# ALGEBRAIC METHODS FOR COMPUTING SMALLEST ENCLOSING AND CIRCUMSCRIBING CYLINDERS OF SIMPLICES

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Abstract. We provide an algebraic framework to compute smallest enclosing and smallest circumscribing cylinders of simplices in Euclidean space $\mathbb{E}^n$. Explicitly, the computation of a smallest enclosing cylinder in $\mathbb{E}^3$ is reduced to the computation of a smallest circumscribing cylinder. We improve existing polynomial formulations to compute the locally extreme circumscribing cylinders in $\mathbb{E}^3$ and exhibit subclasses of simplices where the algebraic degrees can be further reduced. Moreover, we generalize these efficient formulations to the $n$-dimensional case and provide bounds on the number of local extrema. Using elementary invariant theory, we prove structural results on the direction vectors of any locally extreme circumscribing cylinder for regular simplices.

## 1. Introduction

Radii (of various types) belong to the most important functionals of polytopes and general convex bodies in Euclidean space $\mathbb{E}^n$ [3, 16, 18], and they are related to applications in computer vision, robotics, computational biology, functional analysis, and statistics (see [17]). Following the notation in [3], the *outer $j$-radius* $R_j(\mathcal{C})$ of a convex body $\mathcal{C} \subset \mathbb{E}^n$ is the radius of the smallest enclosing $j$-dimensional sphere in an optimal orthogonal projection of $\mathcal{C}$ onto a $j$-dimensional linear subspace. Studying these radii, mainly for regular simplices and regular polytopes, is a classical topic of convex geometry (see [2, 4, 12, 16]).

From the computational point of view, most of the existing algorithms for computing these radii focus on approximation [6, 19]. A major reason is that exact computations lead to algebraic problems of high degree, even for computing, say, the outer $(n-1)$-radius in $\mathbb{E}^n$ (already if $n = 3$). However, since some approaches for computing radii of general polytopes consider the computation of a smallest enclosing or smallest circumscribing cylinder of a simplex as a black box within a larger computation [1, 24], these core problems on simplices are of fundamental importance.

Recently, the authors of [10] demonstrated that using their state-of-the-art numerical polynomial solvers, various problems related to cylinders in $\mathbb{E}^3$ can be solved rather efficiently. In particular, the authors give a polynomial formulation for the smallest circumscribing cylinder of a simplex in $\mathbb{E}^3$, whose Bézout number – the product of the degrees of the polynomial equations – is 60. However, these equations contain certain undesired solutions with multiplicity 4, and as a result of these multiplicities the computation times

2000 *Mathematics Subject Classification.* 51N20, 52B55, 68U05, 68W30, 90C90.

*Key words and phrases.* Smallest enclosing cylinder, circumscribing cylinder, simplex, outer radius, polynomial equations.

(using state-of-the-art numerical techniques) are about a factor 100 larger than those of similar problems in which all solutions occur with multiplicity 1.

Here, we provide a general algebraic framework for computing smallest enclosing and circumscribing cylinders of simplices in $\mathbb{E}^n$. First we reduce the computation of a smallest enclosing cylinder in $\mathbb{E}^3$ to the computation of a smallest circumscribing cylinder, thus combining these two problems. Then we investigate smallest circumscribing cylinders of simplices in $\mathbb{E}^3$. We improve the results of [10] by providing a polynomial formulation for the locally extreme cylinders, whose Bézout bound is 36 and whose solutions generically have multiplicity one. Our formulations use techniques from the paper [22] which studies the lines simultaneously tangent to four unit spheres. These techniques also facilitate to present classes of simplices for which the algebraic degrees in computing smallest circumscribing cylinders can be considerably reduced.

Section 4 contains a generalization of our approach to smallest circumscribing cylinders of simplices in $\mathbb{E}^n$. The Bézout number of this formulation yields a bound on the number of locally extreme cylinders. Since that bound is not tight, we provide better bounds for small dimensions, which are are based on mixed volume computations and Bernstein's Theorem. Moreover, we study in detail the locally extreme circumscribing cylinders of a regular simplex in $\mathbb{E}^n$. To exploit many symmetries in the analysis, a formulation based on symmetric polynomials is provided. Using elementary invariant theory we show that the direction vector of every locally extreme circumscribing cylinder has at most three distinct values in its components. This structural result is then related to the combinatorial results on the number of solutions for general simplices.

As a byproduct of our computational studies, we discovered a subtle but severe mistake in the paper [32] on the explicit determination of the outer $(n-1)$-radius for a regular simplex in $\mathbb{E}^n$ ($n$ even), thus completely invalidating the proof given there. The appendix contains a description of that flaw, including some computer-algebraic calculations illustrating it. We remark that after the present paper had been finished, we found a new way for determining $R_{n-1}$ of a regular simplex in even dimension (see [5]).

## 2. Preliminaries and background

2.1. **$j$-radii and cylinders.** Throughout the paper we work in Euclidean space $\mathbb{E}^n$, i.e., $\mathbb{R}^n$ with the usual scalar product $x \cdot y = \sum_{i=1}^n x_i y_i$ and norm $||x|| = (x \cdot x)^{1/2}$. We write $x^2$ for $x \cdot x$.

