \mathcal{A} is a function of the form

$$f(\mathbf{x}) = \sum_{\alpha \in \mathcal{A}} c_{\alpha} \exp\langle \alpha, \mathbf{x} \rangle,$$

where $c_{\alpha} \in \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ denotes the usual inner product. If all α are nonnegative integer vectors, then the substitution $y_i = \exp(x_i)$ defines a polynomial function $p(\mathbf{y}) = \sum_{\alpha \in \mathcal{A}} c_{\alpha} \mathbf{y}^{\alpha}$ on the nonnegative orthant \mathbb{R}_+^n . To strengthen this connection, every monomial vector $\alpha \in \mathcal{A}$ is associated with a “monomial” basis function e^{α} which takes values $e^{\alpha}(\mathbf{x}) = \exp\langle \alpha, \mathbf{x} \rangle$. From an optimization perspective, for a given signomial $f(\mathbf{x})$ one is interested in the optimization problem

$$\min\{f(\mathbf{x}) \mid \mathbf{x} \in X\},$$

where $X \subseteq \mathbb{R}^n$ is a nonempty feasibility region, given in some reasonable way, say, by a system of explicit inequality constraints. In what follows, we assume that the feasibility region X is convex in a certain non-linear coordinate system (details follow). We refer the reader to [7] and its references for the multifaceted occurrences of signomials in mathematics.

In recent years, investigating sparse settings became an intensely researched area in real algebraic geometry, and in polynomial and signomial optimization. Specifically, in signomial optimization, for given support \mathcal{A} , one considers the space $\mathbb{R}^{\mathcal{A}}$ of all real multivariate signomials supported on \mathcal{A} . The more usual notation for $\mathbb{R}^{\mathcal{A}}$ would be the space of all vectors $(c_{\alpha})_{\alpha \in \mathcal{A}}$, where $c_{\alpha} \in \mathbb{R}$, but we use this notation to denote the respective signomial $f(\mathbf{x})$ defined by these coefficient vectors. An

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initiating moment for modern developments in sparse algebraic geometry was the celebrated BKK theorem, which provides a sparse counterpart of the classical Bézout’s theorem, and the subsequent developments by Gelfand, Kapranov and Zelevinsky, which are summarized in [10], who coined the terminology “ \mathcal{A} -philosophy”, and who investigated the behavior of the space $\mathbb{R}^{\mathcal{A}}$ and structures within like \mathcal{A} -discriminants. Developments on the real algebraic geometry side include the fewnomial theory, initiated by Khovanski, see e.g., [23] for an overview of the recent advances, and exploiting sparsity in a sums of squares context, see e.g., [28], as well as the development of other certificates which perform well in sparse settings; see e.g., [2]. This development is further motivated by the fact that in many real-world applications supports \mathcal{A} of signomials and polynomials are sparse.

As solving signomial optimization problems is NP-hard even in the unconstrained case and notoriously hard to solve in practice, characterizing classes of sparse signomials for which nonnegativity on a convex set X can be decided efficiently and effectively is of ubiquitous interest. In the case of global nonnegativity of polynomials, a prominent classical result of this kind includes Hilbert’s classification [12]. For homogeneous polynomials in two variables, homogeneous quadratic forms or homogeneous ternary quartics, this classification ensures the equality of nonnegative polynomials with sums of squares. From the viewpoint of convex optimization, the global nonnegativity problem can be formulated as a semidefinite program. These techniques, however, do not extend to the case of signomials.

For a given set $\mathcal{A} \subseteq \mathbb{R}^n$ of exponent vectors of a signomial, we say that a set $X \subseteq \mathbb{R}^n$ is \mathcal{A} -convex if there exists $\bar{\alpha} \in \mathcal{A}$ such that the image $\varphi(X)$ of X under the map $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}^n$,

$$\varphi(\mathbf{x}) := \left(\exp\langle \boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}, \mathbf{x} \rangle \right)_{\boldsymbol{\alpha} \in \mathcal{A} \setminus \{\bar{\boldsymbol{\alpha}}\}}$$

is a convex set. If $\mathbf{0} \in \mathcal{A}$ and we choose $\bar{\boldsymbol{\alpha}} = \mathbf{0}$, then the underlying map is simply

$$\varphi(\mathbf{x}) := \left(\exp\langle \boldsymbol{\alpha}, \mathbf{x} \rangle \right)_{\boldsymbol{\alpha} \in \mathcal{A} \setminus \{\mathbf{0}\}},$$

that is, by this φ , we pass from the x_i -variables to the variables $y_i = \exp(x_i)$ and eventually carry out a monomial substitution via $\mathbf{x} \mapsto \mathbf{y} \mapsto (\mathbf{y}^{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathcal{A} \setminus \{\mathbf{0}\}}$.

In this article, we consider the specific case that the Newton polytope of f (that is, the convex hull $\text{conv}(\mathcal{A})$ of its support vectors \mathcal{A}) is a simplex and X is \mathcal{A} -convex. We assume that terms corresponding to non-vertices of $\text{conv}(\mathcal{A})$ have negative coefficients, and all of the coefficients corresponding to vertices of $\text{conv}(\mathcal{A})$ are either positive or unrestricted in sign. Let us investigate the following toy example.

Example 1.1. Consider the signomial

$$f(x, y) = 13 + \exp(4x + 2y) + \exp(2x + 4y) - 12 \exp(x + y) - 3 \exp(2x + 2y),$$

whose support is shown in Figure 1 (upper left). We aim to decide whether f is nonnegative over the triangle $X = \text{conv}\{(-1, 0)^T, (0, -1)^T, (0, 0)^T\}$. It turns out that X is \mathcal{A} -convex. The respective convex set $\varphi(X)$ for $\bar{\boldsymbol{\alpha}} = \mathbf{0}$ is depicted in Figure 1 (upper right). The nonnegativity criterion for this choice of the support and X will be provided in Example 3.4.

Results. In this paper, we provide an exact characterization of nonnegativity of signomials and polynomials on X in the setting described above. In the unconstrained situation $X = \mathbb{R}^n$, the question has been solved in various variants by Ilmanen and

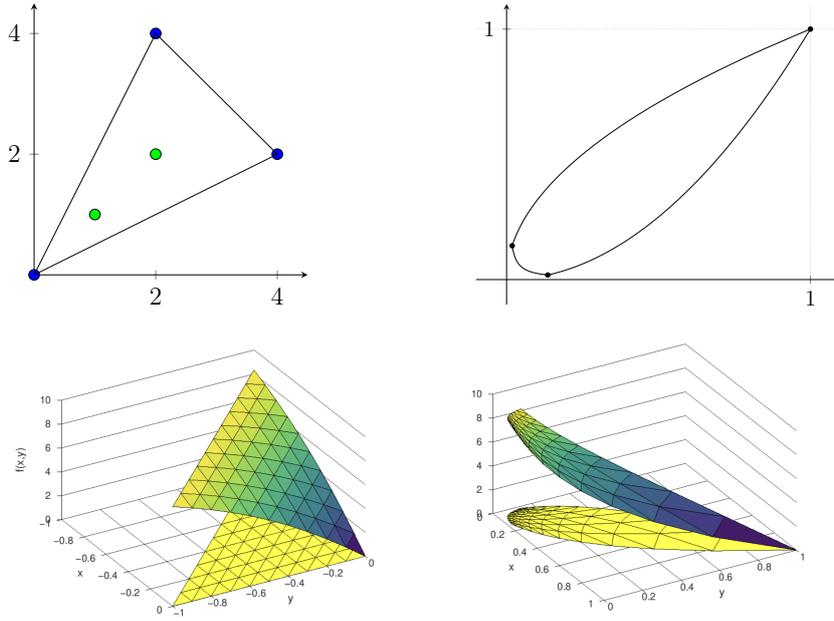


FIGURE 1. The support and the Newton polytope of f (upper left) and the convex set $\varphi(X)$ for $\bar{\alpha} = 0$ (upper right) in Example 1.1, the signomial f over the triangle $X = \text{conv}\{(-1, 0)^T, (0, -1)^T, (0, 0)^T\}$ (lower left) and the respective transformed function $(\exp 4x + 2y, \exp(4x + 2y)) \mapsto f(x, y)$ on $\varphi(X)$.

de Wolff [13], Murray, Chandrasekaran and Wierman [19], and by Wang [27]. All these papers build on methods for nonnegativity certificates based on the arithmetic-geometric inequality (AM/GM inequality), which define a full-dimensional convex subcone of the cone of nonnegative signomials, also known as SONC or SAGE cone; see [13, 5]. Here, we extend this result to the case of \mathcal{A} -convex sets X using a tailored signed variant of the constrained SAGE cone introduced by Murray, Chandrasekaran and Wierman [18]. Signed version means that we distinguish between the positive and the negative coefficients of the terms in the signomials, resulting in replacing \mathcal{A} above by two disjoint support sets \mathcal{A} and \mathcal{B} determining along with X the constrained and sign-restricted SAGE cone $C_X(\mathcal{A}, \mathcal{B})$, which we formally introduce in Section 2. This leads to the following main theorem.

