CONIC STABILITY OF POLYNOMIALS AND POSITIVE MAPS

PAPRI DEY, STEPHAN GARDOLL, AND THORSTEN THEOBALD

Abstract. Given a proper cone $K \subseteq \mathbb{R}^n$, a multivariate polynomial $f \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, \ldots, z_n]$ is called K-stable if it does not have a root whose vector of the imaginary parts is contained in the interior of K. If K is the non-negative orthant, then K-stability specializes to the usual notion of stability of polynomials.

We study conditions and certificates for the K-stability of a given polynomial f, especially for the case of determinantal polynomials as well as for quadratic polynomials. A particular focus is on psd-stability. For cones K with a spectrahedral representation, we construct a semidefinite feasibility problem, which, in the case of feasibility, certifies K-stability of f. This reduction to a semidefinite problem builds upon techniques from the connection of containment of spectrahedra and positive maps.

In the case of psd-stability, if the criterion is satisfied, we can explicitly construct a determinantal representation of the given polynomial. We also show that under certain conditions, for a K-stable polynomial f, the criterion is at least fulfilled for some scaled version of K.

1. Introduction

Recently, there has been wide-spread research interest in stable polynomials and the geometry of polynomials, accompanied by a variety of new connections to other branches of mathematics (including combinatorics [6], differential equations [4], optimization [36], probability theory [5], applied algebraic geometry [40], theoretical computer science [28, 29] and statistical physics [3]). See also the surveys of Pemantle [32] and Wagner [41]. Stable polynomials are strongly linked to matroid theory [6], as delta-matroids arise from support sets of stable polynomials.

In this paper, we concentrate on the generalized notion of K-stability as introduced in [20]. Given a proper cone $K \subseteq \mathbb{R}^n$, a polynomial $f \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, \ldots, z_n]$ is called *K-stable* if $\mathcal{I}(f) \cap \operatorname{int} K = \emptyset$, where $\operatorname{int} K$ is the interior of K and $\mathcal{I}(f)$ denotes the imaginary projection of f (as formally defined in Section 2). Note that $(\mathbb{R}_{\geq 0})^n$-stability coincides with the usual stability, and stability with respect to the positive semidefinite cone on the space of symmetric matrices is denoted as *psd-stability*. In the case of a homogeneous polynomial, K-stability of f is equivalent to the containment of $\operatorname{int} K$ in a hyperbolicity cone of f (see Section 2), which also provides a link to hyperbolic programming.

Here, we study conditions and certificates for the K-stability of a given polynomial $f \in \mathbb{C}[\mathbf{z}]$, especially for the case of determinantal polynomials of the form $f(\mathbf{z}) = \det(A_0 + A_1 z_1 + \cdots + A_n z_n)$ with symmetric or Hermitian matrices $A_0, \ldots, A_n$ as well as for quadratic polynomials. A particular focus is on psd-stability.

Specifically, for cones K with a spectrahedral representation we construct a semidefinite feasibility problem, which, in the case of non-emptiness, certifies K-stability of

f. This reduction to a semidefinite problem builds upon two ingredients. Firstly, we characterize certain conic components in the complement of the imaginary projection of the (not necessarily homogeneous) polynomial f. Secondly, the sufficient criterion employs techniques from [23] on containment problems of spectrahedra and positive maps in order to check whether $\operatorname{int} K \subseteq \mathcal{I}(f)^{\mathsf{c}}$. For the special case of usual stability, we will recover the well-known determinantal stability criterion of Borcea and Brändén (see Proposition 2.6 and Remark 4.4) and thus obtain, as a byproduct, an alternative proof of that statement.

In the case of psd-stability, if the sufficient criterion is satisfied, we can explicitly construct a determinantal representation of the given polynomial, see Corollary 4.9. To this end, the determinantal criterion for psd-stability from [20] can be seen as a special case of our more general results. The procedure enables to check and certify the conic stability for a large subclass of polynomials.

Moreover, we show that under certain preconditions, there always exists a positive scaling factor such that the sufficient criterion applies to a scaled version of the polynomial (or, equivalently, a scaled version of the cone). See Theorem 5.2.

The paper is structured as follows. Section 2 provides relevant background on imaginary projections, conic stability and determinantal representations. In Section 3, we study the conic components in the complement of the imaginary projection for the relevant classes of polynomials. Section 4 develops the sufficient criterion for K-stability based on the techniques from positive maps. The scaling result is contained in Section 5, and Section 6 concludes the paper.

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2. Preliminaries

Throughout the text, bold letters will denote n-dimensional vectors unless noted otherwise.

2.1. Imaginary projections and conic stability.

For a polynomial $f \in \mathbb{C}[\mathbf{z}]$, define its *imaginary projection* $\mathcal{I}(f)$ as the projection of the variety of f onto its imaginary part, i.e.,

$$\mathcal{I}(f) = \{\mathsf{Im}(\mathbf{z}) = (\mathsf{Im}(z_1), \ldots, \mathsf{Im}(z_n)) : f(\mathbf{z}) = 0\}, \tag{1}$$

where $\mathsf{Im}(\cdot)$ denotes the imaginary part of a complex number [22].

Let $\mathcal{S}_d, \mathcal{S}_d^+$ and $\mathcal{S}_d^{++}$ denote the set of symmetric $d \times d$ matrices as well as the subsets of positive semidefinite and positive definite matrices. Moreover, let Herm_d be the set of all Hermitian $d \times d$-matrices.

We consider the following generalization of stability. Let K be a *proper* cone in $\mathbb{R}^n$, that is, a full-dimensional, closed and pointed convex cone in $\mathbb{R}^n$.

Definition 2.1. A polynomial $f \in \mathbb{C}[\mathbf{z}]$ is called *K-stable*, if $f(\mathbf{z}) \neq 0$ whenever $\mathsf{Im}(\mathbf{z}) \in \operatorname{int} K$.

If $f \in \mathbb{C}[Z]$ on the symmetric matrix variables $Z = (z_{ij})_{n \times n}$ is $\mathcal{S}_n^+$-stable, then f is called *positive semidefinite-stable* (for short, *psd-stable*).

A stable or K-stable polynomial with real coefficients is called *real stable* or *real K-stable*, respectively.

Remark 2.2. 1. A set of the form $\mathbb{R}^n + iC$, where C is an open convex cone, is called a *Siegel domain (of the first kind).* Siegel domains provide an important concept in function theory of several complex variables and harmonic analysis, see the books [19, 33, 35].

2. The *Siegel upper half-space* (or *Siegel upper half-plane*) $\mathcal{H}_g$ of degree g (or genus g) is defined as

$$\mathcal{H}_g \ = \{A \in \mathbb{C}^{g \times g} \text{ symmetric } : \mathsf{Im}(A) \text{ is positive definite}\},$$

where $\mathsf{Im}(A) = (\mathsf{Im}(a_{ij}))_{g \times g}$ (see, e.g., [38, §2]). The Siegel upper half-space occurs in algebraic geometry and number theory as the domain of modular forms. Using that notation, psd-stability can be viewed as stability with respect to the Siegel upper half-space.

A form (i.e., a homogeneous polynomial) $f \in \mathbb{R}[\mathbf{z}]$ is *hyperbolic* in direction $\mathbf{e} \in \mathbb{R}^n$ if $f(\mathbf{e}) \neq 0$ and for every $\mathbf{x} \in \mathbb{R}^n$ the univariate polynomial $t \mapsto f(\mathbf{x} + t\mathbf{e})$ has only real roots. The cone $C(\mathbf{e}) = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x} + t\mathbf{e}) = 0 \Rightarrow t < 0\}$ is called the *hyperbolicity cone* of f with respect to $\mathbf{e}$. This cone $C(\mathbf{e})$ is convex, f is hyperbolic with respect to every point $\mathbf{e}' \in C(\mathbf{e})$ and $C(\mathbf{e}) = C(\mathbf{e}')$ (see [11]).

Let f be a hyperbolic polynomial and $C(\mathbf{e})$ denote the hyperbolicity cone containing $\mathbf{e}$. By definition of K-stability, a homogeneous polynomial f is hyperbolic w.r.t. every point $\mathbf{e}' \in C(\mathbf{e})$ if and only if f is $(\operatorname{cl} C(\mathbf{e}))$-stable, where cl denotes the topological closure of a set. The following theorem in [20] reveals the connection between K-stable polynomials and hyperbolic polynomials.