A *$j$-flat* is an affine subspace of dimension $j$. For a convex polytope $\mathcal{P} \subset \mathbb{E}^n$ (or a finite point set $\mathcal{P} \subset \mathbb{E}^n$) and a $j$-flat $E$, we consider

$$\mathcal{RD}(\mathcal{P}, E) := \max_{p \in \mathcal{P}} \operatorname{dist}(p, E),$$

where $\operatorname{dist}(p, E)$ denotes the Euclidean distance from $p$ to $E$. The *outer $j$-radius* of $\mathcal{P}$ is

$$R_j(\mathcal{P}) := \min_{E \text{ is an } (n-j)\text{-flat}} \mathcal{RD}(\mathcal{P}, E)\,.$$

The choice of the indexing in the $j$-radius stems from the fact that it measures the radius of the smallest enclosing $j$-dimensional sphere in an optimal orthogonal projection of $\mathcal{P}$ onto a $j$-dimensional linear subspace (cf. [3, 16]).

One of the most natural representatives of this class is the one with $j = 2$, $n = 3$, i.e., the smallest enclosing (circular) cylinder of a polytope. In $\mathbb{E}^n$, we define a cylinder to be a set of the form

$$\mathrm{bd}(\ell + \rho\mathbb{B}^n),$$

where $\ell$ is a line in $\mathbb{E}^n$, $\mathbb{B}^n$ denotes the unit ball, $\rho > 0$, the addition denotes the Minkowski sum, and $\mathrm{bd}(\cdot)$ denotes the boundary of a set. We say that $P$ *can be enclosed* in a cylinder $\mathcal{C}$ if $P$ is contained in the convex hull of $\mathcal{C}$. Thus the outer $(n-1)$-radius gives the radius of a smallest enclosing cylinder of a polytope.

A simplex in $\mathbb{E}^n$ is the convex hull of $n+1$ affinely independent points. An enclosing cylinder $\mathcal{C}$ of a simplex $\mathcal{P}$ is called a *circumscribing* cylinder of $\mathcal{P}$ if all the vertices of $\mathcal{P}$ are contained in (the hypersurface) $\mathcal{C}$.

2.2. **Smallest circumscribing cylinders and smallest enclosing cylinders.** The following statement connects the computation of a smallest enclosing cylinder of a polytope with the computation of a smallest circumscribing cylinder of a simplex.[1]

**Theorem 1.** *Let $\mathcal{P} = \{p_1, \ldots, p_m\}$ be a set of $m \geq 4$ points in $\mathbb{E}^3$, not all collinear. If $\mathcal{P}$ can be enclosed in a circular cylinder $\mathcal{C}$ of radius $r$, then there exists a circular cylinder $\mathcal{C}'$ of radius $r$ enclosing all elements of $\mathcal{P}$ such that the surface $\mathcal{C}'$ passes through*

(i) *at least four non-collinear points of $\mathcal{P}$, or*
(ii) *three non-collinear points of $\mathcal{P}$, and the axis $\ell$ of $\mathcal{C}'$ is contained in*
   (a) *the cylinder naturally defined by spheres of radius $r$ centered at two of these points;*
   (b) *the double cone naturally defined by spheres of radius $r$ centered at two of these points (and these spheres are disjoint);*
   (c) *or the set of lines which are tangent to the two spheres of radius $r$ centered at two of these points and which are contained in the plane equidistant from these points (and the spheres are non-disjoint).*

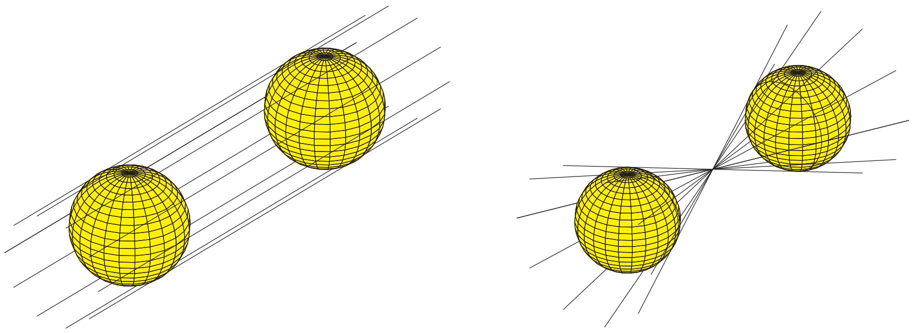
*Moreover, $\mathcal{C}$ can be transformed into $\mathcal{C}'$ by a continuous motion.*

Figures 1 and 2 visualize the three geometric properties in the second possibility.

Since case (ii) in Theorem 1 characterizes the possible special cases, this lemma in particular reduces the computation of a smallest enclosing cylinder of a simplex in $\mathbb{E}^3$ to the computation of a smallest circumscribing cylinder of a simplex. Namely, it suffices to compute the smallest circumscribing cylinder (corresponding to case (i)) as well as the smallest enclosing cylinders whose axes satisfies one of the conditions in (ii); the latter case gives a constant number of problems of smaller algebraic degree (since the positions of the axes are very restricted).