Theorem 1.2. *For the vertex set of a simplex $\mathcal{A} \subseteq \mathbb{R}^n$, a finite set $\mathcal{B} \subseteq \text{conv}(\mathcal{A}) \setminus \mathcal{A}$ and an \mathcal{A} -convex set $X \subseteq \mathbb{R}^n$, consider the signomial*

$$(1) \quad f = \sum_{\alpha \in \mathcal{A}} c_{\alpha} e^{\alpha} + \sum_{\beta \in \mathcal{B}} d_{\beta} e^{\beta} \quad \text{with } c_{\alpha} \in \mathbb{R} \text{ and } d_{\beta} < 0.$$

Then f is nonnegative over X if and only if f has a decomposition of the form

$$f = \sum_{\beta \in \mathcal{B}} f_{\beta} \quad \text{with } f_{\beta}(\mathbf{x}) = \sum_{\alpha \in \mathcal{A}} c_{\alpha}^{(\beta)} e^{\alpha} + d_{\beta} e^{\beta}$$

where, for all $\beta \in \mathcal{B}$, one has $c_{\alpha}^{(\beta)} \in \mathbb{R}$ and f_{β} is nonnegative over X .

Furthermore, if $X = \mathbb{R}^n$, then the assertion remains true with the coefficients $c_{\alpha}^{(\beta)}$ restricted to be nonnegative.

Note that the specific situation $X = \mathbb{R}^n$ is the known unconstrained case as considered above and then, for a nonnegative signomial, necessarily all c_α for $\alpha \in \mathcal{A}$ are nonnegative. As a consequence of the unconstrained case, deciding whether a signomial in this class is nonnegative over \mathbb{R}^n can be formulated as a convex optimization problem, specifically, a relative entropy program. In the constrained case, we present a convex formulation in the exponential moment space in Section 3. Furthermore, the result reveals a hidden convexity structure in the nonnegativity question of our class of constrained signomials. Note that the precondition that X is \mathcal{A} -convex is a significant restriction.

We complement our main result by showing that the condition $\mathcal{B} \subseteq \text{conv}(\mathcal{A}) \setminus \mathcal{A}$ indeed is necessary via providing counterexamples to more general statements; see Lemma 3.11 and Example 3.12. Furthermore, we provide a full characterization of nonnegativity in the univariate case in terms of a separability result; see Theorem 3.8. A formulation of our main result in the language of polynomials is given in Corollary 3.13.

Our result strongly exploits the sign information of the support points. In the study of real polynomials, using sign information of the coefficients has a rich history dating back, in particular, to Descartes' rule of signs. Recently, Bihan and Dickenstein [4] and Feliu and Telek [8, 24] generalized Descartes' rule of signs.

2. PRELIMINARIES

The current study was motivated by the question whether the global SAGE nonnegativity result for Newton simplices can be generalized to a constrained version where X is a subset of \mathbb{R}^n . However, when working over constrained sets X of \mathbb{R}^n , a potential generalization of the unconstrained SAGE characterization in Theorem 1.2 where the SAGE cone is simply replaced by the X -SAGE cone does not hold in general. This is a consequence of Example 2.4 below. To formalize that counterexample and to set up notation, we review some basic concepts on SAGE certificates over a convex set.

2.1. Sparse signomials. From now on, $\mathcal{A} \subseteq \mathbb{R}^n$ is an affinely independent, finite set and $\mathcal{B} \subseteq \mathbb{R}^n$ is finite. Usually \mathcal{A} and \mathcal{B} are disjoint. These sets act as the *positive* and *negative support* of our signomials. We denote by $\mathbb{R}^{\mathcal{A}}$ the $|\mathcal{A}|$ -tuples of \mathbb{R} indexed by \mathcal{A} . The set X is a fixed convex subset of \mathbb{R}^n and without further noticing, we always assume that X is nonempty.

Let f be a general signomial whose coefficients except at most one are positive,

$$(2) \quad f = \sum_{\alpha \in \mathcal{A}} c_\alpha e^\alpha + d e^\beta \text{ with } c_\alpha > 0 \text{ and } d \in \mathbb{R}.$$

For $\beta \in \mathcal{B}$, the set of all these functions which are nonnegative on X form a convex cone called the *constrained β -AGE cone* with respect to \mathcal{A} . We write

$$C_X(\mathcal{A}, \beta) = \left\{ f = \sum_{\alpha \in \mathcal{A}} c_\alpha e^\alpha + d e^\beta \mid f \geq 0 \text{ on } X \text{ and } c_\alpha \geq 0 \text{ for all } \alpha \in \mathcal{A} \right\}.$$

Based on $C_X(\mathcal{A}, \beta)$ we also introduce the *signed X -SAGE cone*

$$(3) \quad C_X(\mathcal{A}, \mathcal{B}) = \sum_{\beta \in \mathcal{B}} C_X(\mathcal{A}, \beta),$$

where the sum denotes the Minkowski sum. By the choice of $C_X(\mathcal{A}, \mathcal{B})$ the exponent vectors of $f \in C_X(\mathcal{A}, \mathcal{B})$ corresponding to negative coefficients form a subset of \mathcal{B} . Signomials in $C_X(\mathcal{A}, \mathcal{B})$ are called *X-SAGE-signomials* with respect to \mathcal{A} and \mathcal{B} . In this terminology, we omit \mathcal{A} and \mathcal{B} if their choice is clear from the context. The signed *X-SAGE* cone provides a convenient notation for expressing nonnegativity certificates based on the AM/GM inequality and used in the context of optimization (see [14, 16, 19]). Note that our definitions are consistent with unsigned versions of the *X-SAGE* cone in the literature (e.g., [18, 26]).

We now explain how to decide membership to the *X-SAGE* cone. The relative entropy function of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}_{>0}^n$ is defined as $D(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^n u_i \ln(u_i/v_i)$. Denote by $\sigma_X(\mathbf{y}) = \sup\{\langle \mathbf{y}, \mathbf{x} \rangle \mid \mathbf{x} \in X\}$ the *support function* of X from classical convex geometry. The function σ_X is a convex function $\mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{\infty\}$. If X is a polyhedron, then σ_X is linear on every normal cone of X . In the following statement, $e\mathbf{c}$ is the usual scalar multiplication of Euler's number e with the vector \mathbf{c} .

Theorem 2.1 ([5, 18]).

1. The signomial f in (2) is nonnegative on \mathbb{R}^n if and only if there exists $\boldsymbol{\nu} \in \mathbb{R}_+^{\mathcal{A}}$ with $\sum_{\alpha \in \mathcal{A}} \nu_\alpha \boldsymbol{\alpha} = (\sum_{\alpha \in \mathcal{A}} \nu_\alpha) \boldsymbol{\beta}$ and $D(\boldsymbol{\nu}, e\mathbf{c}) \leq d$.
2. The signomial f in (2) is nonnegative on X if and only if there exists $\boldsymbol{\nu} \in \mathbb{R}_+^{\mathcal{A}}$ with $\sigma_X(\sum_{\alpha \in \mathcal{A}} \nu_\alpha (\boldsymbol{\beta} - \boldsymbol{\alpha})) + D(\boldsymbol{\nu}, e\mathbf{c}) \leq d$.