Theorem 2.3. *For a homogeneous polynomial $f \in \mathbb{R}[\mathbf{z}]$, the following are equivalent.*

1. *f is K-stable.*
2. *$\mathcal{I}(f) \cap \operatorname{int} K = \emptyset$.*
3. *f is hyperbolic w.r.t. every point in $\operatorname{int} K$.*

By [21], the hyperbolicity cones of a homogeneous polynomial f coincide with the components of $\mathcal{I}(f)^{\mathsf{c}}$, where $\mathcal{I}(f)^{\mathsf{c}}$ denotes the complement of $\mathcal{I}(f)$. This implies:

Corollary 2.4. *A hyperbolic polynomial $f \in \mathbb{R}[\mathbf{z}]$ is K-stable if and only if $\operatorname{int} K \subseteq C(\mathbf{e})$ for some hyperbolicity direction $\mathbf{e}$ of f.*

Proof. This follows from the observation that a hyperbolic polynomial $f \in \mathbb{R}[\mathbf{z}]$ is K-stable if and only if $\operatorname{int} K \subseteq \mathcal{I}(f)^{\mathsf{c}}$. □

It is shown in [21] that the number of hyperbolicity cones of a homogeneous polynomial $f \in \mathbb{R}[\mathbf{z}]$ is at most 2^d for $d \leq n$ and at most $2\sum_{k=0}^{n-1}\binom{d-1}{k}$ for $d > n$.

2.2. **Determinantal representations.** A *determinantal polynomial* is a polynomial of the form $f(\mathbf{z}) = \det(A_0 + \sum_{j=1}^n A_j z_j)$. For our purposes, we always assume that the matrices $A_0, \ldots, A_n$ are Hermitian unless stated otherwise. If the constant coefficient matrix A_0 is positive definite or the identity, then the determinantal polynomial is called *definite* or *monic* determinantal polynomial, respectively. Helton, McCullough and Vinnikov showed that every polynomial $p \in \mathbb{R}[\mathbf{z}]$ with $p(0) \neq 0$ has a symmetric determinantal representation of the form $p(\mathbf{z}) = \det(A_0 + \sum_{j=1}^n A_j z_j)$ with real symmetric matrices $A_0, \ldots, A_n$ ([17, Theorem 14.1], see also Quarez [34, Theorem 4.4] and, for the earlier result of a not necessarily symmetric determinantal representation, Valiant [37] and its exposition in Bürgisser et al. [9]). Note that A_0 is not necessarily positive definite and not even necessarily positive semidefinite.

In [18] and [30], it was shown that several classes of polynomials have monic determinantal representations due to the connection to real zero polynomials. Here, a polynomial $f \in \mathbb{R}[\mathbf{z}]$ is called *real zero*, if the mapping $t \mapsto f(t \cdot \mathbf{z})$ has only real roots. Brändén has constructed a real zero polynomial for which A_0 cannot be taken to be positive definite in a determinantal representation [7]. Recently, Dey and Pillai [10] added a complete characterization of the quadratic case by also using the connection to real zero polynomials.

Proposition 2.5 ([10])**.** *A quadratic polynomial* $f(\mathbf{z}) = \mathbf{z}^T A\mathbf{z} + \mathbf{b}^T\mathbf{z} + 1 \in \mathbb{R}[\mathbf{z}]$ *is a real zero polynomial if and only if* $Q/(1,1) := A - \frac{1}{4}\mathbf{b}\mathbf{b}^T$ *is negative semidefinite. The polynomial* $f(\mathbf{z})$ *has a monic determinantal representation if and only if at least one of the following conditions holds:*

- A *is negative semidefinite.*
- $Q/(1,1)$ *is negative semidefinite and* $\operatorname{rank}(Q/(1,1)) \leq 3$.

2.3. **Real stable polynomials.** As specified in the Introduction and Section 2.1, a real polynomial f is real stable if it is real K-stable with respect to the non-negative orthant $K = \mathbb{R}^n_+$. This holds true if and only if for every $\mathbf{e} \in \mathbb{R}^n_{>0}$ and $\mathbf{x} \in \mathbb{R}^n$, the univariate polynomial $t \mapsto f(t\mathbf{e}+\mathbf{x})$ is real-rooted. Indeed, a particular prominent class of real stable polynomials is generated from determinantal polynomials as follows.

Proposition 2.6. ([2, Thm. 2.4]) Let $A_1, \ldots, A_n$ be positive semidefinite $d \times d$-matrices and A_0 be a Hermitian $d \times d$-matrix. Then

$$f(\mathbf{z}) \; = \; \det(A_0 + \sum_{j=1}^n A_j z_j)$$

is real stable or the zero polynomial.

It is also known that a real polynomial $f \in \mathbb{R}[\mathbf{z}]$ is real stable if and only if the (unique) homogenization polynomial w.r.t. the variable z_0 is hyperbolic w.r.t. every vector $\mathbf{e} \in \mathbb{R}^{n+1}$ such that $e_0 = 0$ and $e_j > 0$ for all $1 \leq j \leq n$ (see [4]).

Example 2.7. The class of homogeneous stable polynomials is contained in the following class of Lorentzian polynomials, see [8, 15]. Let $f \in \mathbb{R}[\mathbf{z}]$ be homogeneous of degree $d \geq 2$ with only positive coefficients. f is called *strictly Lorentzian* if

- $d = 2$ and the *Hessian* $\mathbb{H}(f) = (\partial_i \partial_j f)_{i,j=1}^n$ is non-singular and has exactly one positive eigenvalue (i.e., $\mathbb{H}(f)$ has the *Lorentzian signature* $(1, n-1)$, which

expresses that f has one positive eigenvalue and $n-1$ negative eigenvalues [15]),
- or $d > 2$ and for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = d-2$, the α-th derivative $\partial^\alpha f$ is strictly Lorentzian.

By convention, in degrees 0 and 1, every polynomial with only positive coefficients is strictly Lorentzian. Limits of strictly Lorentzian polynomials are called *Lorentzian.*

Concerning psd-stability, the following variant of Proposition 2.6 is known.

Proposition 2.8. ([20, Thm. 5.3]) Let $A = (A_{ij})_{n\times n}$ be a Hermitian block matrix with $n \times n$ blocks of size $d \times d$. If A is positive semidefinite and A_0 a Hermitian $d\times d$-matrix, then the polynomial $f(Z) = \det(A_0 + \sum_{i,j=1}^n A_{ij} z_{ij})$ on the set of symmetric $n \times n$-matrices is psd-stable or identically zero.

Determinantal representations of complex polynomials which are stable with respect to the unit ball of symmetric matrices have been studied in [13, 14].

In the present paper, for cones K with a spectrahedral representation, we derive a semidefinite problem, which, in the case of feasibility, certifies K-stability of f. For the case of psd-stability, if that criterion is satisfied, we can explicitly construct the determinantal representation of Proposition 2.8. In this respect, the criterion from Proposition 2.8 can be seen as a special case of our treatment.

The following examples serve to pinpoint some relationships between stable, psd-stable and determinantal polynomials.

Example 2.9. a) A quadratic determinantal polynomial does not need to be stable in order to be psd-stable (with respect to a suitable ordering identification between the variables z_i and the matrix variables z_{jk}). Namely, the determinantal polynomial

$$f(z_1, z_2, z_3) = (z_1 + z_3)^2 - z_2^2 = (z_1 + z_3 - z_2)(z_1 + z_3 + z_2)$$

is not stable, because $(1, 2, 1) \in \mathcal{I}(f) \cap \mathbb{R}^3_{>0}$. However, in the matrix variables $Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$, the polynomial $f(Z) = f(z_1, z_2, z_3)$ is psd-stable. To see this, observe that by the arithmetic-geometric mean inequality, every $\mathbf{y} \in \mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^3 : y_1 + y_3 = y_2 \text{ or } y_1 + y_3 = -y_2\}$ satisfies

$$\det \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} = \det \begin{pmatrix} y_1 & \pm(y_1 + y_3) \\ \pm(y_1 + y_3) & y_3 \end{pmatrix} = y_1 y_3 - (y_1 + y_3)^2 \le 0$$

and thus $\mathbf{y} \notin \operatorname{int} \mathcal{S}_2^+$.

b) An example of a non-psd-stable determinantal polynomial on 2×2-matrices, i.e., with matrix variables $Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}$, is $f(Z) = \det \operatorname{Diag}(z_{11}, z_{12}, z_{22}) = z_{11}z_{12}z_{22}$. Namely, since $\mathcal{I}(f) = \{X \in \mathcal{S}_2 : x_{11}x_{12}x_{22} = 0\}$, we have $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{I}(f)$ and thus $\mathcal{I}(f) \cap \operatorname{int} \mathcal{S}_2^+ \neq \emptyset$.

c) Another example of a non-psd-stable determinantal polynomial on 2×2-matrices is the determinant of the spectrahedral representation of the open Lorentz cone $g(\mathbf{z}) = \det \begin{pmatrix} z_1 + z_3 & z_2 \\ z_2 & z_1 - z_3 \end{pmatrix} = z_1^2 - z_2^2 - z_3^2$, where the same variable identification as in a) is used. Note that $g(\mathbf{z}) = 0$ for $\mathbf{z} = (1+2i, 1+i, \sqrt{-3+2i})$ and $\begin{pmatrix} 2 & 1 \\ 1 & \alpha \end{pmatrix} \in \operatorname{int} \mathcal{S}_2^+$ for $\alpha = \mathsf{Im}(\sqrt{-3+2i}) > 1$. Hence, g is not psd-stable.