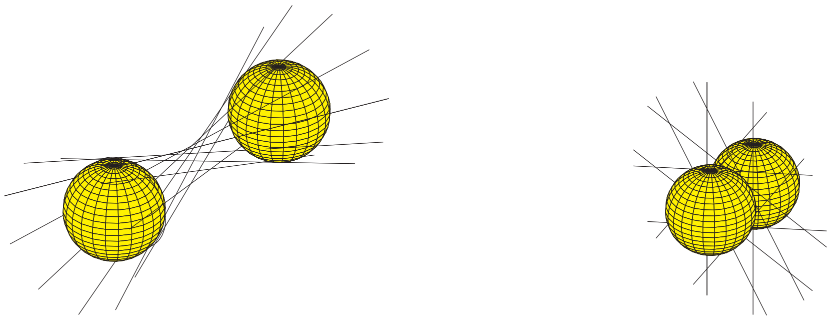
**Remark 2.** Before we start with the proof, we remark that Theorem 1 and its different cases show a quite similar behavior as the well known statement that the (unique) circumsphere of a simplex touches all its vertices, or one of its great $(n-1)$-circles is the circumsphere of one of the $(n-1)$-faces of the simplex (see [2, p. 54]).

[1]We remark that a similar statement has already been used in [24], but the manuscript referenced there does not contain a complete proof.



(a) Cylinder (b) Double cone with apex $(a/2,0,0)^T$

FIGURE 1. Extreme situations of the set of hyperboloids for disjoint spheres



(a) Hyperboloid for $0 < x_h < 2r^2/a$ (b) Degenerated hyperboloid for $x_h = a/2$

FIGURE 2. The left figure shows a general situation for disjoint spheres; the right figure shows an extreme situation for non-disjoint spheres

In the proof we will apply the following geometric equivalence. A point $x \in \mathbb{E}^3$ is enclosed in a cylinder with axis $\ell$ if and only if $\ell$ is a transversal of the sphere with radius $r$ centered at $x$ (i.e., $\ell$ is a line intersecting the sphere).

*Proof of Theorem 1.* Let $\mathcal{C}$ be a cylinder with axis $\ell$ and radius $r$ enclosing $\mathcal{P}$. Then, denoting by $S_i := S(p_i, r)$ the sphere with radius $r$ centered at $p_i$, $\ell$ is a common transversal to $S_1, \dots, S_m$. By continuously translating and rotating $\ell$, we can assume that $\ell$ is tangent to two of the spheres, say $S_1$ and $S_2$. Further, by changing coordinates, we can assume that $S_1$ and $S_2$ have the form $S_1 = S((0,0,0)^T, r)$, $S_2 = S((a,0,0)^T, r)$ for some $a > 0$.

The set of lines tangent to two spheres of radius $r$ constitutes a set of hyperboloids (see, e.g., [9, 20]). Moreover, any of these hyperboloids touches the sphere $S_1$ on a circle lying in a hyperplane parallel to the $yz$-plane. Hence, the set of hyperboloids can be parametrized by the $x$-coordinate of this hyperplane which we denote by $x_h$.

If $S_1 \cap S_2 = \emptyset$ then the boundary values are $x_h = 0$ and $x_h = 2r^2/a$. These two extreme situations yield a cylinder and a double cone with apex $(a/2,0,0)^T$, respectively (see Figure 1). For $0 < x_h < 2r^2/a$ we obtain a hyperboloid of one sheet (see Figure 2(a)).

If $S_1 \cap S_2 \neq \emptyset$ then the boundary values are $x_h = 0$ and $x_h = a/2$. Here, for $0 < x_h < a/2$ we obtain hyperboloids of one sheet, too. For $x_h = a/2$ the hyperboloid degenerates to a set of tangents which are tangents to the circle with radius $r_c = \sqrt{4r^2 - a^2}$ in the hyperplane $x = a/2$ (see Figure 2(b)).

Let $x_{h,0}$ be the parameter value of the hyperboloid containing the line $\ell$. By decreasing the parameter $x_h$ starting from $x_{h,0}$ the hyperboloid changes its shape towards the cylinder around $S_1$ and $S_2$. Let $x_{h,1}$ be the infimum of all $0 \leq x_h < x_{h,0}$ such that the hyperboloid does not contain a generating line tangent to some other sphere $S(p_i, r)$ for some $3 \leq i \leq m$. If $x_{h,1} = 0$, then by choosing any point of $\mathcal{P}$ not collinear to $p_1$ and $p_2$ we are in case (ii) (a).

