THE DUAL CONE OF SUMS OF NON-NEGATIVE CIRCUIT POLYNOMIALS

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Abstract. For a non-empty, finite subset $\mathcal{A} \subseteq \mathbb{N}_0^n$, denote by $C_{\mathrm{sonc}}(\mathcal{A}) \in \mathbb{R}[x_1, \ldots, x_n]$ the cone of sums of non-negative circuit polynomials with support $\mathcal{A}$. We derive a representation of the dual cone $(C_{\mathrm{sonc}}(\mathcal{A}))^*$ and deduce a resulting optimality criterion for the use of sums of non-negative circuit polynomials in polynomial optimization.

1. Introduction

Non-negative polynomials are ubiquitous in real algebraic geometry and occur in many applications (see, e.g., [2, 13]). Whenever a polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ can be written as a sum of squares, then it is clearly non-negative on $\mathbb{R}^n$. Recently, there has been quite some interest in alternative certificates for non-negative polynomials. Iliman and de Wolff introduced the class of sums of non-negative circuit polynomials (SONC) as such an alternative ([9], see also [1, 7, 18]), where non-negativity of a circuit polynomial is characterized in terms of the circuit number (as detailed in Section 2). This approach is closely related to the viewpoint of the arithmetic-geometric inequality and the relative entropy formulation by Chandrasekaran and Shah [4], whose setup is more adapted to the ground set $\mathbb{R}^n_{>0}$ (or, equivalently, to weighted exponential sums). For specific classes of polynomials, testing whether a given polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ can be written as a sum of non-negative circuit polynomials, can be formulated as a geometric program (see [10]) or a relative entropy program (see [18]).

Let $\mathcal{A}$ be a non-empty, finite subset of $\mathbb{N}_0^n$ and $\mathcal{A}^k$ be the set of k-tuples of $\mathcal{A}$. For $k \geq 2$ let

$$\begin{aligned} I_k(\mathcal{A}) \;=\; & \big\{(\alpha(1), \ldots, \alpha(k), \beta) \in \mathcal{A}^{k+1} \,:\, \alpha(1), \ldots, \alpha(k) \in (2\mathbb{N}_0)^n \text{ affinely independent}, \\ & \beta \in \operatorname{relint}(\operatorname{conv}\{\alpha(1), \ldots, \alpha(k)\}) \cap \mathbb{N}_0^n\big\}. \end{aligned}$$

By convention, set $I_1(\mathcal{A}) = \{(\alpha(1)) \in \mathcal{A}^1 \,:\, \alpha(1) \in (2\mathbb{N}_0)^n\}$.

For $A \in I_k(\mathcal{A})$ let $P_{n,A}$ denote the set of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ whose supports are contained in A and which are non-negative on $\mathbb{R}^n$. We can now define the cone of sums of non-negative circuit polynomials (SONC), see [1, 9].

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Definition 1.1. Let $\mathcal{A}$ be a non-empty, finite subset of $\mathbb{N}_0^n$. The Minkowski sum

$$C_{\text{sonc}}(\mathcal{A}) \ = \sum_{A \ \in \ \bigcup_{k=1}^{n+1} I_k(\mathcal{A})} P_{n,A}$$

defines the cone of SONC polynomials whose supports are all contained in $\mathcal{A}$, for short, the *cone of SONC polynomials with support* $\mathcal{A}$.

For any non-empty, finite subset $\mathcal{A} \subseteq \mathbb{N}_0^n$, the set $C_{\text{sonc}}(\mathcal{A})$ is a closed convex cone. Additionally, note that every $p \in \mathbb{R}[x_1, \ldots, x_n]$ which is a sum of non-negative circuit polynomials can indeed be written as a sum $p = \sum_{i=1}^k q_i$ of non-negative circuit polynomials q_i whose supports are all contained in the support of p (Wang [19], cf. also Murray, Chandrasekaran and Wierman [15]).

In this paper, we consider the natural duality pairing between real polynomials $f = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha$ supported on $\mathcal{A}$ and vectors $v \in \mathbb{R}^{\mathcal{A}}$, which is given by

$$v(f) \ = \ \sum_{\alpha \in \mathcal{A}} c_\alpha v_\alpha, \tag{1.1}$$

where the c_α are the coefficients of f. With respect to this pairing, the dual cone $(C_{\text{sonc}}(\mathcal{A}))^*$, is defined as

$$(C_{\text{sonc}}(\mathcal{A}))^* \ = \ \left\{ v \in \mathbb{R}^{\mathcal{A}} \, : \, v(f) \geq 0 \text{ for all } f \in C_{\text{sonc}}(\mathcal{A}) \right\}.$$

We derive a natural description for this dual cone $(C_{\text{sonc}}(\mathcal{A}))^*$, see Theorem 3.1. This description is a variant of the result of Chandrasekaran and Shah who provided a description for the dual SAGE cone (*sums of arithmetic-geometric exponentials* [4]), see Section 2 for a formal definition. For the special case of univariate quartics, we provide a quantifier-free representation in terms of polynomial inequalities (see Corollary 3.8). Building upon the characterization of the dual SONC cone, we then deduce a corresponding sufficient optimality criterion for the SONC approach in polynomial optimization, see Theorem 4.2.

Beyond the specific results, the purpose of the paper is to provide additional understanding of the interplay of the SONC and SAGE cones as well as the interplay of the circuit number in the SONC approach, the relative entropy function underlying the SAGE approach and the exponential cone from the theory of optimization.

We remark that polynomial optimization techniques based on the SONC cone can generally be combined with those based on the cone of sums of squares (see [1] and [11]).

The paper is structured as follows. In Section 2, we review the connection between the circuit number and relative entropy programs. In Section 3, we derive the description of the dual SONC cone and consider in detail the dual cone for the specific case of univariate quartics. Section 4 applies the characterization on the SONC-based lower bounds in optimization and provides a sufficient optimization criterion.

2. The circuit number and relative entropy programs

Non-negative circuit polynomials can be characterized either in terms of circuit numbers or in terms of the relative entropy function. The sets $\mathbb{R}_{>0}$, $\mathbb{R}_+$, $\mathbb{R}_-$, $\mathbb{R}_{\neq 0}$ denote the positive, non-negative, non-positive and non-zero real numbers, respectively.