This statement can be used to express the convex cones $C_X(\mathcal{A}, \mathcal{B})$ and $C_X(\mathcal{A}, \mathcal{B})$ in terms of the relative entropy function and in terms of the support function of X . As another consequence, if the set $(\mathcal{A}')^T X = \{(\langle \boldsymbol{\alpha}, \mathbf{x} \rangle)_{\boldsymbol{\alpha} \in \mathcal{A}'} \in \mathbb{R}^{\mathcal{A}'} \mid \mathbf{x} \in X\}$ with $\mathcal{A}' := \mathcal{A} \cup \{\boldsymbol{\beta}\}$ is a rational polyhedron, then, by [20], the nonnegativity of f on X can be formulated as a second-order program.

Proposition 2.2 (Theorem 1 and Corollary 1 in [18]). *The constrained β -AGE cone can be expressed as*

$$C_X(\mathcal{A}, \boldsymbol{\beta}) = \left\{ f = \sum_{\alpha \in \mathcal{A}} c_\alpha e^\alpha + d e^\beta \mid \text{There exists } \boldsymbol{\nu} \in \mathbb{R}^{\mathcal{A}} \text{ with} \right. \\ \left. \sigma_X \left(\sum_{\alpha \in \mathcal{A}} \nu_\alpha (\boldsymbol{\beta} - \boldsymbol{\alpha}) \right) + D(\boldsymbol{\nu}, e\mathbf{c}) \leq d \text{ and } c_\alpha \geq 0 \right\}.$$

Combining Proposition 2.2 with (3) gives a characterization $C_X(\mathcal{A}, \mathcal{B})$ in terms of the relative entropy function and the support function of X . For convex sets X , there also exists a theory on the dual constrained SAGE cones, see [5, 15].

Remark 2.3. The SAGE language, which we introduce here, has a counterpart called sums of nonnegative circuit functions, which is usually displayed in a polynomial setting. The main difference is that building blocks there are circuits, i.e., minimally affinely dependent support sets with one negative term, but the two resulting cones are identical up to technicalities detailed by Wang [27] and, independently, Murray, Chandrasekaran, and Wierman [19]. In the circuit setting, nonnegativity is decided by an invariant called circuit number, which is the counterpart of the entropy function in the SAGE language.

Also, in the circuit setting there exists a dual version, see, e.g., [11], and a generalized circuit concept, called sublinear circuits, was developed in [20]. In paragraph 3.4.1 of [2], the duality for SAGE/SONC cones is presented in the self-contained fashion and the ‘‘reversed order’’, starting from the dual SAGE/SONC

cones and deducing the SAGE/SONC cones out of them. This reverse order of deduction works equally well. When a pair of dual cones is considered, calling one of them primal (main) and the other dual (subsidiary) is largely a psychological phenomenon – it is a habit rather than necessity. The derivation in [2] is in the spirit of the \mathcal{A} -convexity definition which we introduced above.

The following example shows that the \mathcal{A} -convexity assumption on X in Theorem 1.2 is crucial and that its assertion is not true for general sets X .

Example 2.4. Consider the convex set $X = \mathbb{R} \times \{0\}$ and $\mathcal{A} = \{\boldsymbol{\alpha}^{(0)}, \boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}\} = \{(0, 0), (2, 1), (4, 0)\}$, $\mathcal{B} = \{\boldsymbol{\beta}^{(1)}, \boldsymbol{\beta}^{(2)}\} = \{(1, 0), (3, 0)\}$. That is, we consider signomials of the form

$$(4) \quad f(x, y) = c_1 + c_2 e^{\boldsymbol{\alpha}^{(1)}}(x, y) + c_3 e^{\boldsymbol{\alpha}^{(2)}}(x, y) + d_1 e^{\boldsymbol{\beta}^{(1)}}(x, y) + d_2 e^{\boldsymbol{\beta}^{(2)}}(x, y).$$

Then f is nonnegative on X if and only if the univariate signomial

$$(5) \quad g(x) := f(x, 0) = c_1 + d_1 e^x + c_2 e^{2x} + d_2 e^{3x} + c_3 e^{4x}$$

is nonnegative over \mathbb{R} . For every choice $z_1 < z_2$ of real numbers, consider the univariate signomial $g(x) = (e^x - e^{z_1})^2 (e^x - e^{z_2})^2$, which has a double zero in z_1 and in z_2 and which is of the form (5). Denoting the coefficients of g by c_1, c_2, c_3, d_1, d_2 (as in (5)), then defining the signomial f as in (4) satisfies $f(x, 1) = g(x)$. Since the zeroes of SAGE signomials are linear subspaces, see [9], g cannot be a SAGE signomial and thus f cannot be contained in the X -SAGE cone.

Moreover, there is a fundamental limitation that rules out the possibility of using the X -SAGE cone for completely characterizing non-negativity in this example. The X -SAGE cone is second-order cone representable, but due to the fact that the choice of z_1, z_2 above is arbitrary, it follows from the main result in [1] that the cone of signomials (4) that are non-negative on X is not second-order cone representable. That is, it is not only the X -SAGE that does not characterize non-negativity in this setting, but any approach that stays within the second-order cone programming paradigm. By the main result of [3], it is possible to characterize non-negativity in this example using linear matrix inequalities of size 3×3 .

3. NONNEGATIVITY OF SIGNOMIALS WITH NEWTON SIMPLEX

In this section we prove our main Theorem 1.2, and discuss further examples and extensions.

Let f be a signomial of the form (2). We can assume that $\mathbf{0} \in \mathcal{A}$. Let Y be the image of X under the map $\varphi : \mathbf{x} = (x_1, \dots, x_n)^T \mapsto (\exp\langle \boldsymbol{\alpha}, \mathbf{x} \rangle)_{\boldsymbol{\alpha} \in \mathcal{A} \setminus \{\mathbf{0}\}}$. We call Y the *exponential moment space* of f .

We introduce the *exponential moment variables* $v_{\boldsymbol{\alpha}} = \exp\langle \boldsymbol{\alpha}, \mathbf{x} \rangle$ and express the problem in terms of the $v_{\boldsymbol{\alpha}}$. Note that $v_{\mathbf{0}} = 1$. As described in terms of the mapping φ , any valid choice of $\mathbf{x} \in X$ determines the vector $\mathbf{v} = (v_{\boldsymbol{\alpha}})$ and vice versa, any valid choice of $(v_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathcal{A} \setminus \{\mathbf{0}\}}$ in $\varphi(X)$ uniquely determines \mathbf{x} . Therefore, we can consider f as a function in the variables $(v_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathcal{A}}$.

For every $\boldsymbol{\beta} \in \text{conv}(\mathcal{A})$, we can write $\boldsymbol{\beta}$ in terms of the *barycentric coordinates*, $\boldsymbol{\beta} = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} \lambda_{\boldsymbol{\alpha}}^{(\boldsymbol{\beta})} \boldsymbol{\alpha}$ with $0 \leq \lambda_{\boldsymbol{\alpha}}^{(\boldsymbol{\beta})} \leq 1$, $\sum_{\boldsymbol{\alpha} \in \mathcal{A}} \lambda_{\boldsymbol{\alpha}}^{(\boldsymbol{\beta})} = 1$. Then it suffices to show the following lemma.