3. Conic components in the complement of the imaginary projection

To prepare for the conic stability criteria for determinantal and quadratic polynomials, we characterize particular conic components in the complement of the imaginary projection for these classes. Denote by $X \succ 0$ the positive definiteness of a matrix X.

First consider a determinantal polynomial

$$f(\mathbf{z}) \;=\; \det(A_0 + A_1 z_1 + \cdots + A_n z_n) \tag{2}$$

with $A_0, \ldots, A_n \in \mathsf{Herm}_d$. Note that if $A_0 = I$, then the homogenization of f w.r.t. a variable z_0 is hyperbolic w.r.t. $\mathbf{e} = (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$. Moreover, for a homogeneous determinantal polynomial $f = \det(\sum_{j=1}^n A_j z_j)$, if there exists an $\mathbf{e} \in \mathbb{R}^n$ with $\sum_{j=1}^n A_j e_j \succ 0$, then f is hyperbolic w.r.t. $\mathbf{e}$, and the set

$$\{\mathbf{z} \in \mathbb{R}^n \;:\; A_1 z_1 + \cdots + A_n z_n \succ 0\}$$

as well as its negative are hyperbolicity cones of f, see [26, Prop. 2]. If f is irreducible, then these are the only two hyperbolicity cones (see [25]), whereas in the reducible case there can be more (cf. Section 2.1). Let $A(\mathbf{z})$ be the linear matrix pencil $A(\mathbf{z}) = A_0 + \sum_{j=1}^n A_j z_j$. The *initial form* of f, denoted by $\operatorname{in}(f)$, is defined as $\operatorname{in}(f)(\mathbf{z}) = f_h(0, \mathbf{z})$, where f_h is the homogenization of f w.r.t. the variable z_0.

Theorem 3.1. *If f is a degree d determinantal polynomial of the form* (2) *and there exists an $\mathbf{e} \in \mathbb{R}^n$ with $\sum_{j=1}^n A_j e_j \succ 0$, then $\operatorname{in}(f)$ is hyperbolic and every hyperbolicity cone of $\operatorname{in}(f)$ is contained in $\mathcal{I}(f)^{\mathsf{c}}$.*

Proof. Let $f = \det(A_0 + \sum_{j=1}^n A_j z_j)$ with $A_0, \ldots, A_n \in \mathsf{Herm}_d$. Since f is of degree d, it holds $\operatorname{in}(f) = \det(\sum_{j=1}^n A_j z_j)$. Then $\sum_{j=1}^n A_j e_j \succ 0$ implies that $\operatorname{in}(f)$ is hyperbolic.

First we assume that $\operatorname{in}(f)$ is irreducible. By the precondition $\sum_{j=1}^n A_j e_j \succ 0$, the initial form $\operatorname{in}(f)$ has exactly the two hyperbolicity cones $C_1 = \{\mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^n A_j x_j \succ 0\}$ and $C_2 = \{\mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^n A_j x_j \prec 0\}$.

First we show that $C_1 \subseteq \mathcal{I}(f)^{\mathsf{c}}$. For every $\mathbf{x} \in \mathbb{R}^n$, we have

$$f(\mathbf{x} + t\mathbf{e}) \;=\; \det(A_0 + \sum_{j=1}^n A_j x_j + t \sum_{j=1}^n A_j e_j).$$

Since $\sum_{j=1}^n A_j e_j \succ 0$, we obtain

$$f(\mathbf{x} + t\mathbf{e}) \;=\; \det(\sum_{j=1}^n A_j e_j) \det\Big((\sum_{j=1}^n A_j e_j)^{-1/2} (A_0 + \sum_{j=1}^n A_j x_j) (\sum_{j=1}^n A_j e_j)^{-1/2} + tI \Big).$$

Since $A_0 + \sum_{j=1}^n A_j x_j$ is Hermitian, all the roots of $t \mapsto f(\mathbf{x} + t\mathbf{e})$ are real. Hence, there cannot be a non-real vector $\mathbf{a} + i\mathbf{e}$ with $f(\mathbf{a} + i\mathbf{e}) = 0$, because otherwise setting $\mathbf{x} = \mathbf{a}$ would give a non-real solution to $t \mapsto f(\mathbf{x} + t\mathbf{e})$. Thus, there is a connected component C' in $\mathcal{I}(f)^{\mathsf{c}}$ containing C_1. The case $C_2 \subseteq \mathcal{I}(f)^{\mathsf{c}}$ is symmetric, since $-\mathbf{e} \in C_2$.

To cover also the case of reducible $\mathrm{in}(f)$, it suffices to observe that for reducible $\mathrm{in}(f) = \prod_{j=1}^{k} h_j$ with irreducible $h_1, \ldots, h_k$, every hyperbolicity cone C of $\mathrm{in}(f)$ is of the form $C = \bigcap_{j=1}^{k} C_j$ with some hyperbolicity cones C_j of h_j, $1 \le j \le k$. □

Quadratic polynomials. Now let $f \in \mathbb{R}[\mathbf{z}]$ be a quadratic polynomial of the form

$$f \;=\; \mathbf{z}^T A \mathbf{z} + \mathbf{b}^T \mathbf{z} + c \tag{3}$$

with $A \in \mathcal{S}_n$, $\mathbf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We show that those components of $\mathcal{I}(f)^{\mathsf{c}}$ which are cones, can be described in terms of spectrahedra, as made precise in the following.

First recall the situation of a homogeneous quadratic polynomial $f = \mathbf{z}^T A \mathbf{z}$. By possibly multiplying A with -1, we can assume that the number of positive eigenvalues of A is at least the number of negative eigenvalues. In this setting, it is well known that a non-degenerate quadratic form $f \in \mathbb{R}[\mathbf{z}]$ is hyperbolic if and only if A has signature $(n-1, 1)$ [11].

Specifically, for the normal form

$$f(\mathbf{z}) \;=\; \sum_{j=1}^{n-1} z_j^2 - z_n^2,$$

we have $\mathcal{I}(f) \;=\; \{\mathbf{y} \in \mathbb{R}^n \;:\; y_n^2 \le \sum_{j=1}^{n-1} y_j^2\}$ (see [22]). Hence, there are two unbounded components in the complement $\mathcal{I}(f)^{\mathsf{c}}$, both of which are full-dimensional cones, and these two components are

$$\{\mathbf{y} \in \mathbb{R}^{n-1} \times \mathbb{R}_+ \;:\; \sum_{j=1}^{n-1} y_j^2 < y_n^2\} \;\text{ and }\; \{\mathbf{y} \in \mathbb{R}^{n-1} \times \mathbb{R}_- \;:\; \sum_{j=1}^{n-1} y_j^2 < y_n^2\}.$$

For a general homogeneous quadratic form, this generalizes as follows.

Lemma 3.2. *For a quadratic form $f = \mathbf{z}^T A \mathbf{z} \in \mathbb{R}[\mathbf{z}]$ with A having signature $(n-1, 1)$, the components C of the complement of $\mathcal{I}(f)$ are given by the two components of the set*

$$\{\mathbf{y} \in \mathbb{R}^n \;:\; \mathbf{y}^T A \mathbf{y} < 0\}\,, \tag{4}$$

and the closures of these components are spectrahedra.

The proof makes use of the following property from [22].

Proposition 3.3. *Let $g \in \mathbb{C}[\mathbf{z}]$ and $T \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then, $\mathcal{I}(g(T\mathbf{z})) = T^{-1}\mathcal{I}(g(\mathbf{z}))$.*

Proof of Lemma 3.2. Since $-A$ has Lorentzian signature, there exists $S \in \mathrm{GL}(n, \mathbb{R})$ with $A_I \;:=\; S^T A S \;=\; \mathrm{Diag}(1, \ldots, 1, -1)$. Observing

$$\mathcal{I}(f(S\mathbf{z})) \;=\; \mathcal{I}(\mathbf{z}^T A_I \mathbf{z}) \;=\; \{\mathbf{y} \in \mathbb{R}^n \;:\; y_n^2 \le \sum_{j=1}^{n-1} y_j^2\} \;= \{\mathbf{y} \in \mathbb{R}^n \;:\; \mathbf{y}^T A_I \mathbf{y} \ge 0\}\,,$$

Proposition 3.3 then gives

$$\mathcal{I}(f(\mathbf{z})) \;=\; S \cdot \mathcal{I}(f(S\mathbf{z})) \;=\; \{S \cdot \mathbf{y} \in \mathbb{R}^n \;:\; \mathbf{y}^T A_I \mathbf{y} \ge 0\} \;=\; \{\mathbf{y} \in \mathbb{R}^n \;:\; \mathbf{y}^T A \mathbf{y} \ge 0\}\,.$$

□

For the general, not necessarily homogeneous case, recall that every quadric in $\mathbb{R}^n$ is affinely equivalent to a quadric given by one of the following polynomials,

$$\begin{array}{lll} \text{(I)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 & (1 \le p \le r,\ r \ge 1,\ p \ge \frac{r}{2}), \\ \text{(II)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 + 1 & (0 \le p \le r,\ r \ge 1), \\ \text{(III)} & \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{r} z_j^2 + z_{r+1} & (1 \le p \le r,\ r \ge 1,\ p \ge \frac{r}{2}). \end{array}$$

We refer to [1] as a general background reference for real quadrics. We say that a given quadratic polynomial $f \in \mathbb{R}[\mathbf{z}]$ is of type X if it can be transformed to the normal form X by an affine real transformation.