A *circuit polynomial* $p \in \mathbb{R}[x_1, \ldots, x_n]$ is a polynomial of the form $p(x) = \sum_{i=1}^{k} c_i x^{\alpha(i)} + \delta x^{\beta}$, with $k \leq n+1$, coefficients $c_i \in \mathbb{R}_{>0}$, $\delta \in \mathbb{R}$, and exponents $\alpha(1), \ldots, \alpha(k) \in (2\mathbb{N}_0)^n$ being affinely independent and $\beta \in \mathbb{N}_0^n$, such that $\beta \in \operatorname{relint}(\operatorname{conv}\{\alpha(1), \ldots, \alpha(k)\})$. The *circuit number* Θ_p of p is defined as $\Theta_p = \prod_{i=1}^{k} \left(\frac{c_i}{\mu_i}\right)^{\mu_i}$, where $\mu \in \mathbb{R}_{>0}^k$ denotes the barycentric coordinates of β with respect to $\alpha(1), \ldots, \alpha(k)$, i.e., $\sum_{i=1}^{k} \mu_i = 1$ and $\beta = \sum_{i=1}^{k} \mu_i \alpha(i)$. Note that since $\operatorname{relint}\{\alpha(1)\} = \{\alpha(1)\}$, these definitions formally also make sense for $k = 1$, but notice that in this case $\beta = \alpha(1)$.

The *relative entropy function* D is defined as $\mathbb{R}_{>0}^n \times \mathbb{R}_{>0}^n \to \mathbb{R}$,

$$D(\nu, \lambda) \ = \ \sum_{j=1}^{n} \nu_j \log\left(\frac{\nu_j}{\lambda_j}\right), \quad \nu, \lambda \in \mathbb{R}_{>0}^n$$

and it can be continuously extended to $\mathbb{R}_+^n \times \mathbb{R}_{>0}^n \to \mathbb{R}$ (see [5]).

On the set $\mathbb{R}^n$, non-negativity of a circuit polynomial has been characterized by Iliman and de Wolff in terms of the circuit number [9]. On the set $\mathbb{R}_{>0}^n$, non-negativity of a circuit polynomial has been characterized by Chandrasekaran and Shah [4] in terms of the relative entropy function. Theorems 2.1 and 2.3 review these statements in a uniform way (and thus slightly extend them). In particular, the proofs of these statements exhibit how to transfer from the circuit characterization to the relative entropy characterization and vice versa. Let e be Euler's number and $\mathbf{1}$ denote the all-ones-vector.

Theorem 2.1. *For a circuit polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ with $p(x) = \sum_{i=1}^{k} c_i x^{\alpha(i)} + \delta x^{\beta}$, the following statements are equivalent.*

1. *p is a circuit polynomial which is non-negative on $\mathbb{R}_+^n$.*
2. *$\delta \geq -\Theta_p$.*
3. *There exists some $\nu \in \mathbb{R}_+^k$ such that $\sum_{i=1}^{k} \alpha(i)\nu_i = (\mathbf{1}^T\nu)\beta$ and $D(\nu, e \cdot c) \leq \delta$.*

The existential quantification in condition (3) is essential for its algorithmic use (see [4]). However, for the purpose of our analysis, it is useful to characterize for which $\nu \in \mathbb{R}_+^k$ the entropy function $D(\nu, e \cdot c)$ in the condition of (3) actually takes its minimum.

Lemma 2.2. *Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a circuit polynomial. On the set $\{\nu \in \mathbb{R}_+^k : \sum_{i=1}^{k} \alpha(i)\nu_i = (\mathbf{1}^T\nu)\beta\}$, the function $\nu \mapsto D(\nu, e \cdot c)$ ($\nu \in \mathbb{R}_+^k$) takes its minimum value at $e^{-D(\mu,c)}\mu$, where μ denotes the barycentric coordinates of β w.r.t. $\alpha(1), \ldots, \alpha(k)$.*

Proof. Let μ be the barycentric coordinates of β with respect to $\alpha(1), \ldots, \alpha(k)$, and consider the function $g : \nu \mapsto D(\nu, ec) = \sum_{i=1}^{k} \nu_i \log \frac{\nu_i}{e \cdot c_i}$. We ask for which $\rho \geq 0$ the function

$$h(\rho) \ = \ g(\rho\mu) \ = \ \sum_{i=1}^{k} \rho\mu_i \cdot \log\left(\frac{\rho\mu_i}{e \cdot c_i}\right)$$

is minimized. Its derivative is

$$\begin{aligned} h'(\rho) &= \sum_{i=1}^{k} \left(\mu_i \cdot \log\left(\frac{\rho\mu_i}{e \cdot c_i}\right) + \rho\mu_i \cdot \frac{1}{\rho} \right) \\ &= \log\rho + 1 + \sum_{i=1}^{k} \mu_i \log\left(\frac{\mu_i}{e \cdot c_i}\right) \\ &= \log\rho + D(\mu, c), \end{aligned}$$

where we used $\sum_{i=1}^{k} \mu_i = 1$. The derivative becomes zero for

$$\log\rho = -D(\mu, c),$$

and due to $h''(\rho) = 1/\rho$, we obtain $h''(\rho^*) > 0$ for the root ρ^* of $h'(\rho)$. Hence, ρ^* is a minimum and $\rho^*\mu$ minimizes g. □

Proof of Theorem 2.1. The equivalence of (1) and (3) is well-known (see [4, Lemma 2.2]). We show the equivalence of (2) and (3).

Let μ be the barycentric coordinates of β w.r.t. $\alpha(1), \ldots, \alpha(k)$. By Lemma 2.2, on the set $\{\nu \in \mathbb{R}_+^k : \sum_{i=1}^{k} \alpha(i)\nu_i = (\mathbf{1}^T\nu)\beta\}$, the function $D(\nu, e \cdot c)$ is minimized at $\rho\mu$ where $\rho = e^{-D(\mu,c)}$. Hence, the entropy condition in (3) is equivalent to

$$D(e^{-D(\mu,c)}\mu, e \cdot c) \ \leq \ \delta,$$

which can be rewritten as

$$e^{-D(\mu,c)} \sum_{i=1}^{k} \left(\mu_i \log \frac{e^{-D(\mu,c)}\mu_i}{e \cdot c_i} \right) \ \leq \ \delta. \tag{2.1}$$

Since

$$\sum_{i=1}^{k} \left(\mu_i \log \frac{e^{-D(\mu,c)}\mu_i}{e \cdot c_i} \right) = \sum_{i=1}^{k} \mu_i \left(\log \frac{\mu_i}{c_i} + \log e^{-D(\mu,c)} + \log \frac{1}{e} \right) = D(\mu, c) - D(\mu, c) - 1,$$