If $x_{h,1} > 0$ then let $p_3$ be the corresponding point. Let $T(S_1, S_2, S_3)$ denote the set of lines simultaneously tangent to $S_1$, $S_2$, and $S_3$. Now let $x_{h,2}$ be the infimum of all $0 \leq x_h < x_{h,0}$ such that there exists a continuous function $\ell : (x_{h,2}, x_{h,1}) \to T(S_1, S_2, S_3)$ with $\ell(x_h)$ lying on the hyperboloid with parameter $x_h$. Since the spheres are compact, the infimum is a minimum. If $x_{h,2} > 0$ then one of three hyperboloids involved by the three pairs of spheres must be one of the extreme hyperboloids in that situation and we are in cases (ii) (a), (b), or (c). If $x_{h,2} = 0$ then we distinguish between two possibilities. Either during this process we also reached a tangent to some other sphere $S(p_i, r)$ for some $4 \leq i \leq m$; in this case we are in case (i). Or during the transformation all the points $p_4, \ldots, p_m$ are enclosed in the cylinder with axis $\ell$ and radius $r$, but none of them is contained in it. Then we arrive at situation (ii) (a). $\square$

## 3. Computing the smallest circumscribing cylinders of a simplex in $\mathbb{E}^3$

So far, we have seen how to reduce the computation of a smallest enclosing cylinder of a simplex in $\mathbb{E}^3$ to the computation of a smallest circumscribing cylinder. In order to apply algebraic methods to compute a smallest circumscribing cylinder, there are many different ways to formulate the problem in terms of polynomial equations. It is well-known that the computational costs of solving a system of polynomial equations are mainly dominated by the Bézout number (= product of the degrees) and the mixed volume of the Newton polytopes (the latter one is discussed in Section 4). See [7, 8, 27] for comprehensive introductions and the state-of-the-art. Hence, it is an essential task to find the right formulations. Moreover, we are interested in simplex classes for which the degrees can be further reduced.

3.1. **General simplices in $\mathbb{E}^3$.** In the proof of [10, Theorem 6], a polynomial formulation is given to compute the smallest enclosing cylinders of a simplex in $\mathbb{E}^3$. This formulation describes the problem by three equations in the direction vector $v = (v_1, v_2, v_3)^T$ of the line, one of them normalizing the direction vector $v$ by

$$v_1^2 + v_2^2 + v_3^2 \ = \ 1 \,. \tag{3.1}$$

The equations are of degree 10, 3, and 2, respectively, thus giving a Bézout number of 60. However, as pointed out in that paper, some of the solutions to that system are artificially introduced by the formulation and occur with higher multiplicity, and there are only 18 really different solutions. Even more severely, in the experiments in that paper (using SYNAPS, a state-of-the-art software for numerical polynomial computations), the numerical treatment of these multiple solutions needs much time, roughly a factor 100 compared to similar systems without multiple solutions.

Here, we present an approach, which reflects the true algebraic bound of 18. Namely, we give a polynomial formulation with Bézout bound 36 in which every solution generically has multiplicity one. The additional factor 2 just results from the fact that due to the normalization condition (3.1) every solution $v$ also implies that $-v$ is a solution as well.

Our framework is based on [22] in which the lines simultaneously tangent to four unit spheres are studied. A line in $\mathbb{E}^3$ is represented by a point $u \in \mathbb{E}^3$ lying on the line and a direction vector $v \in \mathbb{E}^3$ with $v^2 = 1$. We can make $u$ unique by requiring that $u \cdot v = 0$. A line $\ell = (u, v)$ has Euclidean distance $r$ from a point $p \in \mathbb{E}^3$ if and only if the quadratic equation $(u + tv - p)^2 = r^2$ has a solution of multiplicity two. This gives the condition

$$\frac{(v \cdot (u-p))^2}{v^2} - (u-p)^2 \ + r^2 \ = \ 0 \, .$$

Expanding this equation yields

$$v^2 u^2 - 2v^2 u \cdot p + v^2 p^2 - (v \cdot p)^2 - r^2 v^2 \ = \ 0 \, . \tag{3.2}$$

Rather than using $v^2 = 1$ to further simplify this equation, we prefer to keep the homogeneous form, in which all terms are of degree 4.

Now let $p_1, \ldots, p_4$ be the affinely independent vertices of the given simplex. Without loss of generality we can choose $p_4$ to be located in the origin. Then the remaining points span $\mathbb{E}^3$. Subtracting the equation for the point in the origin from the equations for $p_1$, $p_2$, $p_3$ gives the following program to compute the square of the radius of a minimal circumscribing cylinder.

$$\begin{array}{rrcll} & \min u^2 & & & \\ \text{s.t.} & u \cdot v & = & 0 \, , & \\ & 2v^2 u \cdot p_i & = & v^2 p_i^2 - (v \cdot p_i)^2 \, , & \quad 1 \le i \le 3 \, , \\ & v^2 & = & 1 \, . & \end{array} \tag{3.3}$$

We remark that the set of admissible solutions is nonempty; a proof of that statement (for general dimension) is contained in Section 4.