Lemma 3.1. *Let Y be nonempty and convex and \mathcal{A} be the vertices of a simplex, $\mathcal{B} \subseteq \text{conv}(\mathcal{A}) \setminus \mathcal{A}$ and $f(\mathbf{v}) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}} c_{\boldsymbol{\alpha}} v_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\beta} \in \mathcal{B}} d_{\boldsymbol{\beta}} \prod_{\boldsymbol{\alpha} \in \mathcal{A}} v_{\boldsymbol{\alpha}}^{\lambda_{\boldsymbol{\alpha}}^{(\boldsymbol{\beta})}}$ with $c_{\boldsymbol{\alpha}} \in \mathbb{R}$ and*

$d_\beta < 0$. Then $f(\mathbf{v})$ is nonnegative on the exponential moment space Y if and only if f can be written as a sum of nonnegative circuit signomials on the exponential moment space Y ,

$$(6) \quad f = \sum_{\beta \in \mathcal{B}} f_\beta \quad \text{with} \quad f_\beta(\mathbf{v}) = \sum_{\alpha \in \mathcal{A}} c_\alpha^{(\beta)} v_\alpha + d_\beta \prod_{\alpha \in \mathcal{A}} v_\alpha^{\lambda_\alpha^{(\beta)}}$$

with coefficients $c_\alpha^{(\beta)} \in \mathbb{R}$ for all $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$.

Note that in the case $X = \mathbb{R}^n$, the nonnegativity of f implies that all coefficients c_α are nonnegative, and similarly, the nonnegativity of f_β implies that all coefficients $c_\alpha^{(\beta)}$ are nonnegative. These implications do not hold anymore in the restricted situations.

Example 3.2. For an example where the summands require some negative coefficient $c_\alpha^{(\beta)}$, let $\mathcal{A} = \{(0, 0, 0)^T, (1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T\}$, $\mathcal{B} = \{(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})^T, (\frac{1}{5}, \frac{3}{5}, \frac{1}{5})^T\}$ and

$$f = \frac{2}{5} + \frac{3}{5} \exp(x_3) - \frac{1}{2} \exp\left(\frac{3}{5}x_1 + \frac{1}{5}x_2 + \frac{1}{5}x_3\right) - \frac{1}{2} \exp\left(\frac{1}{5}x_1 + \frac{3}{5}x_2 + \frac{1}{5}x_3\right).$$

Setting $x_i = \ln y_i$, we consider $X = \{\mathbf{y} \mid x_1 + x_2 - x_3 = 1\} = \{\mathbf{y} \mid \ln y_1 + \ln y_2 - \ln y_3 = 1\}$. Then the image of the set X under the map $\mathbf{x} \mapsto (\exp\langle \alpha, \mathbf{x} \rangle)_{\alpha \in \mathcal{A} \setminus \{\mathbf{0}\}}$ is the convex set $\{(y_1, y_2, y_3)^T \mid y_i > 0 \text{ for } 1 \leq i \leq 3\}$, hence f is \mathcal{A} -convex. Writing f in the y -variables, denoted \bar{f} , gives

$$\bar{f}(\mathbf{y}) = \frac{2}{5} + \frac{3}{5}y_3 - \frac{1}{2}y_1^{3/5}y_2^{1/5}y_3^{1/5} - \frac{1}{2}y_1^{1/5}y_2^{3/5}y_3^{1/5},$$

and thus the function $f(\mathbf{y})$ is convex on \mathbb{R}_+^3 , where \mathbb{R}_+ denotes the set of nonnegative real numbers. Since $\bar{f}(1, 1, 1) = 0$ and the gradient $\nabla \bar{f}(1, 1, 1) = (-2/5, -2/5, 2/5)^T$ is proportional to the coefficients $(1, 1, -1)^T$ of the affine form defining X , we see that $(1, 1, 1)^T$ is the minimizer of \bar{f} and thus $\mathbf{0}$ is the minimizer of f . Since $\mathbf{0}$ is a root of f , the signomial f is nonnegative. Indeed, a possible decomposition allowing negative coefficients is $f = f_1 + f_2$ with the nonnegative functions

$$\begin{aligned} f_1 &= \frac{1}{10} + \frac{1}{5} \exp(x_1) + \frac{1}{5} \exp(x_3) - \frac{1}{2} \exp\left(\frac{3}{5}x_1 + \frac{1}{5}x_2 + \frac{1}{5}x_3\right), \\ f_2 &= \frac{3}{10} - \frac{1}{5} \exp(x_1) + \frac{2}{5} \exp(x_3) - \frac{1}{2} \exp\left(\frac{1}{5}x_1 + \frac{3}{5}x_2 + \frac{1}{5}x_3\right). \end{aligned}$$

Now assume there were also a decomposition $f = f_1 + f_2$ with

$$\begin{aligned} f_1 &= c_0^{(1)} + c_3^{(1)}x_3 - \frac{1}{2} \exp\left(\frac{3}{5}x_1 + \frac{1}{5}x_2 + \frac{1}{5}x_3\right), \\ f_2 &= c_0^{(2)} + c_3^{(2)}x_3 - \frac{1}{2} \exp\left(\frac{1}{5}x_1 + \frac{3}{5}x_2 + \frac{1}{5}x_3\right) \end{aligned}$$

and nonnegative coefficients $c_0^{(1)}, c_3^{(1)}, c_0^{(2)}, c_3^{(2)}$. In the y -coordinates, we obtain $\nabla \bar{f}_1(1, 1, 1) = (-3/10, -1/10, c_3^{(1)} - 1/10)^T$, which shows that the gradient cannot be proportional to $(1, 1, -1)^T$ and thus f_1 cannot be a nonnegative function with a root at $\mathbf{0}$.

We now introduce Fenchel's duality theorem and use it to proof Lemma 3.1. This theorem is concerned with minimizing the sum of two convex functions, or equivalently, with minimizing the difference of a convex function and a concave function. For each $\beta \in \mathcal{B}$, the function $\prod_{\alpha \in \mathcal{A}} v_{\alpha}^{\lambda_{\alpha}^{(\beta)}}$ is concave because each $\lambda_{\alpha}^{(\beta)}$ satisfies $\lambda_{\alpha}^{(\beta)} \leq 1$. Hence, the function $d_{\beta} \prod_{\alpha \in \mathcal{A}} v_{\alpha}^{\lambda_{\alpha}^{(\beta)}}$ is convex due to $d_{\beta} < 0$. We use the following version from [22, Theorem 31.1] of Fenchel's duality theorem, where we follow the convention from convexity theory to set a convex function g outside of its effective domain $\text{dom } g$ to ∞ and then consider g on the whole ground space \mathbb{R}^n . A convex function $g: \mathbb{R} \cup \{-\infty, \infty\}$ is called *proper* if $g(\mathbf{x}) > -\infty$ for all \mathbf{x} and there exists some \mathbf{x}_0 such that $g(\mathbf{x}_0) < \infty$. Similarly, a concave function h is *proper* if $-h$ is a proper convex function and we set $\text{dom } h = \{\mathbf{x} \mid h(\mathbf{x}) > -\infty\}$. For a convex function g , the *convex conjugate function* is $g^*(\mathbf{y}) = \sup_{\mathbf{x}} (\mathbf{y}^T \mathbf{x} - g(\mathbf{x}))$ and for a concave function h , the *concave conjugate function* is $h^*(\mathbf{y}) = \inf_{\mathbf{x}} (\mathbf{y}^T \mathbf{x} - h(\mathbf{x}))$.

Theorem 3.3 (Fenchel's duality theorem). *Let g be a proper convex function on \mathbb{R}^n and let h be a proper concave function on \mathbb{R}^n . Further assume that $\text{relint}(\text{dom } g) \cap \text{relint}(\text{dom } h) \neq \emptyset$. Then*

$$(7) \quad \inf_{\mathbf{x} \in \mathbb{R}^n} (g(\mathbf{x}) - h(\mathbf{x})) = \sup_{\mathbf{y} \in \mathbb{R}^n} (h^*(\mathbf{y}) - g^*(\mathbf{y})),$$

where g^* is the convex conjugate function of g and h^* is the concave conjugate function of h . Moreover, the supremum is attained at some point $\mathbf{y}^* \in \mathbb{R}^n$.