(2.1) is equivalent to

$$-e^{-D(\mu,c)} \ \leq \ \delta$$

and thus to $-\prod_{i=1}^{k} \left(\frac{c_i}{\mu_i}\right)^{\mu_i} \ \leq \ \delta$, which is exactly the circuit condition (2). □

Theorem 2.3. *For a circuit polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$, the following statements are equivalent.*

(1) *p is a non-negative circuit polynomial, i.e., a circuit polynomial which is non-negative on $\mathbb{R}^n$.*
(2) $|\delta| \leq \Theta_p$ *and* $\beta \notin (2\mathbb{N}_0)^n$ *or* $\delta \geq -\Theta_p$ *and* $\beta \in (2\mathbb{N}_0)^n$.
(3) *There exists some $\nu \in \mathbb{R}_+^k$ such that $\sum_{i=1}^{k} \alpha(i)\nu_i = (\mathbf{1}^T\nu)\beta$ and*

$$D(\nu, e \cdot c) \leq -|\delta| \text{ and } \beta \notin (2\mathbb{N}_0)^n \quad \text{or} \quad D(\nu, e \cdot c) \leq \delta \text{ and } \beta \in (2\mathbb{N}_0)^n .$$

The equivalence of (1) and (2) was already shown by Iliman and de Wolff [9]. Here, we deduce Theorem 2.3 as a consequence of Theorem 2.1.

Proof. If $\beta \in (2\mathbb{N}_0)^n$, then the statement coincides with Theorem 2.1. If $\beta \notin (2\mathbb{N}_0)^n$, then there exists at least one index j such that β_j is odd. Fix such an index j. Since $\alpha(1), \ldots, \alpha(k)$ are even, p is non-negative on $\mathbb{R}^n$ if and only if p is non-negative both on $\mathbb{R}^n_+$ and on the orthant $T := \{x \in \mathbb{R}^n \ : \ x_j \leq 0, \ x_i \geq 0 \text{ for all } i \neq j\}$. And this is equivalent to p being non-negative on $\mathbb{R}^n_+ \cup T$. Since p is non-negative on T if and only if $p^- := \sum_{i=1}^k c_i x^{\alpha(i)} - \delta x^\beta$ is non-negative on $\mathbb{R}^n_+$, the equivalence of (1) and (2) (respectively of (1) and (3)) follows by applying the equivalence of (1) and (2) (respectively of (1) and (3)) in Theorem 2.1 twice. □

Example 2.4. Let $p = 1 + x^2y^4 + x^4y^2 + \delta x^2y^2$ with $\delta \in \mathbb{R}$. The circuit number Θ_p of p is

$$\left(\frac{1}{1/3}\right)^{1/3} \cdot \left(\frac{1}{1/3}\right)^{1/3} \cdot \left(\frac{1}{1/3}\right)^{1/3} \ = \ 3\,.$$

By Theorem 2.3, p is non-negative on $\mathbb{R}^n$ if and only if $\delta \geq -3$. In the case $\delta = -3$, the polynomial p is recognized as a Motzkin polynomial (see, e.g., [16]).

The cones SONC and SAGE. The SONC cone, as defined in the Introduction, is closely related to the SAGE cone (sums of arithmetic-geometric exponentials) introduced in [4]. Let $\mathcal{A}$ be a nonempty, finite subset of $\mathbb{R}^n$ (rather than only of $\mathbb{N}_0^n$). For notational consistency with our definition of the SONC cone, we provide here a definition of the SAGE cone in the language of *real-exponent polynomials supported on* $\mathcal{A}$, which are sums of the form $\sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha$ with real coefficients c_α and exponent vectors in $\mathcal{A}$.

Let $Q_{n,\mathcal{A}}$ denote the set of real-exponent polynomials supported on $\mathcal{A}$, which have at most one negative coefficient and which are non-negative on $\mathbb{R}^n_{>0}$. Then we define the SAGE cone $C_{\text{sage}}(\mathcal{A})$ as the set of finite sums of real-exponent polynomials in $Q_{n,\mathcal{A}}$. Note that, in particular, the circuit polynomials satisfying the conditions from Theorem 2.1 are contained in $C_{\text{sage}}(\mathcal{A})$.

Using the duality pairing (1.1) from the Introduction, the following description of the dual SAGE cone is known.

Proposition 2.5. [4] *The dual cone* $(C_{\text{sage}}(\mathcal{A}))^*$ *is the set*

$$\left\{ v \in \mathbb{R}^l_+ \ : \ \exists \tau(j) \in \mathbb{R}^n, j = 1, \ldots, l \ \text{ s.t. } v_i \log \frac{v_i}{v_j} \leq (\alpha(i) - \alpha(j))^T \tau(i) \, \forall i, j \right\},$$

where the settings $0 \cdot \log \frac{0}{y} = 0$, $y \cdot \log \frac{y}{0} = \infty$ *for* $y > 0$ *and* $0 \cdot \log \frac{0}{0} = 0$ *are used.*

3. The dual cone

We study the dual SONC cone $(C_{\text{sonc}}(\mathcal{A}))^*$. Let $\operatorname{cl} S$ be the topological closure of a set S. We show:

Theorem 3.1. *The dual cone* $(C_{\text{sonc}}(\mathcal{A}))^*$ *is*

$$\Big\{ (v_\alpha)_{\alpha \in \mathcal{A}} \ \big| \ v_\alpha \geq 0 \text{ for } \alpha \in \mathcal{A} \cap (2\mathbb{N}_0)^n \ \wedge \ \forall k \geq 2 \text{ and } (\alpha(1), \ldots, \alpha(k), \beta) \in I_k(\mathcal{A}) \ : \\ \exists v^* \geq |v_\beta| \ \exists \tau \in \mathbb{R}^n \text{ with } v^* \log \frac{v^*}{v_{\alpha(j)}} \leq (\beta - \alpha(j))^T \tau, \ 1 \leq j \leq k \Big\},$$

where we use the settings $0 \cdot \log \frac{0}{y} = 0$, $y \cdot \log \frac{y}{0} = \infty$ *for* $y > 0$ *and* $0 \cdot \log \frac{0}{0} = 0$.