Since the points $p_1, p_2, p_3$ are linearly independent, the matrix $M := (p_1, p_2, p_3)^T$ is invertible, and we can solve the equations in the penultimate line of (3.3) for $u$:

$$u \ = \ \frac{1}{2v^2} M^{-1} \begin{pmatrix} v^2 p_1^2 - (v \cdot p_1)^2 \\ v^2 p_2^2 - (v \cdot p_2)^2 \\ v^2 p_3^2 - (v \cdot p_3)^2 \end{pmatrix} . \tag{3.4}$$

Now substitute this expression for $u$ into the objective function and into the first constraint of the system (3.3). After setting $v^2 = 1$ in the denominator of the first constraint, this gives a homogeneous cubic equation which we denote by $g_1(v_1, v_2, v_3) = 0$. Hence, we arrive at the following polynomial optimization formulation in terms of the variables $v_1$,

$v_2$, and $v_3$.

$$
\begin{array}{rrcl}
 & \min \left( \frac{1}{2} M^{-1} \begin{pmatrix} v^2 p_1^2 - (v \cdot p_1)^2 \\ v^2 p_2^2 - (v \cdot p_2)^2 \\ v^2 p_3^2 - (v \cdot p_3)^2 \end{pmatrix} \right)^2 & & \\
\text{s.t.} & g_1(v_1, v_2, v_3) & = & 0\,, \\
 & g_2(v_1, v_2, v_3) := v^2 - 1 & = & 0\,.
\end{array}
\tag{3.5}
$$

Note that the objective function is a homogeneous polynomial of degree 4. We denote this polynomial by $f$.

Using Lagrange multipliers $\lambda_1$ and $\lambda_2$, a necessary local optimality condition is

$$
\operatorname{grad} f = \lambda_1 \operatorname{grad} g_1 + \lambda_2 \operatorname{grad} g_2\,. \tag{3.6}
$$

By thinking of an additional factor $\lambda_0$ before grad $f$ and considering (3.6) as a system of linear equations in $\lambda_0$, $\lambda_1$, $\lambda_2$, we see that if (3.6) is satisfied for some vector $v$ then the determinant

$$
\det \begin{pmatrix} -\frac{\partial f}{\partial v_1} & \frac{\partial g_1}{\partial v_1} & \frac{\partial g_2}{\partial v_1} \\ -\frac{\partial f}{\partial v_2} & \frac{\partial g_1}{\partial v_2} & \frac{\partial g_2}{\partial v_2} \\ -\frac{\partial f}{\partial v_3} & \frac{\partial g_1}{\partial v_3} & \frac{\partial g_2}{\partial v_3} \end{pmatrix} \tag{3.7}
$$

vanishes. Thus the following lemma characterizes those circumscribing cylinders, within the space of all circumscribing cylinders, whose radius is locally extreme.

**Lemma 3.** *(a) For any normalized direction vector $(v_1, v_2, v_3)^T \in \mathbb{E}^3$ of the axis of a locally extreme circumscribing cylinder, the determinant* (3.7) *vanishes. If there are only finitely many locally extreme, normalized direction vectors than that number is bounded by 36.*

*(b) For a generic simplex the number of solutions is indeed finite, and all solutions have multiplicity one.*

*Proof.* Let $v$ be the direction vector of an axis of a locally extreme circumscribing cylinder. Then $v$ satisfies the first constraint of (3.5), and the determinant (3.7) vanishes. Since these are homogeneous equations of degree 3 and 6, respectively, Bézout's Theorem implies that in connection with $v^2 = 1$ we obtain at most 36 isolated solutions.

For the second statement it suffices to check that for one specific simplex there are only finitely many (complex) solutions and that all solutions are pairwise distinct. E.g., choose the vertices $(2, 3, 5)^T$, $(7, 11, -13)^T$, $(17, -19, -23)^T$, $(0, 0, 0)^T$. □

3.2. **Special simplex classes in $\mathbb{E}^3$.** In this section, we investigate conditions under which the degree of the resulting equations is reduced. Moreover, we show that for equifacial simplices, the minimal circumscribing radius can be computed quite easily.

We use the following classification from [22, 23].

**Proposition 4.** *Let $T$ be a simplex in $\mathbb{E}^3$ with vertices $p_1, \ldots, p_4$. The polynomial $g_1$ in the cubic equation factors into a linear polynomial and an irreducible quadratic polynomial if and only if the four faces of $T$ can be partitioned into two pairs of faces $\{F_1, F_2\}$, $\{F_3, F_4\}$*

*with area($F_1$) = area($F_2$) $\neq$ area($F_3$) = area($F_4$). Moreover, $g_1$ factors into three linear terms if and only if the areas of all four faces of $T$ are equal.*

First let us consider the case where $g_1$ decomposes into a linear polynomial and an irreducible quadratic polynomial. By optimizing separately over the linear and the quadratic constraint, the degrees of our equations are smaller than for the general case. Namely, analogously to the derivation in Section 3.1, for the quadratic constraint we obtain a Bézout bound of

$$(3+1+1) \cdot 2 \cdot 2 = 20\,,$$

and for the linear constraint we obtain

$$(3+0+1) \cdot 1 \cdot 2 = 8\,.$$

Thus, we can conclude:

**Lemma 5.** *If the four faces of the simplex can be partitioned into two pairs of faces $\{F_1, F_2\}$, $\{F_3, F_4\}$ with area($F_1$) = area($F_2$) $\neq$ area($F_3$) = area($F_4$) then there are at most 28 isolated local extrema for the minimal circumscribing cylinder. They can be computed from two polynomial systems with Bézout numbers 20 and 8, respectively.*

**Equifacial simplices.** A simplex in $\mathbb{E}^3$ is called *equifacial* if all four faces have the same area. By Proposition 4, for an equifacial simplex the cubic polynomial $g_1$ factors into three linear terms. Hence, we obtain at most $3 \cdot 8 = 24$ local extrema. Somewhat surprisingly, using a characterization from [29], it is even possible to compute smallest circumscribing cylinder of an equifacial simplex essentially without any algebraic computation.