The homogeneous case, case (I), has already been treated, and by [22], it is known that in case (III), the imaginary projection does not contain a full-dimensional component in $\mathcal{I}(f)^{\mathsf{c}}$.

By [22], in case (II), unbounded components only exist in the cases $p = 1$ and $p = r - 1$, so we can restrict to these cases. We list these relevant two cases from [22].

Theorem 3.4. *Let $n \ge r \ge 3$ and $f \in \mathbb{R}[\mathbf{z}]$ be a quadratic polynomial. If f is of type* (II)*, then*

$$\mathcal{I}(f) \;=\; \begin{cases} \{\mathbf{y} \in \mathbb{R}^n \;:\; y_1^2 - \sum_{j=2}^{r} y_j^2 \le 1\} & \text{if } p = 1, \\ \{\mathbf{y} \in \mathbb{R}^n \;:\; \sum_{j=1}^{r-1} y_j^2 > y_r^2\} \cup \{\mathbf{0}\} & \text{if } p = r - 1. \end{cases} \tag{5}$$

For the proof see [22]. Since the proofs of the case $p = 1$ and of the case $p = r - 1$ differ in some important details, which are not carried out there, we include a proof here for the convenience of the reader.

Proof. Without loss of generality we can assume $r = n$. Writing $z_j = x_j + iy_j$, we have $f(\mathbf{z}) = \sum_{j=1}^{p} z_j^2 - \sum_{j=p+1}^{n} z_j^2 + 1 = 0$ if and only if

$$\sum_{j=1}^{p} x_j^2 - \sum_{j=p+1}^{n} x_j^2 - \sum_{j=1}^{p} y_j^2 + \sum_{j=p+1}^{n} y_j^2 + 1 \;=\; 0 \tag{6}$$

$$\text{and} \sum_{j=1}^{p} x_j y_j - \sum_{j=p+1}^{n} x_j y_j \;=\; 0. \tag{7}$$

Set $\alpha := -\sum_{j=1}^{p} y_j^2 + \sum_{j=p+1}^{n} y_j^2 + 1$, and let $\mathbf{y} \in \mathbb{R}^n$ be fixed. Note that in both cases $p = 1$ and $p = n - 1$, we have $\mathbf{0} \in \mathcal{I}(f)$, since $f(\mathbf{x} + i \cdot \mathbf{0}) = 0$ for $\mathbf{x} = (0, \ldots, 0, 1)$. Hence, we can assume $\mathbf{y} \ne \mathbf{0}$.

Case $p = 1$: Write $\mathbf{x} = (x_1, \mathbf{x}') = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, \mathbf{y}') = (y_1, y_2, \ldots, y_n)$. Observe the rotational symmetry of (6) w.r.t. $\mathbf{x}'$ and $\mathbf{y}'$ and the invariance of the standard scalar product $(\mathbf{x}', \mathbf{y}') \mapsto \sum_{j=2}^{n} x_j y_j$ under orthogonal transformations. Hence, if $((x_1, \mathbf{x}'), (y_1, \mathbf{y}'))$ is a solution of (6) and (7), then for any $T \in \mathrm{SO}(n-1)$, the point $((x_1, T\mathbf{x}'), (y_1, T\mathbf{y}'))$ is a solution as well, where $\mathrm{SO}(n-1)$ denotes the special orthogonal group of order $n - 1$. Thus, we can assume $y_3 = \cdots = y_n = 0$, and α simplifies to $\alpha = -y_1^2 + y_2^2 + 1$. Solving (7) for x_1 (by assuming, without loss of generality, $y_1 \ne 0$) yields $x_1 = \frac{x_2 y_2}{y_1}$ and substituting this into (6) then

$$0 \;=\; \left(\frac{y_2^2}{y_1^2} - 1\right) x_2^2 - \sum_{j=3}^{n} x_j^2 + \alpha \;=\; \frac{(\alpha - 1) x_2^2}{y_1^2} - \sum_{j=3}^{n} x_j^2 + \alpha.$$

This equation has a real solution $(x_2, \ldots, x_n)$ if and only if $\alpha \geq 0$, which shows $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^n \,:\, y_1^2 - \sum_{j=2}^n y_j^2 \leq 1\}$.

Case $p = n-1$: Following the same proof strategy, we now write $x = (\mathbf{x}', x_n) = (x_1, x_2, \ldots, x_n)$ and $y = (\mathbf{y}', y_n) = (y_1, y_2, \ldots, y_n)$. Then the symmetry of the problem allows to assume $y_2 = \cdots = y_{n-1} = 0$, and α simplifies to $\alpha = -y_1^2 + y_n^2 + 1$. If $y_1 \neq 0$, solving (7) for x_1 gives $x_1 = \frac{x_n y_n}{y_1}$, and a substitution into (6)

$$0 \;=\; \left(\frac{y_n^2}{y_1^2} - 1\right) x_n^2 + \sum_{j=2}^{n-1} x_j^2 + \alpha \;=\; \frac{(\alpha-1)x_2^2}{y_1^2} + \sum_{j=2}^{n-1} x_j^2 + \alpha.$$

There exists a real solution $(x_2, \ldots, x_n)$ if and only if $\alpha < 1$, which, taking also into account the special case $y_1 = 0$, gives $\mathcal{I}(f) = \{\mathbf{y} \in \mathbb{R}^n \,:\, \sum_{j=1}^{n-1} y_j^2 > y_n^2\} \cup \{\mathbf{0}\}$. □

For the inhomogeneous case, we use the following lemma to reduce it to the homogeneous case.

Lemma 3.5. *Let* $n \geq 3$ *and* $f \in \mathbb{R}[\mathbf{z}]$ *be quadratic of the form* (3).

If f *is of type (II) with* $p = 1$*, then* $\mathcal{I}(f)^{\mathsf{c}}$ *does not have connected components whose closures contain full-dimensional cones.*

If f *is of type (II) with* $p = n-1$ *then every full-dimensional cone which is contained in* $\mathcal{I}(f)^{\mathsf{c}}$ *is contained in the closure of a hyperbolicity cone of* $\mathrm{in}(f)$.

Note, that in particular, that $\mathcal{I}(f)^{\mathsf{c}}$ does not contain a point at all if and only if $\mathrm{in}(f)$ is not hyperbolic.

Proof. If f is of type (II) with $p = 1$, then the statement is a consequence of (5).

Now consider the case that f is of type (II) with $p = n-1$ and let C be full-dimensional cone which is contained in a component of $\mathcal{I}(f)^{\mathsf{c}}$. By [21, Theorem 4.2 and Lemma 4.3], $\operatorname{int} C$ is contained in a hyperbolicity cone of $\mathrm{in}(f)$. □

Hence, among the quadratic polynomials of type (II), only the ones with $p = n-1$ might possibly be K-stable.

Theorem 3.6. *Let* $n \geq 3$ *and* $f \in \mathbb{R}[\mathbf{z}]$ *be quadratic of the form* (3) *and of type (II) with* $p = n-1$*. Then there exists a linear form* $\ell(\mathbf{z})$ *in* $\mathbf{z}$ *such that* $-\ell(\mathbf{z})^{n-2}\mathrm{in}(f)$ *has a determinantal representation. In particular, the closure of each unbounded component of* $\mathcal{I}(f)^{\mathsf{c}}$ *is a spectrahedral cone.*

The theorem can be seen as an adaption of the well-known result that hyperbolic quadratic forms have determinantal representations. See, e.g., [39, Section 2] or [30, Example 2.16] for the determinantal representations which underlie that result and which are utilized in the subsequent proof.