We immediately obtain the following corollary for the dual of the cone of non-negative polynomials of total degree at most d, where we set $\mathcal{A}_d = \{\alpha \in \mathbb{N}_0^n : |\alpha| \leq d\}$ and $C_{\text{sonc}}(d) = C_{\text{sonc}}(\mathcal{A}_d)$.

Corollary 3.2. *For any number of variables* n *and any even* $d \geq 2$*, the dual cone* $(C_{\text{sonc}}(d))^*$ *is*

$$\Big\{(v_\alpha)_{|\alpha|\leq d} \mid v_\alpha \geq 0 \text{ for } \alpha \in \mathcal{A}_d \cap (2\mathbb{N}_0)^n \wedge \forall k \geq 2 \text{ and } (\alpha(1), \ldots, \alpha(k), \beta) \in I_k(\mathcal{A}_d) : \\ \exists v^* \geq |v_\beta| \ \exists \tau \in \mathbb{R}^n \text{ with } v^* \log \frac{v^*}{v_{\alpha(j)}} \leq (\beta - \alpha(j))^T \tau,\ 1 \leq j \leq k\Big\}.$$

A main step in the proof of Theorem 3.1 is to capture the case of a single circuit, this is the content of Lemma 3.6 below. The remaining part of the proof of Theorem 3.1 will then follow from elementary convex geometry. Let us also point out that the main difference of the proof of Theorem 3.1 in comparison to Theorem 2.5 does not only come from the circuits (which are not present in the definition of the SAGE cone), but also from the non-negativity on the ground set $\mathbb{R}^n$ (consisting of 2^n orthants) rather than only on the orthant $\mathbb{R}^n_{>0}$, which leads to the more involved statement.

To prepare the proof of the circuit case, we start from the well-known exponential cone (see, e.g., [3, §6.3.4]). Setting

$$K = \{(x, y, z) \in \mathbb{R}^3 : y \cdot e^{x/y} \leq z,\ y > 0\},$$

the *exponential cone* is defined as

$$K_{\exp} = \operatorname{cl} K = K \cup (\mathbb{R}_- \times \{0\} \times \mathbb{R}_+).$$

$K_{\exp}$ is a closed convex cone with nonempty interior.

The following characterization for the relative entropy function D is well known, and it shows that the *relative entropy cone* $\operatorname{cl}\{(\nu, \lambda, \delta) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R} : D(\nu, \lambda) \leq \delta\}$ can be viewed as a reparametrization of the exponential cone (see, e.g., [5]).

Proposition 3.3. *For* $\nu, \lambda > 0$ *and* $\delta \in \mathbb{R}$ *we have* $D(\nu, \lambda) \leq \delta$ *if and only if* $(-\delta, \nu, \lambda) \in K_{\exp}$.

For the sake of completeness, we provide a short proof.

Proof. By definition of the entropy function, $D(\nu, \lambda) \leq \delta$ if and only if $\nu \log \frac{\nu}{\lambda} \leq \delta$. Applying the exponential function on both sides and taking the ν-th root on both sides gives

$$\frac{\nu}{\lambda} \leq (e^\delta)^{1/\nu} = \exp\left(\frac{\delta}{\nu}\right).$$

This is equivalent to $\nu \exp(-\delta/\nu) \leq \lambda$, i.e., to $(-\delta, \nu, \lambda) \in K_{\exp}$. □

The dual of the exponential cone is

$$(3.1)\quad \begin{aligned} (K_{\exp})^* &= \operatorname{cl}\{(a, b, c) \in \mathbb{R}_{<0} \times \mathbb{R} \times \mathbb{R}_+,\ c \geq -a \cdot e^{b/a-1}\} \\ &= \{(a, b, c) \in \mathbb{R}_{<0} \times \mathbb{R} \times \mathbb{R}_+,\ c \geq -a \cdot e^{b/a-1}\} \cup (\{0\} \times \mathbb{R}_+ \times \mathbb{R}_+) \end{aligned}$$

(see, e.g., [6, Theorem 4.3.3]).

Lemma 3.4. (1) *The dual cone of*

$$\begin{aligned} C &= \mathrm{cl}\{(\nu, c, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{>0} \times \mathbb{R} \,:\, D(\nu, ec) \leq \delta\} \\ &= \{(\nu, c, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{>0} \times \mathbb{R} \,:\, D(\nu, ec) \leq \delta\} \cup (\{0\} \times \mathbb{R}_+ \times \mathbb{R}_+) \end{aligned}$$

is the convex cone

$$C^* = \mathrm{cl}\left\{(r, s, t) \in \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \,:\, t \log \frac{t}{s} \leq r\right\}.$$

(2) *The dual cone of* $\mathrm{cl}\,\{(\nu, c, \delta) \in \mathbb{R}_+ \times \mathbb{R}_{>0} \times \mathbb{R} \,:\, D(\nu, ec) \leq -|\delta|\}$ *is the convex cone*

$$\mathrm{cl}\left\{(r, s, t) \in \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}_{\neq 0} \,:\, \exists t^* \geq |t| \text{ with } t^* \log \frac{t^*}{s} \leq r\right\}. \tag{3.2}$$

Proof. By Proposition 3.3, we have $C = \{(\nu, c, \delta) \,:\, (-\delta, \nu, ec) \in K_{\exp}\}$. Hence, by (3.1), the dual cone is

$$\begin{aligned} C^* &= \left\{(r, s, t) \,:\, \left(-t, r, \frac{s}{e}\right) \in K^*_{\exp}\right\} \\ &= \mathrm{cl}\left\{(r, s, t) \in \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \,:\, \frac{s}{e} \geq t e^{\frac{r}{-t} - 1}\right\}. \end{aligned}$$

Since $\frac{s}{e} \geq t e^{\frac{r}{-t} - 1}$ is equivalent to $\log \frac{s}{t} \geq \frac{r}{-t}$, and thus equivalent to $t \log \frac{t}{s} \leq r$, the statement (1) follows.