Namely, it is well-known that the vertices of an equifacial simplex $T$ can be regarded as four pairwise non-adjacent vertices of a rectangular box (see, e.g., [21]). Hence, there exists a representation $p_1 = (w_1, w_2, w_3)^T$, $p_2 = (w_1, -w_2, -w_3)^T$, $p_3 = (-w_1, w_2, -w_3)^T$, $p_4 = (-w_1, -w_2, w_3)^T$ with $w_1, w_2, w_3 > 0$.

Assuming $v^2 = 1$ as before, (3.2) gives

$$(v \cdot p_i)^2 + 2u \cdot p_i = \sum_{j=1}^{3} w_j^2 + u^2 - r^2, \qquad 1 \leq i \leq 4\,. \tag{3.8}$$

Subtracting these equations pairwise gives

$$4(w_2u_2 + w_3u_3) = -4(w_1w_3v_1v_3 + w_1w_2v_1v_2)$$

(for indices 1, 2) and analogous equations, so that

$$w_1u_1 = -w_2w_3v_2v_3, \quad w_2u_2 = -w_1w_3v_1v_3, \quad w_3u_3 = -w_1w_2v_1v_2.$$

Since $u \cdot v = 0$, this yields $v_1v_2v_3 = 0$. Without loss of generality we can assume $v_1 = 0$. In this case,

$$u = \left( -\frac{w_2w_3}{w_1} v_2v_3, 0, 0 \right)^T.$$

So we can express (3.8) in terms of the direction vector $v$,

$$w_2^2 v_2^2 + w_3^2 v_3^2 = \sum_{j=1}^{3} w_j^2 + \left( -\frac{w_2 w_3}{w_1} v_2 v_3 \right)^2 - r^2,$$

which, by using $v_2^2 + v_3^2 = 1$, gives

$$r^2 \;=\; -\frac{w_2^2 w_3^2}{w_1^2} v_2^4 - \left( w_2^2 - w_3^2 - \frac{w_2^2 w_3^2}{w_1^2} \right) v_2^2 + w_1^2 + w_2^2 \,. \tag{3.9}$$

Thus, by computing the derivative of this expression $r^2 = r^2(v_2)$ and taking into account the three cases $v_i = 0$, we can reduce the computation of the minimal circumscribing cylinders to solving three univariate equations of degree 3. However, we can still do better. Substitute $z_2 := v_2^2$, and let $\rho$ be the expression for $r^2$ in terms of $z_2$,

$$\rho(z_2) \;=\; -\frac{w_2^2 w_3^2}{w_1^2} z_2^2 - \left( w_2^2 - w_3^2 - \frac{w_2^2 w_3^2}{w_1^2} \right) z_2 + w_1^2 + w_2^2 \,.$$

Since the second derivative of that quadratic function is negative, $\rho(z_2)$ is a concave function. Hence, within the interval $z_2 \in [0,1]$, the minimum is attained at one of the boundary values $z_2 \in \{0,1\}$. Consequently, two of the components of $(v_1, v_2, v_3)^T$ must be zero and therefore $v$ is perpendicular to two opposite edges. Since the latter geometric characterization is independent of our specific choice of coordinates, we can conclude:

**Lemma 6.** *If all four faces of the simplex $T$ have the same area then the axis of a minimum circumscribing cylinder is perpendicular to two opposite edges.*

Hence, for an equifacial simplex it suffices to investigate the cross products of the three pairs of opposite edges (equipped with an orientation), and we do not need to solve a system of polynomial equations at all.

In order to illustrate how these three solutions relate to the 18 solutions of the general approach above, we consider the regular simplex in $\mathbb{E}^3$. In the general approach, as already pointed out in [10], the six edge directions $p_i p_j$ $(1 \le i < j \le 4)$ all have multiplicity 1, and each of the three directions in Lemma 6, $p_1p_2 \times p_3p_4$, $p_1p_3 \times p_2p_4$, $p_1p_4 \times p_2p_3$, have multiplicity 4.

## 4. Smallest circumscribing cylinders in higher dimensions

In Section 3 we have given polynomial formulations with small Bézout numbers for computing smallest circumscribing cylinders of a simplex in $\mathbb{E}^3$. Using the characterization in [26] of lines simultaneously tangent to $2n-2$ spheres in $\mathbb{E}^n$, we generalize these formulations to smallest circumscribing cylinders of a simplex in $\mathbb{E}^n$, $n \ge 2$. Analogous to the three-dimensional case let $p_1, \ldots, p_{n+1}$ be the affinely independent vertices of a simplex in $\mathbb{E}^n$, and let $p_{n+1}$ be located in the origin.