Proof of Lemma 3.1. First we assume that $|\mathcal{B}| = 2$, that is, f is of the form

$$f(\mathbf{v}) = \sum_{\alpha \in \mathcal{A}} c_{\alpha} v_{\alpha} + d_1 \prod_{\alpha \in \mathcal{A}} v_{\alpha}^{\lambda_{\alpha}} + d_2 \prod_{\alpha \in \mathcal{A}} v_{\alpha}^{\mu_{\alpha}},$$

where we distinguish the two elements of \mathcal{B} by using λ_{α} and μ_{α} as their respective barycentric coordinates. Since $\lambda_{\alpha}, \mu_{\alpha} \in [0, 1]$, the functions $\prod_{\alpha \in \mathcal{A}} v_{\alpha}^{\lambda_{\alpha}}$ and $\prod_{\alpha \in \mathcal{A}} v_{\alpha}^{\mu_{\alpha}}$ are proper concave functions on Y . We write f as a difference of a proper convex and of a proper concave function, $f = g - h$, with

$$g(\mathbf{v}) = \sum_{\alpha \in \mathcal{A}} c_{\alpha} v_{\alpha} + d_1 \prod_{\alpha \in \mathcal{A}} v_{\alpha}^{\lambda_{\alpha}}, \quad h(\mathbf{v}) = -d_2 \prod_{\alpha \in \mathcal{A}} v_{\alpha}^{\mu_{\alpha}}$$

and $\text{dom } g = \text{dom } h = Y$. Let $\inf_{\mathbf{v}} (g(\mathbf{v}) - h(\mathbf{v})) \geq 0$. By Fenchel's duality theorem 3.3, we obtain $\sup_{\mathbf{y}} (h^*(\mathbf{y}) - g^*(\mathbf{y})) = \inf_{\mathbf{v}} (g(\mathbf{v}) - h(\mathbf{v})) \geq 0$, where $g^*(\mathbf{y})$ denotes the convex conjugate and $h^*(\mathbf{y})$ is the concave conjugate.

Since $\text{dom } g = \text{dom } h = Y \neq \emptyset$, we have $\text{relint}(\text{dom } g) = \text{relint}(\text{dom } h) \neq \emptyset$ as well, and hence the supremum is attained in (7). Thus, there exists a vector \mathbf{y}^* such that $h^*(\mathbf{y}^*) - g^*(\mathbf{y}^*) \geq 0$. Hence,

$$\inf_{\mathbf{v}} \left\{ \sum_{\alpha \in \mathcal{A} \setminus \{0\}} y_{\alpha}^* v_{\alpha} - h(\mathbf{v}) \right\} = h^*(\mathbf{y}^*) \geq g^*(\mathbf{y}^*) = \sup_{\mathbf{v}} \left\{ \sum_{\alpha \in \mathcal{A} \setminus \{0\}} y_{\alpha}^* v_{\alpha} - g(\mathbf{v}) \right\}.$$

Setting γ to the finite value $\gamma := -h^*(\mathbf{y}^*)$, we have for all \mathbf{v} that

$$\sum_{\alpha \in \mathcal{A} \setminus \{0\}} y_{\alpha}^* v_{\alpha} + \gamma \geq h(\mathbf{v}).$$

Further, since $\inf_{\mathbf{v}} (-\sum_{\alpha \in \mathcal{A} \setminus \{0\}} y_{\alpha}^* v_{\alpha} + g(\mathbf{v})) \geq \gamma$, we obtain for all \mathbf{x} that

$$g(\mathbf{v}) \geq \sum_{\alpha \in \mathcal{A} \setminus \{0\}} y_{\alpha}^* v_{\alpha} + \gamma.$$

Combining the inequalities gives

$$g(\mathbf{v}) \geq \sum_{\alpha \in \mathcal{A}} y_{\alpha}^* v_{\alpha} + \gamma \geq h(\mathbf{v}).$$

Hence, we have for all \mathbf{v} that

$$f_1(\mathbf{v}) := \sum_{\alpha \in \mathcal{A} \setminus \{\mathbf{0}\}} y_{\alpha}^* v_{\alpha} + \gamma - h(\mathbf{v}) \geq 0$$

and

$$0 \leq g(\mathbf{v}) - \sum_{\alpha \in \mathcal{A} \setminus \{\mathbf{0}\}} y_{\alpha}^* v_{\alpha} - \gamma = \sum_{\alpha \in \mathcal{A} \setminus \{\mathbf{0}\}} (c_{\alpha} - y_{\alpha}^*) v_{\alpha} + (c_{\mathbf{0}} - \gamma) + \prod_{\alpha \in \mathcal{A}} v_{\alpha}^{\lambda_{\alpha}} =: f_2(\mathbf{v}).$$

Since $f_1 + f_2 = f$, this provides the desired decomposition of our nonnegative signomial into two nonnegative circuit signomials.

The case of more than two summands can be successively reduced to two summands by splitting f via $f = g - h$ into a convex function g and a concave function h of the form

$$g(\mathbf{v}) = \sum_{\alpha \in \mathcal{A}} c_{\alpha} v_{\alpha} + \sum_{\beta \in \mathcal{B} \setminus \{\beta_0\}} d_{\beta} \prod_{\alpha \in \mathcal{A}} v_{\alpha}^{\lambda_{\alpha}^{(\beta)}}, \quad h(\mathbf{v}) = -d_{\beta_0} \prod_{\alpha \in \mathcal{A}} v_{\alpha}^{\lambda_{\alpha}^{(\beta_0)}}$$

for some $\beta_0 \in \mathcal{B}$ and then repeatedly splitting g in the same way. \square

Note that the proof of Lemma 3.1 does not require explicit expressions of the conjugate functions.

Using the description in terms of the exponential moment variables, the minimal value of f can be expressed in terms of a convex optimization problem. For $\lambda_1, \dots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$, we denote by $\text{Pow}_{n+1}^{\lambda_1, \dots, \lambda_n}$ a *general power cone*

$$\text{Pow}_{n+1}^{\lambda_1, \dots, \lambda_n} = \{(\mathbf{x}, z) \mid x_1^{\lambda_1} \cdots x_n^{\lambda_n} \geq |z|, x_1, \dots, x_n \geq 0\},$$

see, e.g., [6, 20, 21]. $\text{Pow}_{n+1}^{\lambda_1, \dots, \lambda_n}$ is a convex cone in $(n+1)$ -dimensional space.

The minimal value of f over \mathbb{R}^n can be expressed in terms of the following convex program, where $n = |\mathcal{A} \setminus \{\mathbf{0}\}|$ and $\boldsymbol{\lambda}^{(\beta)} = (\lambda_{\alpha}^{(\beta)})_{\alpha \in \mathcal{A}}$.

$$\begin{aligned} & \inf_{\mathbf{v}=(v_{\alpha})_{\alpha \in \mathcal{A}}, (z_{\beta})_{\beta \in \mathcal{B}}} \sum_{\alpha \in \mathcal{A}} c_{\alpha} v_{\alpha} + \sum_{\beta \in \mathcal{B}} d_{\beta} z_{\beta} \\ & ((v_{\alpha})_{\alpha \in \mathcal{A}}, (z_{\beta})_{\beta \in \mathcal{B}}) \in \text{Pow}_{n+2}^{\boldsymbol{\lambda}^{(\beta)}}, \quad \beta \in \mathcal{B}, \\ & v_{\mathbf{0}} = 1. \end{aligned}$$

If a description of the moment space Y is known, then the decomposition can be computed by expressing the minimal value of f over X as the convex program

$$(8) \quad \begin{aligned} & \inf_{\mathbf{v}=(v_{\alpha})_{\alpha \in \mathcal{A}}, (z_{\beta})_{\beta \in \mathcal{B}}} \sum_{\alpha \in \mathcal{A}} c_{\alpha} v_{\alpha} + \sum_{\beta \in \mathcal{B}} d_{\beta} z_{\beta} \\ & ((v_{\alpha})_{\alpha \in \mathcal{A}}, (z_{\beta})_{\beta \in \mathcal{B}}) \in \text{Pow}_{n+2}^{\boldsymbol{\lambda}^{(\beta)}}, \quad \beta \in \mathcal{B}, \\ & v_{\mathbf{0}} = 1, \\ & (v_{\alpha})_{\alpha \in \mathcal{A} \setminus \{\mathbf{0}\}} \in Y. \end{aligned}$$