Applying (1) with respect to $-\delta$ rather than δ gives the auxiliary dual cone

$$(C_-)^* = \mathrm{cl}\left\{(r, s, t) \in \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}_{<0} \,:\, (-t) \log \frac{-t}{s} \leq r\right\}. \tag{3.3}$$

Using the general formula $(C_1 \cap C_2)^* = C_1^* + C_2^*$ for two closed convex cones C_1 and C_2 (see, e.g., [17]), it then remains to show that the Minkowski sum of C^* and $(C_-)^*$ equals (3.2). Since $C^* := \{(r_1, s_1, t_1) \,:\, (r_1, s_1, -t_1) \in (C_-)^*\}$, the Minkowski sum $C^* + (C_-)^*$ consists of the closure of all the points $(r, s, t) \in \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R}$ such that

$$\begin{aligned} &(t > 0 \text{ and } \exists t_1 \geq t \text{ with } t_1 \log \tfrac{t_1}{s} \leq r) \\ \text{or} \quad &(t < 0 \text{ and } \exists t_2 \leq t \text{ with } (-t_2) \log \tfrac{-t_2}{s} \leq r). \end{aligned}$$

This gives the desired dual in (3.2). □

We obtain the following multivariate version:

Lemma 3.5. (1) *The dual cone of* $\mathrm{cl}\{(\nu, c, \delta) \in \mathbb{R}^n_+ \times \mathbb{R}^n_{>0} \times \mathbb{R} \,:\, D(\nu, ec) \leq \delta\}$ *is*

$$\mathrm{cl}\left\{(r, s, t) \in \mathbb{R}^n \times \mathbb{R}^n_{>0} \times \mathbb{R}_{>0} \,:\, t \log \frac{t}{s_j} \leq r_j \text{ for } 1 \leq j \leq n\right\}.$$

(2) *The dual cone of* $\mathrm{cl}\{(\nu, c, \delta) \in \mathbb{R}^n_+ \times \mathbb{R}^n_{>0} \times \mathbb{R} \,:\, D(\nu, ec) \leq -|\delta|\}$ *is*

$$\mathrm{cl}\left\{(r, s, t) \in \mathbb{R}^n \times \mathbb{R}^n_{>0} \times \mathbb{R}_{\neq 0} \,:\, \exists t^* \geq |t| \text{ with } t^* \log \frac{t^*}{s_j} \leq r_j \text{ for } 1 \leq j \leq n\right\}.$$

Proof. The cone $C = \mathrm{cl}\{(\nu, c, \delta) : D(\nu, ec) \le \delta\}$ can be interpreted as a Minkowski sum $\sum_{i=1}^n C_i$, where $C_i \subseteq \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}$ is given by embedding $\{(\nu_i, c_i, \delta) : D(\nu_i, ec_i) \le \delta\}$ into the corresponding coordinates of (ν, c, δ), that is,

$$C_i \;=\; \mathrm{cl}\left\{(\nu_i e^{(i)}, c_i e^{(i)}, \delta) \in \mathbb{R}_+^n \times \mathbb{R}_{>0}^n \times \mathbb{R} \;:\; \nu_i \log\left(\frac{\nu_i}{ec_i}\right) \le \delta\right\},$$

where $e^{(i)}$ is the i-th unit vector. Hence, C_i^* is known from Lemma 3.4(1), and using $(\sum_{i=1}^n D_i)^* = \bigcap_{i=1}^n D_i^*$ for any closed convex cones D_i (see [17]) proves the first statement.

For the second statement, combine the first statement with Lemma 3.4(2). $\square$

For affinely independent $\alpha(1), \ldots, \alpha(k) \in (2\mathbb{N}_0)^n$ and $\beta \in \mathrm{relint}(\mathrm{conv}\{\alpha(1), \ldots, \alpha(k)\}) \cap \mathbb{N}_0^n$, denote by

$$C_{nc}(\alpha(1), \ldots, \alpha(k), \beta) \;=\; \left\{(c_1, \ldots, c_k, \delta) \in \mathbb{R}_+^k \times \mathbb{R} \;:\; \sum_{i=1}^k c_i x^{\alpha(i)} + \delta x^\beta \ge 0 \text{ on } \mathbb{R}^n\right\}$$

the cone of non-negative circuit polynomials with support contained in $(\alpha(1), \ldots, \alpha(k), \beta)$.

Lemma 3.6. *Let $k \ge 2$, $\alpha(1), \ldots, \alpha(k) \in (2\mathbb{N}_0)^n$ be affinely independent and $\beta \in \mathrm{relint}(\mathrm{conv}\{\alpha(1), \ldots, \alpha(k)\}) \cap \mathbb{N}_0^n$. The dual cone of $C_{\mathrm{nc}}(\alpha(1), \ldots, \alpha(k), \beta)$ is*

$$\begin{cases} \mathrm{cl}\left\{(v, v_0) \in \mathbb{R}_{>0}^k \times \mathbb{R}_{>0} \;:\; \exists \tau \in \mathbb{R}^n \text{ with } v_0 \log \frac{v_0}{v_j} \le (\beta - \alpha(j))^T \tau \;\forall j\right\} & \text{if } \beta \in (2\mathbb{N}_0)^n, \\ \mathrm{cl}\left\{(v, v_0) \in \mathbb{R}_{>0}^k \times \mathbb{R}_{\neq 0} \;:\; \exists v_0^* \ge |v_0| \;\exists \tau \in \mathbb{R}^n \text{ with} \right. \\ \qquad \left. v_0^* \log \frac{v_0^*}{v_j} \le (\beta - \alpha(j))^T \tau \;\forall j\right\} & \text{if } \beta \notin (2\mathbb{N}_0)^n. \end{cases}$$

Proof. First assume $\beta \in (2\mathbb{N}_0)^n$ and consider the lifted version

$$\widehat{C_{\mathrm{nc}}}(\alpha(1), \ldots, \alpha(k), \beta) := \mathrm{cl}\left\{(\nu, c, \delta) \in \mathbb{R}_+^k \times \mathbb{R}_{>0}^k \times \mathbb{R} \;:\; D(\nu, ec) \le \delta, \sum_{j=1}^k \alpha(j)\nu_j = \beta \sum_{j=1}^k \nu_j\right\}.$$

By the convexity of the function D, this is a convex cone. Let H be the linear subspace in $\mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}$ defined by

$$\begin{aligned} H \;&=\; \left\{(\nu, c, \delta) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \;:\; \sum\nolimits_{j=1}^k \alpha(j)\nu_j = \beta \sum\nolimits_{j=1}^k \nu_j\right\} \\ &=\; \left\{\nu \in \mathbb{R}^k \;:\; \sum\nolimits_{j=1}^k (\beta_t - \alpha(j)_t)\nu_j = 0,\, 1 \le t \le n\right\} \times \mathbb{R}^k \times \mathbb{R}\,. \end{aligned}$$