First note that (3.3) also holds in general dimension $n$ if we replace the index 3 by the index $n$. Since the points $p_1, \ldots, p_n$ are linearly independent, the matrix $M := (p_1, \ldots, p_n)^T$

is invertible, and we can solve for $u$:

$$u \;=\; \frac{1}{2v^2} M^{-1} \begin{pmatrix} v^2 p_1^2 - (v \cdot p_1)^2 \\ \vdots \\ v^2 p_n^2 - (v \cdot p_n)^2 \end{pmatrix} . \tag{4.1}$$

Hence, by generalizing the formulation for the three-dimensional case, we obtain the program

$$\begin{array}{rrcl} \min & \multicolumn{3}{l}{\left( \tfrac{1}{2} M^{-1} \begin{pmatrix} v^2 p_1^2 - (v \cdot p_1)^2 \\ \vdots \\ v^2 p_n^2 - (v \cdot p_n)^2 \end{pmatrix} \right)^2} \\ \text{s.t.} & g_1(v_1, \ldots, v_n) & = & 0 \, , \\ & g_2(v_1, \ldots, v_n) := v^2 - 1 & = & 0 \, , \end{array} \tag{4.2}$$

where $g_1$ denotes the cubic equation as before. In order to show that the set of admissible solutions for our optimization problem is nonempty, we record the following result.

**Lemma 7.** *For any simplex in $\mathbb{E}^n$ the $\binom{n+1}{2}$ edge directions of the simplex are direction vectors of circumscribing cylinders.*

*Proof.* Since the edge directions $p_i - p_j$ have a simple description in the basis $p_1, \ldots, p_n$, we express the cubic equation $g_1(v) = 0$ in that basis. Let $v$ be an arbitrary direction vector, and let the representation of $v$ in the basis $p_1, \ldots, p_n$ be

$$v \;=\; \sum_{i=1}^{n} t_i p_i \, .$$

Further, let $p'_1, \ldots, p'_n$ be a dual basis to $p_1, \ldots, p_n$; i.e., let $p'_1, \ldots, p'_n$ be defined by $p'_i \cdot p_j = \delta_{ij}$, where $\delta_{ij}$ denotes Kronecker's delta function. By elementary linear algebra, we have $t_i = p'_i \cdot v$.

When expressing $u$ in this dual basis, $u = \sum u'_i p'_i$, the second constraint of (3.3) gives

$$u'_i \;=\; \frac{1}{2v^2} \left( v^2 p_i^2 - (v \cdot p_i)^2 \right) .$$

Substituting this representation of $u$ into the equation $g_1(v) = 0$ gives

$$0 \;=\; g_1(v) = \; v^2 (u \cdot v) \;=\; v^2 \left( \sum_{i=1}^{n} u'_i p'_i \right) \cdot v \;=\; v^2 \sum_{i=1}^{n} u'_i t_i \, ,$$

where the last step uses the duality of the bases. Hence, we obtain the cubic equation

$$\frac{1}{2} \sum_{i=1}^{n} (v^2 p_i^2 - (v \cdot p_i)^2) t_i \;=\; 0 \, .$$

Expressing $v$ in terms of the $t$-variables yields

$$\frac{1}{2} \sum_{1 \le i \ne j \le n} \alpha_{ij} t_i^2 t_j + \sum_{1 \le i < j < k \le n} \beta_{ijk} t_i t_j t_k \;=\; 0 \, ,$$

where

$$\begin{aligned} \alpha_{ij} &= (\mathrm{vol}_2(p_i,p_j))^2 = \det\begin{pmatrix} p_i\cdot p_i & p_i\cdot p_j \\ p_j\cdot p_i & p_j\cdot p_j \end{pmatrix}, \\ \beta_{ijk} &= \det\begin{pmatrix} p_i\cdot p_j & p_i\cdot p_k \\ p_k\cdot p_j & p_k\cdot p_k \end{pmatrix} + \det\begin{pmatrix} p_i\cdot p_k & p_i\cdot p_j \\ p_j\cdot p_k & p_j\cdot p_j \end{pmatrix} \\ &\quad + \det\begin{pmatrix} p_j\cdot p_k & p_j\cdot p_i \\ p_i\cdot p_k & p_i\cdot p_i \end{pmatrix}, \end{aligned}$$

and $\mathrm{vol}_2(p_i,p_j)$ denotes the oriented area of the parallelogram spanned by $p_i$ and $p_j$. In terms of the $t$-coordinates, the $\binom{n+1}{2}$ edges of the simplex are $t = e_i$, $1 \le i \le n$, and $t = e_i - e_j$, $1 \le i < j \le n$, where $e_i$ denotes the $i$-th standard unit vector. For all these edges, the cubic equation is satisfied. □

Considering Lagrange multipliers $\lambda_1$ and $\lambda_2$ yields the following necessary optimality condition.