Example 3.4. With the main theorem proven, we answer the question from Example 1.1 whether the function

$$f(x, y) = 13 + \exp(4x + 2y) + \exp(2x + 4y) - 12 \exp(x + y) - 3 \exp(2x + 2y)$$

is nonnegative over $X = \text{conv}\{(-1, 0)^T, (0, -1)^T, (0, 0)^T\}$. All the negative support points of f are contained in the convex hull of the positive support points. Therefore,

it suffices to calculate a decomposition as seen in Theorem 1.2. For any $\gamma \in \mathbb{R}$ such that $f - \gamma$ has a decomposition $f - \gamma = f_1 + f_2$ with

$$\begin{aligned} f_1 &= c_0^{(1)} + c_1^{(1)} \exp(4x + 2y) + c_2^{(1)} \exp(2x + 4y) - 12 \exp(x + y) \text{ and} \\ f_2 &= c_0^{(2)} + c_1^{(2)} \exp(4x + 2y) + c_2^{(2)} \exp(2x + 4y) - 3 \exp(2x + 2y) \end{aligned}$$

and $c_j^{(1)} + c_j^{(2)} = c_j$ for $j \in \{1, 2, 3\}$, we can choose f_1 and f_2 to have the same global minimizer. The smallest value of γ for which $f - \gamma$ is globally nonnegative is $\gamma = 7$. The resulting f_1 and f_2 are given by

$$\begin{aligned} f_1(x, y) &= 16 + 0.5 \exp(4x + 2y) + 0.5 \exp(2x + 4y) - 12 \exp(x + y) \text{ and} \\ f_2(x, y) &= 4 + 0.5 \exp(4x + 2y) + 0.5 \exp(2x + 4y) - 3 \exp(2x + 2y). \end{aligned}$$

These signomials share the same global minimizer $(\ln(\sqrt{2}), \ln(\sqrt{2}))$. By calculating the derivative of f_1 in x ,

$$\partial_x f_1(x, y) = 2 \exp(4x + 2y) + \exp(2x + 4y) - 12 \exp(x + y),$$

we can check that for every $x, y \in [-1, 0]$ this is a negative value. This is also the case for the derivative in y since f is symmetric in (x, y) . Therefore, the minimum of f_1 over X is attained at $(0, 0)$. By the same argument, it follows that f_2 and therefore also f attains its minimum over X at $(0, 0)$. This means $\min_{(x,y) \in X} f(x, y) = 0$ and thus f is nonnegative on X .

In the one-dimensional case, the set X is \mathcal{A} -convex if and only if X is convex. The following theorem shows that in the one-dimensional case with positive coefficients c_α , the coefficients $c_\alpha^{(\beta)}$ in the decomposition of Theorem 1.2 can be restricted to be nonnegative. Since in our main setup, \mathcal{A} is the vertex set of a simplex, we have in the one-dimensional case that $|\mathcal{A}| \leq 2$, w.l.o.g. we can assume $|\mathcal{A}| = 2$.

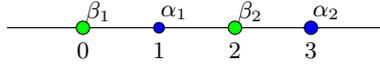
Theorem 3.5. *Let $\mathcal{A} = \{\alpha_1, \alpha_2\} \subseteq \mathbb{R}$, $\mathcal{B} \subseteq \text{conv}(\mathcal{A}) \setminus \mathcal{A}$ and $X \subseteq \mathbb{R}$ convex. Further let f be a univariate signomial of the form (1) with $c_\alpha > 0$ for $\alpha \in \mathcal{A}$ and $d_\beta < 0$ for $\beta \in \mathcal{B}$. Then f is nonnegative over X if and only if there exists a decomposition of the form as in Theorem 1.2 such that the coefficients $c_\alpha^{(\beta)}$ are nonnegative for $\beta \in \mathcal{B}$, that is,*

$$(9) \quad f = \sum_{\beta \in \mathcal{B}} f_\beta \quad \text{with} \quad f_\beta(x) = \sum_{\alpha \in \mathcal{A}} c_\alpha^{(\beta)} e^\alpha + d_\beta e^\beta$$

with coefficients $c_\alpha^{(\beta)} \geq 0$ for all $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ and f_β is nonnegative over X . Hence, f is nonnegative if and only if f is contained in the X -SAGE cone $C_X(\mathcal{A}, \mathcal{B})$.

Proof. We can assume that $\alpha_1 = 0$ and $\alpha_1 < \alpha_2$. The unconstrained case $X = \mathbb{R}$ is well known from [13, 19, 27], therefore we can assume that $\inf X > -\infty$ or $\sup X < \infty$. Without loss of generality it suffices to consider one of these cases, say, $a := \inf X > -\infty$. We can also assume that f has a zero z in the topological closure $\text{cl } X$. By Lemma 3.1, there exists a decomposition $f = \sum_{\beta \in \mathcal{B}} f_\beta$ of the form (6) with real coefficients $c_\alpha^{(\beta)}$. If z is contained in the interior of X , then the multiplicity of z must be at least two and even. Hence, in every f_β , the point z must be a zero of even multiplicity at least two. However, if there exists $\beta \in \mathcal{B}$ and $\alpha \in \mathcal{A}$ with $c_\beta^{(\alpha)} < 0$, then f_β cannot have a zero of order at least two by Descartes' rule of signs.

It remains to consider the case where z is a boundary point of X . Without loss of generality, we can assume $z = \inf X$. We claim that f must be monotone increasing

FIGURE 2. The support points of f in Example 3.6

on X . If f were not monotone increasing on X , then there exists a point $z' \in X \setminus \{z\}$ such that the derivative $f'(z')$ vanishes. Applying Descartes' rule on f' shows that z' must be a simple root of f' . Hence, using also $f(a) = 0$ and $\lim_{x \rightarrow \infty} f(x) = \infty$, there exists some constant $\gamma \in \mathbb{R}$ such that the signomial $f - \gamma$ has at least three zeroes on \mathbb{R} . Since $f - \gamma$ has at most two sign changes in its coefficient sequence, this contradicts Descartes' rule. Hence, f is monotone increasing on X . We now can choose coefficients

$$(10) \quad c_i^{(\beta)} = c_i \frac{-d_\beta \exp(\beta a)}{\sum_{\beta \in \mathcal{B}} -d_\beta \exp(\beta a)} > 0$$

for every $\beta \in \mathcal{B}$ and $i \in \{1, 2\}$. The decomposition $f = \sum_{\beta \in \mathcal{B}} f_\beta$ with

$$(11) \quad f_\beta = c_1^\beta e^{\alpha_1} + c_2^\beta e^{\alpha_2} + d_\beta e^\beta$$

gives the desired decomposition, because, by the same arguments as before, all f_β are monotone and

$$\begin{aligned} f_\beta(a) &= \frac{-d_\beta \exp(\beta a)}{\sum_{\beta \in \mathcal{B}} -d_\beta \exp(\beta a)} (c_1 \exp(\alpha_1 a) + c_2 \exp(\alpha_2 a)) + d_\beta \exp(\beta a) \\ &= \frac{d_\beta \exp(\beta a)}{\sum_{\beta \in \mathcal{B}} d_\beta \exp(\beta a)} \left(f(a) + \sum_{\beta \in \mathcal{B}} (-d_\beta \exp(\beta a)) \right) + d_\beta \exp(\beta a) \\ &= \frac{d_\beta \exp(\beta a)}{\sum_{\beta \in \mathcal{B}} d_\beta \exp(\beta a)} f(a) = 0. \end{aligned}$$

□

We consider the generalization of Theorem 3.5 where the set \mathcal{B} is possibly not contained in $\text{conv}(\mathcal{A}) \setminus \mathcal{A} \subseteq \mathbb{R}$. This also provides a generalization of Theorem 1.2 in the one-dimensional case where positive support points in the interior of the Newton simplex are allowed. If we only consider signomials with at most one negative term, the nonnegativity cone over X clearly coincides with the signed X -SAGE cone by definition.