Proof. First consider the normal form of type (II) with $p = n-1$,

$$g(\mathbf{z}) \;=\; \sum_{j=1}^{n-1} z_j^2 - z_n^2 + 1\,.$$

By (5), the complement of $\mathcal{I}(g)$ has the two unbounded conic components

$$\{\mathbf{y} \in \mathbb{R}^{n-1} \times \mathbb{R}_+ \,:\, \sum_{j=1}^{n-1} y_j^2 \le y_n^2\} \setminus \{0\} \;\text{ and }\; \{\mathbf{y} \in \mathbb{R}^{n-1} \times \mathbb{R}_- \,:\, \sum_{j=1}^{n-1} y_j^2 \le y_n^2\} \setminus \{0\},$$

which (up to the origin) are the open Lorentz cone and its negative. Their closures are exactly the closures of the hyperbolicity cones of the initial form $\mathrm{in}(g)$ of g. It is well-known that the open Lorentz cone has the spectrahedral representation

$$L(\mathbf{z}) \;:=\; \left(\begin{array}{ccc|c} & & & z_1 \\ & z_n I & & \vdots \\ & & & z_{n-1} \\ \hline z_1 & \cdots & z_{n-1} & z_n \end{array}\right) \;\succ\; 0\,, \tag{8}$$

and thus we also have $z_n^{n-2}\,\mathrm{in}(g) = -\det(L(\mathbf{z}))$. Since g results from f by an affine transformation, the initial form $\mathrm{in}(g)$ results from the initial form $\mathrm{in}(f)$ by a linear transformation,

$$\mathrm{in}(g)(T\mathbf{z}) \;=\; \mathrm{in}(f)(\mathbf{z})$$

for some matrix $T \in \mathrm{GL}(n, \mathbb{R})$. Hence, we obtain the spectrahedral representation for one of the unbounded conic components in $\mathcal{I}(f)^{\mathsf{c}}$,

$$F(\mathbf{z}) \;:=\; \left(\begin{array}{ccc|c} & & & (T\mathbf{z})_1 \\ & (T\mathbf{z})_n I & & \vdots \\ & & & (T\mathbf{z})_{n-1} \\ \hline (T\mathbf{z})_1 & \cdots & (T\mathbf{z})_{n-1} & (T\mathbf{z})_n \end{array}\right) \;\succ\; 0\,,$$

as well as its negative. Moreover,

$$-\det F(\mathbf{z}) \;=\; ((T\mathbf{z})_n)^{n-2}\,\mathrm{in}(f)\,,$$

so that $(T\mathbf{z})_n$ provides the desired linear form $\ell(\mathbf{z})$. ☐

Remark 3.7. Concerning $L(\mathbf{z})$ in (8), by subtracting $\frac{z_j}{z_n}$ times the j-th row from its n-th row for every $j \in \{1, \ldots, n-1\}$, we obtain

$$\det(L(\mathbf{z})) = \det \left(\begin{array}{ccc|c} & & & z_1 \\ & z_n I & & \vdots \\ & & & z_{n-1} \\ \hline 0 & \cdots & 0 & z_n - \frac{1}{z_n}\sum_{i=1}^{n-1} z_i^2 \end{array}\right) = z_n^{n-2}\left(z_n^2 - \sum_{i=1}^{n-1} z_i^2\right).$$

By (3), in the proof we have $\mathrm{in}(f) = \mathbf{z}^T A \mathbf{z}$. Let $A = LDL^T$ be an LDL^T decomposition of A with $D = \mathrm{Diag}(d_1, \ldots, d_{n-1}, d_n)$ such that $d_1, \ldots, d_{n-1} > 0$ and $d_n < 0$. Then the variable transformation T in the proof is

$$T \;=\; \mathrm{Diag}(\sqrt{d_1}, \ldots, \sqrt{d_{n-1}}, \sqrt{|d_n|}) \cdot L^T$$

and we derive

$$A = T^T \cdot \left(\begin{array}{ccc|c} & & & 0 \\ & I & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & -1 \end{array}\right) \cdot T.$$

Example 3.8. Consider $f(z_1, z_2, z_3, z_4) = -15z_1^2 - 12z_1z_4 + z_2^2 + z_3^2 = \mathbf{z}^T A\mathbf{z}$ with

$$A = \begin{pmatrix} -15 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & 0 & 0 & 0 \end{pmatrix}.$$

For $\ell(\mathbf{z}) = 4z_1 + 2z_4$, a representation from Theorem 3.6 is

$$-\ell(\mathbf{z})^2 \cdot f(\mathbf{z}) = \det \begin{pmatrix} 4z_1 + 2z_4 & 0 & 0 & z_1 + 2z_4 \\ 0 & 4z_1 + 2z_4 & 0 & z_2 \\ 0 & 0 & 4z_1 + 2z_4 & z_3 \\ z_1 + 2z_4 & z_2 & z_3 & 4z_1 + 2z_4 \end{pmatrix}.$$

□

Remark 3.9. A quadratic polynomial $f \in \mathbb{R}[\mathbf{z}]$ is of the form (3) and of type (II) with $p = n - 1$ (i.e., $-f$ has Lorentzian signature) if and only if $f \in \mathbb{R}[\mathbf{z}]$ is a real zero polynomial, see for example [10].

Remark 3.10. For the case of homogeneous polynomials, Theorem 3.6 recovers the known fact that hyperbolicity cones defined by homogeneous quadratic polynomials f are spectrahedral [30]. In the affine setting, we can homogenize the type (II) polynomial f w.r.t. variable z_0 and get a quadratic polynomial of type (I) in $n + 1$ variables with $p = n$. Then, using $\text{in}(f_h) = f_h$, Theorem 3.6 recovers that *the rigidly convex sets* (introduced by Helton-Vinnikov [18]) defined by real zero polynomials f are spectrahedra [30].

Remark 3.11. The proof of Theorem 3.6 explicitly explains a technique to compute a suitable linear factor $\ell(\mathbf{z})$ as well as a determinantal representation to get a spectrahedral structure.

4. Conic stability and positive maps

Based on the characterizations of the conic components in the complement of $\mathcal{I}(f)$, we now study the problem whether f is K-stable, in particular, whether it is psd-stable.

In order to decide whether the cone K is contained in one of the components of $\mathcal{I}(f)^{\mathsf{c}}$, observe that in the case of spectrahedral representations of K and of the components of $\mathcal{I}(f)^{\mathsf{c}}$, the problem of K-stability can be phrased as a containment problem for spectrahedra. The theory of positive and completely positive maps (as detailed in [31]) provides a sufficient condition for the containment problem of spectrahedra, see [16, 23, 24].

Definition 4.1. Given two linear subspaces $\mathcal{U} \subseteq \mathsf{Herm}_k$ and $\mathcal{V} \subseteq \mathsf{Herm}_l$ (or $\mathcal{U} \subseteq \mathcal{S}_k$ and $\mathcal{V} \subseteq \mathcal{S}_l$), a linear map $\Phi : \mathcal{U} \to \mathcal{V}$ is called *positive* if $\Phi(U) \succeq 0$ for any $U \in \mathcal{U}$ with $U \succeq 0$.

For $d \geq 1$, define the *d-multiplicity map* Φ_d on the set of all Hermitian $d \times d$ block matrices with symmetric $n \times n$-matrix entries by

$$(A_{ij})_{i,j=1}^d \;\mapsto\; \left(\Phi\left(A_{ij}\right)\right)_{i,j=1}^d.$$

The map Φ is called *d-positive* if the d-multiplicity map Φ_d (viewed as a map on a Hermitian matrix space) is a positive map. Φ is called *completely positive* if Φ_d is a positive map for all $d \geq 1$.

Let $U(\mathbf{x}) = \sum_{j=1}^n U_j x_j$ and $V(\mathbf{x}) = \sum_{j=1}^n V_j x_j$ be homogeneous linear pencils with symmetric matrices of size $k \times k$ and $l \times l$, respectively (since the matrices are symmetric, we prefer to denote the variables by $\mathbf{x}$ rather than $\mathbf{z}$). Then the spectrahedra $S_U := \{\mathbf{x} \in \mathbb{R}^n : U(\mathbf{x}) \succeq 0\}$, and $S_V := \{\mathbf{x} \in \mathbb{R}^n : V(\mathbf{x}) \succeq 0\}$ are cones. Further, let $\mathcal{U} = \operatorname{span}(U_1, \ldots, U_n) \subseteq \mathcal{S}_k$ and $\mathcal{V} = \operatorname{span}(V_1, \ldots, V_n) \subseteq \mathcal{S}_l$.

If $U_1, \ldots, U_n$ are linearly independent, then the linear mapping $\Phi_{UV} : \mathcal{U} \to \mathcal{V}$, $\Phi_{UV}(U_i) := V_i$, $1 \leq i \leq n$, is well defined.