By the general duality statement $(\{\nu \in \mathbb{R}^k : u^T \nu = 0\})^* = \mathrm{span}\, u$ for any vector $u \in \mathbb{R}^k$, we obtain

$$\begin{aligned} H^* \;&=\; \mathrm{span}\{(\beta_t - \alpha(1)_t, \ldots, \beta_t - \alpha(k)_t)^T \;:\; 1 \le t \le n\} \times \{0\} \times \{0\} \\ &=\; \mathrm{span}\{(\beta_t - \alpha(1)_t, \ldots, \beta_t - \alpha(k)_t, 0, \ldots, 0)^T \;:\; 1 \le t \le n\}\,. \end{aligned}$$

Hence, applying Lemma 3.5 and using again that $(C_1 \cap C_2)^* = (C_1^* + C_2^*)$ for two closed convex cones C_1, C_2, the dual of $\widehat{C_{\text{nc}}} := \widehat{C_{\text{nc}}}(\alpha(1), \ldots, \alpha(k), \beta)$ is

$$
\begin{aligned}
(3.4) \qquad (\widehat{C_{\text{nc}}})^* \; = \; & \operatorname{cl}\left\{(w, v, v_0) \in \mathbb{R}^k \times \mathbb{R}^k_{>0} \times \mathbb{R}_{>0} \, : \, v_0 \log \frac{v_0}{v_j} \le w_j, \, 1 \le j \le k\right\} \\
& + \operatorname{span}\{(\beta_t - \alpha(1)_t, \ldots, \beta_t - \alpha(k)_t, 0, \ldots, 0, 0)^T \, : \, 1 \le t \le n\}.
\end{aligned}
$$

In order to obtain the projection π of $(\widehat{C_{\text{nc}}})^*$ on the (v, v_0)-coordinates, we substitute w into the inequalities in (3.4) and obtain

$$
\pi((\widehat{C_{\text{nc}}})^*) \; = \; \operatorname{cl}\left\{(v, v_0) \in \mathbb{R}^k_{>0} \times \mathbb{R}_{>0} \, : \, \exists \tau \in \mathbb{R}^n \text{ with } v_0 \log \frac{v_0}{v_j} \le (\beta - \alpha(j))^T \tau, \, 1 \le j \le k\right\}.
$$

This is the desired dual cone $(C_{\text{nc}}(\alpha(1), \ldots, \alpha(k), \beta))^*$.

In the case $\beta \notin (2\mathbb{N}_0)^n$, analogous to Lemma 3.5, the dual cone is given by the Minkowski sum of $\widehat{C_{\text{nc}}}$ and of the dual of

$$
\operatorname{cl}\left\{(\nu, c, \delta) \in \mathbb{R}^k_+ \times \mathbb{R}^k_{>0} \times \mathbb{R} \, : \, D(\nu, ec) \le -\delta, \, \sum_{j=1}^k \alpha(j)\nu_j = \beta \sum_{j=1}^k \nu_j\right\}.
$$

This yields the dual cone for the case $\beta \notin (2\mathbb{N}_0)^n$. $\square$

Remark 3.7. In the situation of Lemma 3.6, $(C_{\text{nc}}(\alpha(1), \ldots, \alpha(k), \beta))^*$ can also be expressed as the closure of the conic hull of the image of $\mathbb{R}^n$ under the map $x \mapsto (x^{\alpha(1)}, \ldots, x^{\alpha(k)}, x^\beta)$. This is because for a single circuit, the SONC cone coincides with the cone of non-negative polynomials (see, for example, [2] for the dual cone of non-negative polynomials).

We can now provide the proof of Theorem 3.1.

Proof of Theorem 3.1. From the definition of $C_{\text{sonc}}(\mathcal{A})$, we infer

$$
\begin{aligned}
(C_{\text{sonc}}(\mathcal{A}))^* \; &= \; \bigcap_{A \in \bigcup_{k=1}^{n+1} I_k(\mathcal{A})} P^*_{n,A} \\
&= \bigcap_{A \in \bigcup_{k=2}^{n+1} I_k(\mathcal{A})} \left\{(C_{\text{nc}}(\alpha(1), \ldots, \alpha(k), \beta))^* \, : \, (\alpha(1), \ldots, \alpha(k), \beta) \in I_k(\mathcal{A})\right\} \cap \bigcap_{A \in I_1(\mathcal{A})} P^*_{n,A}.
\end{aligned}
$$

Observing $\bigcap_{A \in I_1(\mathcal{A})} P^*_{n,A} = \{(v_\alpha)_{\alpha \in \mathcal{A}} \mid v_\alpha \ge 0 \text{ for } \alpha \in \mathcal{A} \cap (2\mathbb{N}_0)^n\}$ and using Lemma 3.6, we obtain for the dual cone $(C_{\text{sonc}}(\mathcal{A}))^*$:

$$
\begin{aligned}
&\Big\{(v_\alpha)_{\alpha \in \mathcal{A}} \mid v_\alpha \ge 0 \text{ for } \alpha \in \mathcal{A} \cap (2\mathbb{N}_0)^n \wedge \forall k \ge 2 \text{ and } (\alpha(1), \ldots, \alpha(k), \beta) \in I_k(\mathcal{A}) \, : \\
&\qquad \exists v^* \ge |v_\beta| \; \exists \tau \in \mathbb{R}^n \text{ with } v^* \log \frac{v^*}{v_{\alpha(j)}} \le (\beta - \alpha(j))^T \tau, \, 1 \le j \le k\Big\},
\end{aligned}
$$

where the degenerate cases of taking the logarithm are interpreted as described in the statement of the theorem.

Note that for even β, the values v_β are always non-negative, so that taking the absolute value of v_β is just done to allow a convenient notation by avoiding the case distinction. $\square$

The case of univariate quartics. We illustrate Theorem 3.1 by considering the case of univariate quartics ($d = 4$). In particular, we derive a representation of the dual cone in terms of polynomial inequalities (without any quantification such as the variables τ in Theorem 3.1) and explicate this description in terms of duality theory of plane algebraic curves.