Example 3.6. We examine the case with two negative terms to show that, even if all $c_\alpha > 0$ for $\alpha \in \mathcal{A} \subseteq \mathbb{R}$, the nonnegativity property of f does not coincide with the existence of a decomposition of the form (1) when there is at least one $\beta \notin \text{conv}(\mathcal{A})$.

Let $\mathcal{A} = \{1, 3\}$ and $f = \sum_{i=0}^3 c_i e^i$ with $c_0, c_2 \leq 0$ and $c_1, c_3 > 0$ on the set $X = \mathbb{R}_+$. The support points of f are shown in Figure 2. We can assume $c_3 = 1$. Consider a subfamily of nonnegative signomials of the previous form which have a double root, say, at $\ln(2)$. This family is a one-dimensional family, and parametrizing it in terms of c_2 gives $f = \exp(3x) + c_2 \exp(2x) + (-12 - 4c_2) \exp(x) + (16 + 4c_2)$. For $c_2 \in (-5, -4)$, the signs of the coefficients are as desired and $f(0) > 0$. Hence, for $c_2 \in (-5, -4)$, the signomial f describes a family of nonnegative signomials satisfying the sign constraints.

However, none of the signomials in this family has a decomposition of the desired form (with $c_\alpha^{(i)} \in \mathbb{R}$). If one of these signomials had such a certificate, then there were a decomposition $f = f_1 + f_2$ with $\text{supp } f_1 \subseteq \{1, 2, 3\}$ and negative coefficient for the exponent 2, as well as $\text{supp } f_2 \subseteq \{0, 1, 3\}$ and negative coefficient for the exponent 0. By Descartes' rule, f_2 cannot be nonnegative with a double root at $\ln(2)$. Hence, for $c_2 \in (-5, -4)$, the signomial f with the double root at $\ln(2)$ does not have a decomposition of the desired form.

Even though, by Example 3.6, in general the signed X -SAGE cone (with X convex and \mathcal{A} -convex) does not coincide with the signed nonnegativity cone over X when $\beta \notin \text{conv}(\mathcal{A})$, there are cases in which those two cones coincide. In the one-dimensional case, we can exactly characterize these cases. In the case $|X| = 1$ (i.e., say, $X = \{a\}$ with some $a \in \mathbb{R}$), a decomposition (10) shows that the nonnegativity cone coincides with the SAGE cone, so we can assume $|X| \geq 2$. We start with the following definition.

Definition 3.7. Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}$ be two sets. We say \mathcal{A} separates \mathcal{B} if there exists $\alpha \in \mathcal{A}$ and $\beta_1, \beta_2 \in \mathcal{B}$ with $\beta_1 < \alpha < \beta_2$.

Let $\text{rec}(X) = \{y \in \mathbb{R} \mid x + \lambda y \in X \text{ for all } x \in X \text{ and for all } \lambda \geq 0\}$ denote the recession cone of X and $\text{rec}(X)^*$ its dual cone. For a convex set X in \mathbb{R} , there are only four possibilities (recall that X is assumed to be nonempty): either $\text{rec}(X) = \mathbb{R}$ (if $X = \mathbb{R}$) or $\text{rec}(X) = \{0\}$ (if X is bounded) or $\text{rec}(X) = [0, \infty)$ or $\text{rec}(X) = (-\infty, 0]$.

Theorem 3.8. For arbitrary $\alpha_1, \alpha_2 \in \mathbb{R}$ and $X \subseteq \mathbb{R}$ convex with $|X| \geq 2$, the X -SAGE cone $C_X(\{\alpha_1, \alpha_2\}, \mathcal{B})$ coincides with the nonnegativity cone if and only if $\beta \in \text{conv}\{\alpha_1, \alpha_2\} - \text{rec}(X)^*$ for all $\beta \in \mathcal{B}$ and $\{\alpha_1, \alpha_2\}$ does not separate \mathcal{B} .

The minus sign in $\beta \in \text{conv}\{\alpha_1, \alpha_2\} - \text{rec}(X)^*$ denotes the Minkowski difference. Note that this precondition on every β is a necessary condition for f to be nonnegative on X [25].

We prove the theorem through the following two lemmas. The first one shows that the condition on \mathcal{B} is sufficient for a nonnegative signomial to be X -SAGE. The second lemma states that if \mathcal{A} separates \mathcal{B} and $\beta \in \text{conv}\{\alpha_1, \alpha_2\} - \text{rec}(X)^*$ for all $\beta \in \mathcal{B}$ then there are nonnegative signomials which are not X -SAGE. Note that in the one-dimensional case, X is convex if and only if X is \mathcal{A} -convex.

Lemma 3.9. Let $f = c_1 e^{\alpha_1} + c_2 e^{\alpha_2} + \sum_{\beta \in \mathcal{B}} d_\beta e^\beta$ be a signomial in one variable with $c_i \geq 0$ for $i \in 1, 2$ and $d_\beta < 0$ for $\beta \in \mathcal{B}$ such that $\{\alpha_1, \alpha_2\}$ does not separate \mathcal{B} . Then f is nonnegative on a convex set X if and only if f is X -SAGE.

This lemma does not only cover the univariate case of Theorem 1.2, but also the case where the negative exponents are not contained in the convex hull of the positive ones.

Proof. The ‘‘if’’ direction is clear, and we only have to show the ‘‘only if’’ direction. The case $\alpha_1 < \beta < \alpha_2$ for every $\beta \in \mathcal{B}$ holds by Lemma 3.5. We only need to consider the case where $\beta < \alpha_1 < \alpha_2$ for every $\beta \in \mathcal{B}$. The case $\alpha_1 < \alpha_2 < \beta$ is analogous by considering $f(-x)$ instead of $f(x)$. For f to be nonnegative on X , the set X needs to be bounded in the direction of $-\infty$. Then $a := \inf(X) \in \mathbb{R}$. By the remarks at the beginning of the section, we can choose without loss of generality

$\alpha_1 = 0$. To calculate the minimum of f on X we use the derivative

$$f' = c_2 \alpha_2 e^{\alpha_2} + \sum_{\beta \in \mathcal{B}} d_\beta \beta e^\beta > 0,$$

because $d_\beta < 0$ and $\beta < 0$. Hence, the infimum is attained at a . We now can choose coefficients as in (10) for every $\beta \in \mathcal{B}$ and $i \in \{1, 2\}$. The decomposition $f = \sum_{\beta \in \mathcal{B}} f_\beta$ with f_β defined as in (11) gives the desired decomposition because all f_β are monotone and we can use the derivation from (10). \square

Remark 3.10. Not all the functions considered in Lemma 3.9 are convex. For example, $f(x) = 1 + \exp(x) - \exp(-x) - \exp(-2x)$ with the set $X = [0, b]$ for any $b \in \mathbb{R}_{>0}$ satisfies all the necessary conditions, but $f''(0) = -4$.

Then the “only if” direction in Theorem 3.8 follows from the following lemma.

Lemma 3.11. *Let $X \subseteq \mathbb{R}$ be convex with $|X| \geq 2$ and assume $\beta \in \text{conv}\{\alpha_1, \alpha_2\} - \text{rec}(X)^*$ for every $\beta \in \mathcal{B}$. If $\{\alpha_1, \alpha_2\}$ separates \mathcal{B} then there exist coefficients $c_1, c_2 > 0$ and $d_\beta \leq 0$ for $\beta \in \mathcal{B}$ such that the univariate signomial*

$$(12) \quad f = c_1 e^{\alpha_1} + c_2 e^{\alpha_2} + \sum_{\beta \in \mathcal{B}} d_\beta e^\beta$$

is nonnegative on X , but $f \notin C_X(\{\alpha_1, \alpha_2\}, \mathcal{B})$.