Proposition 4.2 ([23]). *Let $U_1, \ldots, U_n \in \mathsf{Herm}_k$ (or, $U_1, \ldots, U_n \in \mathcal{S}_k$, respectively) be linearly independent and $S_U \neq \emptyset$. Then for the properties*

(1) *the semidefinite feasibility problem*

$$C = (C_{ij})_{i,j=1}^k \succeq 0 \text{ and } V_p = \sum_{i,j=1}^k (U_p)_{ij} C_{ij} \text{ for } p = 1, \ldots, n \tag{9}$$

has a solution with Hermitian (respectively symmetric) matrix C,

(2) *Φ_{UV} is completely positive,*

(3) *Φ_{UV} is positive,*

(4) *$S_U \subseteq S_V$,*

the implications and equivalences (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longleftrightarrow$ (4) *hold, and if $\mathcal{U}$ contains a positive definite matrix,* (1) $\Longleftrightarrow$ (2).

Note that the statement (1) $\Longrightarrow$ (4) (which does not involve the definition of Φ_{UV}) is also valid without the assumption of linear independence of $U_1, \ldots, U_n$ (see [16, 23]).

So, in case the cone K and the conic components of $\mathcal{I}(f)^{\mathsf{c}}$ can be described in terms of spectrahedra, we can approach the conic stability problem in terms of the block matrix $C \succeq 0$ in (9), the so-called *Choi matrix*, corresponding to an appropriate positive map Φ, which maps the underlying pencils of those spectrahedra onto each other certifying their containment. This sufficient condition is provided by a certain semidefinite feasibility problem whose non-emptiness of its feasible domain thus provides a sufficient criterion for psd-stability.

Moreover, if we know a spectrahedral description of some of the components of $\mathcal{I}(f)^{\mathsf{c}}$ (as in the quadratic case or the determinantal case), the sufficient containment criterion is based on writing a matrix pencil for these components using linear combinations of the matrices of a linear matrix pencil for K. As formalized in Theorem 4.8 and Corollary 4.9, taking the determinant of a matrix pencil for a suitable component of $\mathcal{I}(f)^{\mathsf{c}}$ provides a particular determinantal description for the homogeneous part of the given polynomial f. That description has exactly the structure of the sufficient determinantal criterion for psd-stability and thus provides an elegant determinantal representation that certifies the psd-stability of a homogeneous polynomial f.

Let K be a cone which is given as the positive semidefiniteness region of a linear matrix pencil $M(\mathbf{x}) = \sum_{j=1}^n M_j x_j$ with symmetric $l \times l$-matrices (since K is a cone in $\mathbb{R}^n$, we prefer to denote the variables by $\mathbf{x}$ rather than $\mathbf{z}$). In the case of usual stability, the cone K is the positive semidefiniteness region of the linear matrix pencil

$$M^{\geq 0}(\mathbf{x}) = \sum_{j=1}^{n} M_j^{\geq 0} x_j \tag{10}$$

with $M_j^{\geq 0} = E_{jj}$, where E_{ij} is the matrix with a one in position (i,j) and zeros elsewhere. In the case of psd-stability, the matrix pencil is

$$M^{\mathrm{psd}}(X) = \sum_{i,j=1}^{n} M_{ij}^{\mathrm{psd}} x_{ij} \tag{11}$$

with symmetric matrix variables $X = (x_{ij})$ and $M_{ij}^{\mathrm{psd}} = \frac{1}{2}(E_{ij} + E_{ji})$, i.e., $M^{\mathrm{psd}}(X)$ is the matrix pencil $M^{\mathrm{psd}}(X) = (x_{ij})_{ij}$ in the symmetric matrix variables x_{ij}.

Theorem 4.3. *Let $f = \det(A_0 + \sum_{j=1}^n A_j z_j)$ with Hermitian matrices $A_0, \ldots, A_n$ be a degree d determinantal polynomial of the form* (2) *such that* $\mathrm{in}(f)$ *is irreducible and there exists $\mathbf{e} \in \mathbb{R}^n$ with $\sum_{j=1}^n A_j e_j \succ 0$. Let $M(\mathbf{x}) = \sum_{j=1}^n M_j x_j$ with symmetric $l \times l$-matrices be a pencil of the cone K. If there exists a Hermitian block matrix $C = (C_{ij})_{i,j=1}^l$ with blocks C_{ij} of size $d \times d$ and*

$$C = (C_{ij})_{i,j=1}^l \succeq 0, \quad \forall p = 1, \ldots, n \,:\, \sigma A_p = \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij} \tag{12}$$

for some $\sigma \in \{-1, 1\}$, then f is K-stable. Deciding whether such a block matrix C exists is a semidefinite feasibility problem.

Note that a necessary condition of K-stability of f is obtained as follows. Fix any vector $\mathbf{v}$ in the interior of the cone K. Then a necessary condition for K-stability is that $\mathbf{v}$ is contained in the complement of $\mathcal{I}(f)$.

Proof. Let C be a block matrix $C = (C_{ij})_{i,j=1}^l$ with $d \times d$-blocks and which satisfies (12) for some $\sigma \in \{-1, 1\}$. The initial form $\mathrm{in}(f)$ is hyperbolic and, by Theorem 3.1, every hyperbolicity cone of $\mathrm{in}(f)$ is contained in $\mathcal{I}(f)^{\mathsf{c}}$. So, in order to show K-stability of f, it suffices to show that K is contained in the closure of a hyperbolicity cone of $\mathrm{in}(f)$, i.e., in the closure of a component of $\mathcal{I}(\mathrm{in}(f))^{\mathsf{c}}$.

As recorded at the beginning of Section 3, since $\mathrm{in}(f)$ is irreducible, $\mathrm{in}(f)$ has exactly two hyperbolicity cones, and these are given by $A^h(\mathbf{x}) = \sum_{j=1}^n A_j x_j \succ 0$ as well as $A^h(\mathbf{x}) = \sum_{j=1}^n A_j x_j \prec 0$.

By Proposition 4.2, if (12) is satisfied, say with $\sigma = 1$, then the spectrahedron given by the matrix pencil $M(\mathbf{x})$ is contained in the closure of $\mathcal{I}(\mathrm{in}(f))^{\mathsf{c}}$. For the service of the reader, we provide an explicit derivation of this step in our setting. Namely, for $\mathbf{x}$ in the spectrahedron defined by $M(\mathbf{x})$, we have

$$A^h(\mathbf{x}) = \sum_{p=1}^{n} A_p x_p = \sum_{p=1}^{n} x_p \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij} \tag{13}$$

$$= \sum_{i,j=1}^{l} (M(\mathbf{x}))_{ij} C_{ij}. \tag{14}$$

Apply the Khatri-Rao product (where the blocks of $M(\mathbf{x})$ are of size 1×1 and the blocks of C are of size $d \times d$). Since $M(\mathbf{x})$ and C are positive semidefinite, the Khatri-Rao product

$$M(\mathbf{x}) * C \ := \ ((M(\mathbf{x}))_{ij} \otimes C_{ij})_{i,j=1}^{l} \ = \ ((M(\mathbf{x}))_{ij} C_{ij})_{i,j=1}^{l}$$

is positive semidefinite as well; see Liu [27], where this property is stated on the space of symmetric positive semidefinite matrices. Since $M(\mathbf{x})$ is a real symmetric pencil, Liu's result carries over to our situation of a Hermitian positive semidefinite matrix C by employing that a Hermitian matrix $Z = X + iY$ with $X \in \mathcal{S}_k$ and Y skew-symmetric is positive semidefinite if and only if the real symmetric matrix

$$\begin{pmatrix} X & -Y \\ Y & Z \end{pmatrix} \ \in \ \mathcal{S}_{2k}$$

is positive semidefinite (see, e.g., [12]).

Altogether, since

$$A^h(\mathbf{x}) \ = \ (I \cdots I)(M(\mathbf{x}) * C) \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix} ,$$

$A^h(\mathbf{x})$ is positive semidefinite as well. Hence, $\mathbf{x}$ is contained in the spectrahedron defined by $A^h(\mathbf{x})$. Since $A^h(\mathbf{x})$ is the matrix pencil of the closure of a component of $\mathcal{I}(\mathrm{in}(f))^{\mathsf{c}}$, the claim follows. ☐

Note that the constant coefficient matrix A_0 does not play any role for the criterion in Theorem 4.3. This comes from Theorem 3.1 and its proof, where only the Hermitian property of A_0 matters rather than the exact values of the coefficients themselves.

Remark 4.4. In the special case of usual stability, Theorem 4.3 provides a new proof for Borcea and Brändén's determinantal criterion from Proposition 2.6. Namely, for usual stability, K is given by (10) and thus, a matrix C satisfying the hypothesis of Theorem 4.3 can be viewed as a block diagonal matrix $C = (C_{ij})_{i=1}^{l}$ with diagonal blocks C_{ii} of size $d \times d$ and vanishing non-diagonal blocks C_{ij} $(i \neq j)$. Since the condition (12) specializes to

$$A_p = C_{pp} \quad \text{for } p = 1, \ldots, n,$$

the stability criterion in Theorem 4.3 is satisfied if and only if the matrices $A_1, \ldots, A_n$ are positive semidefinite.