The dual cone of non-negative univariate polynomials of degree at most 4 is given by $(\mathcal{P}[x]_{\leq 4})^* = \{v = (v_0, \ldots, v_4) \in \mathbb{R}^5 : H_4(v) \succeq 0\}$, where

$$H_4(v) = \begin{pmatrix} v_0 & v_1 & v_2 \\ v_1 & v_2 & v_3 \\ v_2 & v_3 & v_4 \end{pmatrix}$$

is a Hankel matrix (see, e.g., [13, 14]), that is,

$$(\mathcal{P}[x]_{\leq 4})^* = \{v \in \mathbb{R}^5 : v_0, v_2, v_4 \geq 0,\ v_0v_2 - v_1^2 \geq 0,\ v_0v_4 - v_2^2 \geq 0, \\ v_2v_4 - v_3^2 \geq 0,\ v_0v_2v_4 + 2v_1v_2v_3 - v_2^3 - v_0v_3^2 - v_1^2v_4 \geq 0\}\ . \tag{3.5}$$

For the dual of the univariate SONC cone of univariate quartics, an inequality representation can be obtained as a corollary of Theorem 3.1:

Corollary 3.8. *The dual of the univariate SONC cone $C_{\text{sonc}}(4)$ is*

$$(C_{\text{sonc}}(4))^* = \{v \in \mathbb{R}^5 : v_0, v_2, v_4 \geq 0,\ v_0v_2 - v_1^2 \geq 0,\ v_0^3v_4 - v_1^4 \geq 0, \\ v_0v_4 - v_2^2 \geq 0,\ v_0v_4^3 - v_3^4 \geq 0,\ v_2v_4 - v_3^2 \geq 0\}\,. \tag{3.6}$$

Proof. For the SONC cone and its dual, we have $\mathcal{A}_4 = \{0, \ldots, 4\}$ and

$$I_2(\mathcal{A}_4) = \{(0,2,1), (0,4,1), (0,4,2), (0,4,3), (2,4,3)\}. \tag{3.7}$$

Specializing Theorem 3.1 to univariate quartics gives the conditions $v_0, v_2, v_4 \geq 0$ as well as, say, for the circuit $(0,2,1)$ in (3.7):

$$\exists v_1^* \geq |v_1| \quad v_1^* \log \frac{v_1^*}{v_0} \leq \tau \text{ and } v_1^* \log \frac{v_1^*}{v_2} \leq -\tau,$$

which gives the condition $v_0v_2 - (v_1^*)^2 \geq 0$. This is equivalent to $v_0v_2 - v_1^2 \geq 0$. Similarly, the other circuits in (3.7) yield

$$v_0^3v_4 - v_1^4,\ v_0v_4 - v_2^2 \geq 0,\ v_0v_4^3 - v_3^4,\ v_2v_4 - v_3^2 \geq 0.$$

□

We illustrate the situation from the viewpoint of duality of plane algebraic curves. For the dual of the cone $C_{\text{nc}}(0,2,1)$ (and analogously, for $C_{\text{nc}}(0,4,2)$, $C_{\text{nc}}(2,4,3)$), the structure of the dual is reflected by the facts that a polynomial $p_0 + p_1x + p_2x^2$ is non-negative if and only if the matrix $\begin{pmatrix} p_0 & p_1/2 \\ p_1/2 & p_2 \end{pmatrix}$ is positive semidefinite and that the cone of positive semidefinite matrices is self-dual (see [13]). This gives the well-known positive semidefiniteness condition on the moment sequence (see, e.g., [13]).

For the case $C_{\mathrm{nc}}(0,4,1)$, we start from the fact that the polynomial $p = p_0 + p_1 x + p_4 x^4$ is non-negative if and only if the conditions on the circuit number

$$p_1 \;\leq\; \left(\frac{p_0}{3/4}\right)^{3/4} \cdot \left(\frac{p_4}{1/4}\right)^{1/4} \tag{3.8}$$

as well as $-p_1 \;\leq\; \left(\frac{p_0}{3/4}\right)^{3/4} \cdot \left(\frac{p_4}{1/4}\right)^{1/4}$ are satisfied. Now consider the case of equality within the inequality (3.8), which defines a planar projective curve in the homogeneous variables p_0, p_1, p_4, given by the polynomial

$$G(p_0,p_1,p_4) \;=\; \left(\frac{4}{3}p_0\right)^3 (4p_4) - p_1^4 \,.$$

The dual curve of this projective plane algebraic curve can be computed by considering the equations

$$x \;=\; \lambda\frac{dG}{dx}(r,s,t), \quad y \;=\; \lambda\frac{dG}{dy}(r,s,t), \quad z \;=\; \lambda\frac{dG}{dz}(r,s,t), \quad xr + ys + zt = 0\,,$$

and eliminating r, s, t, λ. In our situation, we have

$$x \;=\; \lambda \cdot 3 \cdot \left(\frac{4}{3}r\right)^2 \cdot \frac{4}{3} \cdot 4s, \quad y \;=\; \lambda \cdot \left(-4s^3\right), \quad z \;=\; \lambda \cdot \left(\frac{4}{3}r\right)^3 \cdot 4.$$

Using a computer algebra system, the elimination provides the desired equation $x^3 z - y^4$, which confirms the inequality $v_0^3 v_4 - v_1^4 \geq 0$ in (3.6). Since v_1 occurs with even exponent, considering instead of (3.8) the version for $-v_1$ leads to the same equation in the curve viewpoint of the dual.

As every polynomial in $C_{\mathrm{sonc}}(4)$ is non-negative, it is clear that $(\mathcal{P}[x]_{\leq 4})^* \subseteq (C_{\mathrm{sonc}}(4))^*$. To see this inclusion directly from the inequalities in (3.5) and (3.6), first observe that there are only two inequalities that appear in the representation of $(C_{\mathrm{sonc}}(4))^*$ but not in $(\mathcal{P}[x]_{\leq 4})^*$: $v_0^3 v_4 - v_1^4 \geq 0$ and $v_0 v_4^3 - v_3^4 \geq 0$. In order to see that for every v satisfying the inequalities of (3.5), these two inequalities are satisfied, we use the inequality $v_0 v_4 - v_2^2 \geq 0$ from (3.5) to deduce

$$\begin{aligned} v_0^3 v_4 - v_1^4 \;&\geq\; v_0^2 v_2^2 - v_1^4 \;=\; (v_0 v_2 - v_1^2)(v_0 v_2 + v_1^2) \\ \text{and}\;\; v_0 v_4^3 - v_3^4 \;&\geq\; v_2^2 v_4^2 - v_3^4 \;=\; (v_2 v_4 - v_3^2)(v_2 v_4 + v_3^2). \end{aligned}$$

Since the inequalities $v_0 v_2 - v_1^2 \geq 0$ and $v_2 v_4 - v_3^2 \geq 0$ appear in (3.6), the validity of $v_0^3 v_4 - v_1^4 \geq 0$ and $v_0 v_4^3 - v_3^4 \geq 0$ follows.