Proof. Since $\{\alpha_1, \alpha_2\}$ separates \mathcal{B} , we have $|\mathcal{B}| \geq 2$ and we can restrict to a two-element set $|\mathcal{B}| = \{\beta_1, \beta_2\}$ by setting all other d_β to be zero.

We can exclude the case $X = \mathbb{R}$, since in that case the condition $\beta \in \text{conv}\{\alpha_1, \alpha_2\} - \text{rec}(X)^* = \text{conv}\{\alpha_1, \alpha_2\}$ for $\beta \in \mathcal{B}$ implies that $\{\alpha_1, \alpha_2\}$ does not separate \mathcal{B} . Hence, either $\inf(X)$ or $\sup(X)$ is finite. Without loss of generality assume that $a := \inf(X)$ is finite. The separation precondition then allows to assume $\beta_1 < \alpha_1 < \beta_2 < \alpha_2$. Moreover, we can assume $\alpha_1 = 0$.

Let $b \in X \setminus \{a\}$ be an arbitrary point. Set $c_2 = \frac{-1}{\exp(\alpha_2 a) - \exp(\alpha_2 b)} > 0$, $d_{\beta_2} = \frac{2}{\exp(\beta_2 a) - \exp(\beta_2 b)} < 0$ and choose some

$$d_{\beta_1} < \min \left\{ \frac{-(c_2 \exp(\alpha_2 a) + d_{\beta_2} \exp(\beta_2 a))}{\exp(\beta_1 a)}, 0 \right\}.$$

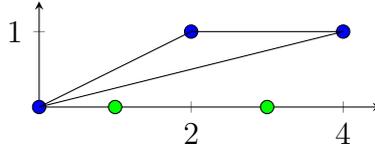
Further set

$$(13) \quad c_1 = - \inf_{x \in X} \{c_2 \exp(\alpha_2 x) + d_{\beta_1} \exp(\beta_1 x) + d_{\beta_2} \exp(\beta_2 x)\} > 0,$$

where the nonnegativity follows from considering the point $x = a$. Since the set X is bounded from below and $\beta_1 < \beta_2 < \alpha_2$, the expression in the argument for the infimum in (13) is finite. Hence, the signomial f in (12) is nonnegative with a zero on $\text{cl } X$.

Now assume that f has an X -SAGE decomposition $f = f_{\beta_1} + f_{\beta_2}$ with $f_{\beta_i} = c_1^{(\beta_i)} + c_2^{(\beta_i)} e^{\alpha_2} + d_{\beta_i} e^{\beta_i}$ and coefficients $c_1^{(\beta_i)}, c_2^{(\beta_i)} > 0$, $1 \leq i \leq 2$. Regardless of the specific values of $c_i^{(\beta_1)}$, the function f_{β_1} is monotone increasing over X , because its derivative is strictly positive. Hence, the signomial f_{β_1} must attain its minimal value on $\text{cl } X$ at the point a . Since f has a zero on $\text{cl } X$, f_{β_1} must also have a zero on $\text{cl } X$ and thus $f_{\beta_1}(a) = 0$. Hence, f_{β_2} must also satisfy $f_{\beta_2}(a) = 0$. However, since

$$f_{\beta_2}(a) > f_{\beta_2}(b) \quad \text{iff} \quad c_2^{(\beta_2)} (\exp(\alpha_2 a) - \exp(\alpha_2 b)) + d_{\beta_2} (\exp(\beta_2 a) - \exp(\beta_2 b)) > 0,$$

FIGURE 3. The support points of f in Example 3.12

we obtain $0 = f_{\beta_2}(a) > f_{\beta_2}(b)$, because every possible choice for $c_2^{(\beta_2)}$ must satisfy $c_2^{(\beta_2)} \leq c_2$. Hence, f_{β_2} cannot be nonnegative. This contradiction shows that f cannot have an X -SAGE-decomposition. \square

An open question is whether and how Theorem 3.8 can be generalized to the multivariate case. The following counterexample shows that in general it does not suffice that the negative support points are contained in the same polyhedral cell of the natural polyhedral subdivision of \mathbb{R}^n , which is given by the facet-defining hyperplanes of the convex hull of \mathcal{A} .

Example 3.12. Let

$$f = \exp(4x + y) - 10 \exp(3x) + 37 \exp(2x + y) - 60 \exp(x) + 36$$

and $X = \mathbb{R}_+^2$. Then X and $\varphi(X)$ are convex. The support of f is depicted in Figure 3. For $y = 0$ we obtain

$$f_{y=0} = (\exp(x) - 2)^2(\exp(x) - 3)^2,$$

so that $f_{y=0}$ has double roots at $\ln(2)$ and at $\ln(3)$. For fixed x , the function f is strictly monotonically increasing in y , so the minimum of f on X must be located on the nonnegative x -axis. Hence, the two zeroes of f are the global minimizers. Since f has two isolated roots, it cannot be a SAGE signomial.

Finally, we provide a formulation of Theorem 1.2 in terms of polynomials. Let

$$(14) \quad p = \sum_{\alpha \in \mathcal{A}} c_{\alpha} \mathbf{x}^{\alpha} + \sum_{\beta \in \mathcal{B}} d_{\beta} \mathbf{x}^{\beta}$$

where $\mathcal{A} \subseteq \mathbb{N}^n$ is an affinely independent finite set and $\mathcal{B} \subseteq \mathbb{N}^n$ is finite and disjoint from \mathcal{A} . Furthermore, let $c_{\alpha} \in \mathbb{R}$, $d_{\beta} < 0$ and X be a logarithmically convex subset of $\mathbb{R}_{>0}^n$. We say that X is \mathcal{A} -monomial convex if there exists some $\bar{\alpha} \in \mathcal{A}$ such that the image of X under the map $\psi : \mathbf{y} \mapsto (\mathbf{y}^{\alpha - \bar{\alpha}})_{\alpha \in \mathcal{A} \setminus \{\bar{\alpha}\}}$ is convex as well. If we know a description of the image Y of X under ψ , then the optimization problem to minimize f over X can be formulated using exactly the convex program (8), where $n = |\mathcal{A} \setminus \{0\}|$ and $\boldsymbol{\lambda}^{(\beta)} = (\lambda_{\alpha}^{(\beta)})_{\alpha \in \mathcal{A}}$ and the $(\lambda_{\alpha}^{(\beta)})$ are the barycentric coordinates given by $\boldsymbol{\beta} = \sum_{\alpha \in \mathcal{A}} \lambda_{\alpha}^{(\beta)} \alpha$.

These constraints coming from the power cones are convex constraints. In the language of polynomials, Theorem 1.2 can be stated as follows.

Corollary 3.13. *Let $\mathcal{A} \subseteq \mathbb{N}^n$ be the vertex set of a simplex, $\mathcal{B} \subseteq \text{conv}(\mathcal{A}) \setminus \mathcal{A}$ finite and $X \subseteq \mathbb{R}_{>0}^n$ be \mathcal{A} -monomial convex. Then the polynomial p in (14) with $c_{\alpha} \in \mathbb{R}$ and $d_{\beta} < 0$ is nonnegative on X if and only if p has a decomposition of the form*

$$p = \sum_{\beta \in \mathcal{B}} p_{\beta} \quad \text{with} \quad p_{\beta}(\mathbf{x}) = \sum_{\alpha \in \mathcal{A}} c_{\alpha}^{(\beta)} \mathbf{x}^{\alpha} + d_{\beta} \mathbf{x}^{\beta}$$

with coefficients $c_{\alpha}^{(\beta)} \in \mathbb{R}$ for all $\beta \in \mathcal{B}$ such that each p_{β} is nonnegative over X .

4. OUTLOOK

In this work, we have provided an exact nonnegativity characterization for a class of signomials whose Newton polytope is a simplex. It remains a general open question to extend our and other existing classes of signomials regarding nonnegativity results. We mention that the papers [9, 19] and, in the symmetric setting [17], study other classes of polynomials or signomials with respect to exactness in the unconstrained case. One specific question is whether those results also extend to constrained settings.

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