Remark 4.5. Theorem 4.3 gives a sufficient criterion, but it is not necessary. As a counterexample, consider the following adaption from an example in [16, Example 3.1, 3.4] and [23, Section 6.1]. Let $K \subseteq \mathbb{R}^3$ be the Lorentz cone as given by (8). The polynomial

$$f \ = \ \det \begin{pmatrix} z_1 + z_3 & z_2 \\ z_2 & -z_1 + z_3 \end{pmatrix} \ = \ z_3^2 - z_1^2 - z_2^2$$

(whose underlying matrix pencil provides an alternative matrix pencil for the Lorentz cone) has all its zeroes on the boundary of the Lorentz cone or on its negative. Hence, f is K-stable, but by the results in [16] and [23], the condition (12) is not satisfied.

Example 4.6. i) Let $g(z_1, z_2, z_3) := 31z_1^2+32z_1z_3+8z_3^2-8z_1z_2-16z_2^2$. A determinantal representation of g is given by $\det \begin{pmatrix} 4z_1 + 2z_3 & z_1 + 4z_2 \\ z_1 + 4z_2 & 8z_1 + 4z_3 \end{pmatrix}$, and at $\mathbf{z} = (0, 0, 1)^T$, the matrix polynomial is positive definite. Let $M(\mathbf{x})$ denote the linear matrix pencil of the psd cone $\mathcal{S}_2^+$. Then the psd-stability of g follows from Theorem 4.3 and by the matrix

$$C = \begin{pmatrix} 4 & 1 & 0 & 2 \\ 1 & 8 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 4 \end{pmatrix} \succeq 0.$$

ii) Let $f = \sum_{i,j=1}^2 M_{ij}^{\text{psd}} x_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_{11} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x_{12} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x_{22}$ be the canonical matrix polynomial of the 2×2-psd cone. Clearly, f is psd-stable, and the following consideration shows that this is also recognized by the sufficient criterion. For symmetric 2×2-matrices, the condition in Theorem 4.3 requires to find a block matrix $C \succeq 0$ with 2×2 blocks of size 2×2 such that

$$M_{pq}^{\text{psd}} = \sum_{i,j=1}^{2} (M_{pq}^{\text{psd}})_{ij} C_{ij} \quad \text{for } 1 \le p, q \le 2. \tag{15}$$

This yields $C_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $C_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $C_{12} + C_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ is symmetric, C_{12} must be of the form $\begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix}$ with $\gamma, \delta \in \mathbb{R}$. Positive semidefiniteness of C then implies $\delta = 0$, and further, the condition on $C_{12} + C_{21}$ gives $\gamma = 1$. Hence, the matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

satisfies (15) and thus certifies the psd-stability of f in view of the sufficient criterion in Theorem 4.3.

For quadratic polynomials, we can provide the following criterion. As in the proof of Theorem 3.6, for a homogeneous quadratic polynomial $f(\mathbf{z}) = \mathbf{z}^T A \mathbf{z}$ of signature $(n-1, 1)$, we consider

$$F(\mathbf{x}) := \sum_{p=1}^{n} F_p x_p \ := \left(\begin{array}{ccc|c} & (T\mathbf{x})_n I & & \begin{matrix} (T\mathbf{x})_1 \\ \vdots \\ (T\mathbf{x})_{n-1} \end{matrix} \\ \hline (T\mathbf{x})_1 & \cdots & (T\mathbf{x})_{n-1} & (T\mathbf{x})_n \end{array} \right) \succ 0\,, \tag{16}$$

where T is as in that proof.

Theorem 4.7. *Let $n \ge 3$ and f be a quadratic polynomial of the form* (3), *let f be of type (II) with A having signature $(n-1, 1)$ and $\text{in}(f)$ be irreducible. Let $M(\mathbf{x})$ be a matrix pencil for the cone K, and let T and $F(\mathbf{x}) := \sum_{p=1}^{n} F_p x_p$ be defined as in* (16) *w.r.t.* $\mathrm{in}(f)$. *If there exists a block matrix* $C = (C_{ij})_{i=1}^{l}$ *with blocks* C_{ij} *of size* $d \times d$ *and*

$$C = (C_{ij})_{i,j=1}^{l} \ \succeq \ 0, \quad \forall p = 1, \ldots, n \ : \ \sigma F_p \ = \ \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij} \tag{17}$$

for some $\sigma \in \{-1, 1\}$, *then* f *is* K*-stable. Deciding whether such a block matrix* C *exists is a semidefinite feasibility problem.*

Proof. By Theorem 3.6 and its proof, the unbounded components of $\mathcal{I}(f)^{\mathsf{c}}$ which are full-dimensional cones are exactly the hyperbolicity cones of $\mathrm{in}(f)$. For $\mathbf{x}$ in the spectrahedron defined by $M(\mathbf{x}) \succeq 0$, we have

$$F(\mathbf{x}) \ = \ \sum_{p=1}^{n} F_p x_p \ = \ \sum_{p=1}^{n} x_p \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij} \ = \ \sum_{i,j=1}^{l} (M(\mathbf{x}))_{ij} C_{ij}.$$

Analogous to the application of the Khatri-Rao product in the proof of Theorem 4.3, this yields $F(\mathbf{x}) \succeq 0$. Hence, f is K-stable. □

Theorem 4.8. *Let* $n \geq 3$ *and* $f(\mathbf{z}) = \mathbf{z}^T A \mathbf{z}$ *be an irreducible homogeneous quadratic polynomial of signature* $(n-1, 1)$, $M(\mathbf{z})$ *be a matrix pencil for the cone* K, *and let* T *and* $F(\mathbf{z}) := \sum_{p=1}^{n} F_p z_p$ *be defined as in* (16). *If there exists a block matrix* $C = (C_{ij})_{i=1}^{l}$ *with blocks* C_{ij} *of size* $d \times d$ *satisfying*

$$C = (C_{ij})_{i,j=1}^{l} \ \succeq \ 0, \quad \forall p = 1, \ldots, n \ : \ \sigma F_p \ = \ \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij} \tag{18}$$

for some $\sigma \in \{-1, 1\}$, *then there exists a linear form* $\ell(\mathbf{z})$ *such that* $-\ell(\mathbf{z})^{n-2} f$ *has a determinantal representation*

$$-\sigma \ell(\mathbf{z})^{n-2} f = \det\Big(\sum_{p=1}^{n} z_p \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij}\Big)$$

with positive semidefinite matrices C_{ij}. *The representation provides a certificate for the* K*-stability of* f.

Proof. The K-stability was shown in Theorem 4.7. By (18) and the definition of $F(\mathbf{z})$, we have

$$\sigma \det F(\mathbf{z}) = \det\Big(\sum_{p=1}^{n} z_p \sum_{i,j=1}^{l} (M_p)_{ij} C_{ij}\Big).$$

Since $\det F(\mathbf{z}) = -((T\mathbf{z})_n)^{n-2} f$, the choice $\ell(\mathbf{z}) := (T\mathbf{z})_n$ provides the desired representation. This provides a certificate for the K-stability of f. □

Corollary 4.9. *Let* $n \geq 2$ *and* $f(Z)$ *be a homogeneous quadratic polynomial on symmetric* $n \times n$*-variables, in the linearized vector* $\mathbf{z} = (z_1, \ldots, z_N)$ *let* $f = \mathbf{z}^T A \mathbf{z}$ *with* $A \in \mathbb{R}^{N \times N}$ *of signature* $(N-1, 1)$. *If* $M(\mathbf{z})$ *is a matrix pencil for the psd-cone and* C

is a block matrix satisfying (18)*, then for some linear form $\ell(\mathbf{z})$ in $\mathbf{z}$, the polynomial $-\ell(\mathbf{z})^{N-2}f$ has a determinantal representation of the form*

$$-\ell(\mathbf{z})^{N-2}f \;=\; \det\big(\sum_{i,j=1}^{l} C_{ij}z_{ij}\big)$$

with positive semidefinite matrices C_{ij}. This representation provides a certificate for the psd-stability of f in the sense of the sufficient criterion for psd-stability.

Proof. This is a consequence of Theorem 4.8. □

5. Certifying K-stability with respect to scaled cones

The sufficient criterion does not capture all the cases of K-stable polynomials. Here, we extend our techniques to scaled versions of the cone. To this end, we will reduce a scaled version of the K-stability problem to the situation of the following statement.