Moreover, $(\mathcal{P}[x]_{\leq 4})^* \subsetneq (C_{\mathrm{sonc}}(4))^*$. A specific point in $(C_{\mathrm{sonc}}(4))^* \setminus (\mathcal{P}[x]_{\leq 4})^*$ is, for example, $v = (v_0, \ldots, v_4) = (2,0,1,1,1)^T$. And the inequalities in (3.5) and (3.6) give the representation

$$(\mathcal{P}[x]_{\leq 4})^* \;=\; (C_{\mathrm{sonc}}(4))^* \cap \{v \in \mathbb{R}^5 \,:\, v_0 v_2 v_4 + 2 v_1 v_2 v_3 - v_2^3 - v_0 v_3^2 - v_1^2 v_4 \geq 0\}\,.$$

Remark 3.9. Since the dual of the cone of non-negative polynomials, in n variables and of degree at most d is a moment cone, we can interpret the cones in Theorems 3.1, 3.2 and Corollary 3.8 as supersets of moment cones.

4. Dual programs of SONC programs

Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ with $p(x) = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha$. Set $c = (c_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$ and identify c with the corresponding vector in $\mathbb{R}^{|\mathcal{A}|}$. For the global optimization problem

$$\inf_{x \in \mathbb{R}^n} p(x) \, ,$$

the general strategy to obtain lower bounds is to consider a conic relaxation (see, e.g., [13]). For the SONC cone, the relaxation is given by the conic program

$$\begin{array}{rcl} p_{\text{sonc}} & = & \sup_{\gamma \in \mathbb{R}} \gamma \\ & & \text{s.t. } p - \gamma \in C_{\text{sonc}}(\mathcal{A}) \, . \end{array} \tag{4.1}$$

Its dual is the program

$$\begin{array}{rcl} p^*_{\text{sonc}} & = & \inf_{v \in \mathbb{R}^{\mathcal{A}}} c^T v \\ & & \text{s.t. } v \in (C_{\text{sonc}}(\mathcal{A}))^*. \end{array} \tag{4.2}$$

Note that for specific subclasses of polynomials, the optimization problem (4.1) can be formulated as a geometric program ([10], for the class of so-called ST-polynomials) or a relative entropy program [18]. In these cases, the duality theories of geometric programming and relative entropy programming then also yield formulations for the corresponding duals. For the SAGE cone, Chandrasekaran and Shah have given a sufficient optimality criterion [4]. By transferring their result to the dual SONC cone derived in Theorem 3.1, we provide a sufficient optimality criterion in terms of the underlying primal-dual pair of optimization problems over the full SONC cone. We first observe that any point $x \in \mathbb{R}^n$ naturally induces a point in the dual cone $(C_{\text{sonc}}(\mathcal{A}))^*$:

Lemma 4.1. *For any $x \in \mathbb{R}^n$, we have $(x^\alpha)_{\alpha \in \mathcal{A}} \in (C_{\text{sonc}}(\mathcal{A}))^*$.*

Proof. First consider the case $x \in (\mathbb{R} \setminus \{0\})^n$ and set $v = (v_\alpha)_{\alpha \in \mathcal{A}} = (x^\alpha)_{\alpha \in \mathcal{A}}$. Clearly $v_\alpha \geq 0$ for $\alpha \in \mathcal{A} \cap (2\mathbb{N}_0)^n$.

Now let $k \geq 2$ and $(\alpha(1), \ldots, \alpha(k), \beta) \in I_k(\mathcal{A})$. Consider

$$|v_\beta| \log \frac{|v_\beta|}{v_{\alpha(j)}} \;=\; |x^\beta| \left(\log |x^\beta| - \log x^{\alpha(j)} \right) \;=\; |x^\beta| \sum_{i=1}^n \log |x_i| (\beta_i - \alpha(j)_i) \, ,$$

so that setting $\tau_i = |x^\beta| \log |x_i|$, $1 \leq i \leq n$, and $v^* = |v_\beta|$ gives

$$v^* \log \frac{v^*}{v_{\alpha(j)}} \;\leq\; (\beta - \alpha(j))^T \tau \, .$$

Hence, $v \in (C_{\text{sonc}}(\mathcal{A}))^*$.

If one of the components of x is zero, we still have $v = (x^\alpha)_{\alpha \in C} \in (C_{\text{sonc}}(\mathcal{A}))^*$, because $(C_{\text{sonc}}(\mathcal{A}))^*$ is closed. □

We obtain the following sufficient optimization criterion.

Theorem 4.2. *Let $v \in \mathbb{R}^{\mathcal{A}}$ be an optimal solution of (4.2), and assume that there exists $z \in \mathbb{R}^n$ with $v = (z^\alpha)_{\alpha \in \mathcal{A}}$. Then z is an optimal solution of p, and hence $p^*_{\text{sonc}} = \inf_{x \in \mathbb{R}^n} p(x)$.*

Proof. Let $z \in \mathbb{R}^n$ such that $v = (z^\alpha)_{\alpha \in \mathcal{A}}$ is an optimal point for (4.2). Lemma 4.1 then implies that z is a minimizer for p. Hence,

$$\inf_{x \in \mathbb{R}^n} p(x) \;=\; p(z) \;=\; c^T v \;=\; p^*_{\text{sonc}}\,,$$

which implies the claim. □

5. Final remarks and open questions

In the setup of sums of squares based relaxations for polynomial optimization, the dual view of moments plays a central role, in particular in the situation of constrained optimization (see, e.g., [13, 14]). In [7], hierarchical relaxation techniques for SONC-based constrained optimization have been developed. It remains a future task to extend our results on the dual cone and the duality aspects to these constrained settings.

Moreover, after the preprint of the present paper had been posted, various other works have recently appeared which also pose further research challenges on the dual SONC cone and its relatives. In particular, [15] gives a necessary condition for the extreme rays of the SONC cone and of the SAGE cone, [12] provides a common generalization of the SONC cone and the SAGE cone and provides an exact characterization of the extremals of the SONC cone and of the SAGE cone, and [8] characterizes the algebraic boundary of the SONC cone and its connection to discriminants and to the Horn-Kapranov uniformization. It would be interesting to understand those aspects also for the duals of theses cones, for example to characterize the extremals of the dual SONC cone and of the dual SAGE cone.

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