Proposition 5.1 (Proposition 6.2 in [23])**.** *Let $A(\mathbf{z})$ and $B(\mathbf{z})$ be monic linear matrix pencils of size $k \times k$ and $l \times l$, respectively, and such that $S_A := \{\mathbf{z} \in \mathbb{R}^n : A(\mathbf{z}) \succeq 0\}$ is bounded. Then there exists a constant $\nu > 0$ such that for the scaled spectrahedron νS_A the inclusion $\nu S_A \subseteq S_B$ is certified by the system*

$$C = (C_{ij})_{i,j=1}^{k} \;\succeq\; 0, \quad \forall p = 1,\ldots,n \;:\; B_p \;=\; \sum_{i,j=1}^{k}\Big(\frac{1}{\nu}A_p\Big)_{ij}C_{ij}.$$

As before, let K be a proper cone which is given by a linear matrix pencil $M(\mathbf{z}) = \sum_{j=1}^{n} M_j z_j$ with $l\times l$-matrices, and assume that there exists a hyperplane H not passing through the origin and such that $K\cap H$ is bounded. For notational convenience, assume that $H = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_1 = 1\}$ and that $M_1 = I_n$. In particular, then the first unit vector $\mathbf{e}^{(1)}$ is contained in the interior of the full-dimensional cone K.

Theorem 5.2. *Let $f \in \mathbb{R}[\mathbf{z}]$ and $M(\mathbf{z})$ be as described before. Let $N(\mathbf{z})$ be the matrix pencil of a spectrahedral, conic set contained in* $\mathrm{cl}(\mathcal{I}(f)^{\mathrm{c}})$*, and assume that $N_1 = I_n$ as well.*

Then there exists a constant $\nu > 0$ such that $g_\nu(z_1,\ldots,z_n) := f(z_1,\nu z_2,\ldots,\nu z_n)$ is K-stable and such that the K-stability of g is certified by the system

$$C = (C_{ij})_{i,j=1}^{l} \succeq 0, \quad \forall p = 1,\ldots,n \;:\; \nu N_p \;=\; \sum_{i,j=1}^{l}(M_p)_{ij}\,C_{ij}, \tag{19}$$

where the variable matrix C is a block matrix with $l \times l$ blocks.

As a consequence, f is $\hat{K}$-stable with respect to $\hat{K} = \mathrm{cone}(\{1\} \times \nu(K\cap H))$, where the multiplication of ν with the set $K\cap H$ is done in the $(n-1)$-dimensional space with variables $\mathbf{z}' = (z_2,\ldots,z_n)$ and cone *denotes the conic hull.*

Since the scaling variable ν occurs linearly in (19), its optimal value can be expressed by a semidefinite program. Further note that the preconditions $M_1 = I_n$ and $N_1 = I_n$ imply that the induced matrix pencils of the conic spectrahedra of $M(\mathbf{z})$ and of $N(\mathbf{z})$ give monic pencils within the hyperplane H.

Proof. Let $N'(\mathbf{z}'), M'(\mathbf{z}')$ be the matrix pencils in the $n-1$ variables $\mathbf{z}' = (z_2, \ldots, z_n)$ defined by

$$N'(\mathbf{z}') = N(\mathbf{z})\Big|_{z_1=1} \text{ and } M'(\mathbf{z}') = M(\mathbf{z})\Big|_{z_1=1}.$$

$N'(\mathbf{z}')$ and $M'(\mathbf{z}')$ are monic linear matrix pencils and the spectrahedron $S_{M'(\mathbf{z}')} = \{\mathbf{z}' = (z_2, \ldots, z_n) \in \mathbb{R}^n : M'(\mathbf{z}') \succeq 0\}$ is bounded. By Proposition 5.1, the inclusion $\nu S_{M'(\mathbf{z}')} \subseteq S_{L'(\mathbf{z}')}$ is certified by the system

$$C = (C_{ij})_{i,j=1}^{l} \succeq 0, \quad \forall p = 1, \ldots, n \ : \ N'_p = \sum_{i,j=1}^{l} \left(\frac{1}{\nu} M'_p\right)_{ij} C_{ij} \tag{20}$$

with some block matrix $C = (C_{ij})_{i,j=1}^{l}$. Since $M'_p = M_p$ and $N'_p = N_p$ for $p \geq 1$, this is equivalent to (19).

Moreover, $\nu S_{M'(\mathbf{z}')} \subseteq S_{N'(\mathbf{z}')}$ implies that $\nu S_{M(\mathbf{z})} \subseteq S_{N(\mathbf{z})}$ and also that for any $\mathbf{z}$ with $z_1 = 1$ and $f(\mathbf{z}) = 0$, we have $(1, \frac{z_2}{\nu}, \ldots, \frac{z_n}{\nu}) \notin \operatorname{int} S_{M'(\mathbf{z}')}$, or, equivalently, $g_\nu(\mathbf{z})$ is K-stable. Finally, this also gives the reformulation that f is $\hat{K}$-stable. □

Theorem 5.2 can also be applied to such polynomials f which meet the requirements of the theorem after applying a invertible linear transformation, since those preserve the containment of sets.

Example 5.3. Setting $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix}$, the polynomial $f = \det \begin{pmatrix} z_1 & 2z_2 \\ 2z_2 & z_3 \end{pmatrix}$ $= z_1 z_3 - 4z_2^2$ is not psd-stable. To fit the requirements of Theorem 5.2, let Q be the rotation matrix $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}$ and consider the rotated versions of the underlying matrix pencils

$$\begin{aligned} N_Q(\mathbf{y}) &= N(Q^{-1}\mathbf{z}) = \frac{1}{\sqrt{2}} \begin{pmatrix} y_1 - y_3 & \sqrt{8} y_2 \\ \sqrt{8} y_2 & y_1 + y_3 \end{pmatrix} \\ \text{and } M_Q(\mathbf{y}) &= M(Q^{-1}\mathbf{z}) = \frac{1}{\sqrt{2}} \begin{pmatrix} y_1 - y_3 & \sqrt{2} y_2 \\ \sqrt{2} y_2 & y_1 + y_3 \end{pmatrix}. \end{aligned}$$

For $N_{Q,\nu}(\mathbf{y}) := N_Q(y_1, \nu y_2, \nu y_3)$ and $M_Q(\mathbf{y})$, (19) leads to the equations

$$C_{11} + C_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_{12} + C_{21} = \begin{pmatrix} 0 & 2\nu \\ 2\nu & 0 \end{pmatrix}, \quad -C_{11} + C_{22} = \begin{pmatrix} -\nu & 0 \\ 0 & \nu \end{pmatrix}.$$

Hence, the set of matrices $C = C_\nu$ satisfying (19) is given by the system

$$C = \frac{1}{2} \begin{pmatrix} 1+\nu & 0 & 0 & 4\lambda\nu \\ 0 & 1-\nu & 4(1-\lambda)\nu & 0 \\ 0 & 4(1-\lambda)\nu & 1-\nu & 0 \\ 4\lambda\nu & 0 & 0 & 1+\nu \end{pmatrix}, \quad C \succeq 0 \quad \text{with } \lambda \in \mathbb{R}. \tag{21}$$

The largest ν satisfying (21) is given by $\nu = \frac{1}{2}$ with $\lambda = \frac{3}{4}$. When rotating back, this certifies the psd-stability of

$$f_{\frac{1}{2}}(\mathbf{z}) := \det\left(N_{Q,\frac{1}{2}}(Q\mathbf{z})\right) = \frac{1}{16} \cdot (3z_1^2 + 10z_1 z_3 + 3z_3^2 - 16z_2^2).$$

In addition to that, we obtain that f is $\hat{K}$-stable with respect to the cone

$$\hat{K} \ = \ \left\{ \mathbf{y} \in \mathbb{R}^3 \ : \ \frac{1}{2} \begin{pmatrix} 3y_1 - y_3 & 4y_2 \\ 4y_2 & -y_1 + 3y_3 \end{pmatrix} \succeq 0 \right\}.$$

6. Conclusion and open questions

In this paper, we have shown how techniques from the theory of positive maps and from the containment of spectrahedra can be used to provide a sufficient criterion for the K-stability of a given polynomial f. In particular, we have considered quadratic and determinantal polynomials. Beyond that, our approach generally applies whenever (for a polynomial of arbitrary degree) some spectrahedral components in the complement of $\mathcal{I}(f)$ are known.

It would be interesting to understand whether this or related techniques can be effectively exploited also for classes of polynomials beyond the ones studied in the paper. In particular, with regard to the recent development of a theory of Lorentzian polynomials [8], which provides a superset of the set of homogeneous stable polynomials, it would be of interest to understand the connection of Lorentzian polynomials to conic stability and to the effective methods presented in our paper.

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P. Dey: Department of Mathematics, University of Missouri, Columbia, MO, USA

S. Gardoll and T.Theobald: Goethe-Universität, FB 12 – Institut für Mathematik, Postfach 11 19 32, 60054 Frankfurt am Main, Germany