$$\begin{aligned} \mathrm{grad}\, f &= \lambda_1 \mathrm{grad}\, g_1 + \lambda_2 \mathrm{grad}\, g_2\,, \\ g_1(v_1,\ldots,v_n) &= 0\,, \\ g_2(v_1,\ldots,v_n) &= 0\,. \end{aligned} \tag{4.3}$$

Since the Bézout bound of this system is $3^n \cdot 3 \cdot 2 = 2 \cdot 3^{n+1}$, we have:

**Lemma 8.** *For $n \ge 2$, the number of isolated local extrema for the minimal circumscribing cylinder is bounded by $2 \cdot 3^{n+1}$.*

This bound is not tight. Trying to reduce this upper bound of isolated solutions like in the three-dimensional case, we can eliminate the linear occurrences of the Lagrange variables $\lambda_1$ and $\lambda_2$. Generalizing (3.7), we have to consider the vanishing of all $3 \times 3$-subdeterminants of the matrix

$$\begin{pmatrix} -\frac{\partial f}{\partial v_1} & \frac{\partial g_1}{\partial v_1} & \frac{\partial g_2}{\partial v_1} \\ -\frac{\partial f}{\partial v_2} & \frac{\partial g_1}{\partial v_2} & \frac{\partial g_2}{\partial v_2} \\ \vdots & \vdots & \vdots \\ -\frac{\partial f}{\partial v_n} & \frac{\partial g_1}{\partial v_n} & \frac{\partial g_2}{\partial v_n} \end{pmatrix}. \tag{4.4}$$

Thus, for $n \ge 4$ we arrive at a non-complete intersection of equations where we have more equations than variables. Hence, we cannot apply our Bézout bound on these systems.

However, for small dimensions we can improve Lemma 8 by directly working on the formulation (4.3). In order to provide better bounds, we use well-known characterizations of the number of zeroes of a polynomial equation by the mixed volume of a Minkowski sum of polytopes (for an easily accessible introduction into this topic we refer to [8]). Here, let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

**Lemma 9.** *For $2 \le n \le 7$, the number of isolated solutions of the system* (4.3) *in $(v_1,\ldots,v_n,\lambda_1,\lambda_2) \in (\mathbb{C}^*)^{n+2}$ is bounded by*

$$6\left\{ {n+1 \atop 3} \right\},$$

*where* $\left\{ {n \atop k} \right\}$ *denotes the Stirling number of the second kind (see, e.g.,* [14, 25]*).*

The sequence $6\left\{ {n+1 \atop 3} \right\}$ starts as follows.

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $6\left\{ {n+1 \atop 3} \right\}$ | 6 | 36 | 150 | 540 | 1806 | 5796 |

*Proof.* For a polynomial $h = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha x^\alpha \in \mathbb{C}[x_1, \ldots, x_n]$, let

$$\mathrm{NP}(h) := \mathrm{conv}\{\alpha \in \mathbb{N}_0^n \, : \, c_\alpha \neq 0\}$$

denote the Newton polytope of $h$ (see, e.g., [8, §7.1]). Let $h_1, \ldots, h_n$ be the polynomials of the gradient equation in (4.3). Further let $P_1, \ldots, P_n, Q_1, Q_2$ be the Newton polytopes of $h_1, \ldots, h_n, g_1, g_2$ for generic instances of these equations.

Recall that the mixed volume $\mathrm{MV}(P_1, \ldots, P_n, Q_1, Q_2)$ is the coefficient of the monomial $\lambda_1 \cdot \lambda_2 \cdots \lambda_n \cdot \mu_1 \cdot \mu_2$ in the $(n+2)$-dimensional volume $\mathrm{Vol}_{n+2}(\lambda_1 P_1 + \ldots + \lambda_n P_n + \mu_1 Q_1 + \mu_2 Q_2)$ (which is a polynomial expression in $\lambda_1, \ldots, \lambda_n, \mu_1, \mu_2$). By Bernstein's Theorem, the number of isolated common zeroes in $(\mathbb{C}^*)^{n+2}$ of the set of polynomials $h_1, \ldots, h_n, g_1, g_2$ is bounded from above by

$$\mathrm{MV}(P_1, \ldots, P_n, Q_1, Q_2)$$

(see [8, Chapter 8, Theorem 5.4]). For every given $n$ this volume can be computed using software for computing mixed volumes (see, e.g, [13, 30]). □

We conjecture that for any $n \geq 2$, the number of isolated solutions in $(\mathbb{C}^*)^{n+2}$ is bounded by $6\left\{ {n+1 \atop 3} \right\}$. With regard to the different values in Lemmas 8 and 9, note that $\lim_{n \to \infty} (2 \cdot 3^{n+1}) / (6\left\{ {n+1 \atop 3} \right\}) = 2$.

4.1. **The regular simplex in $\mathbb{E}^n$.** Here, we analyze the local extrema of circumscribing cylinders for the regular simplex. Our aim is both to illustrate the algebraic formulations given before and to relate our investigations to classical investigations on the regular simplex in convex geometry. In order to achieve many symmetries in the algebraic formulation, we use a slightly modified coordinate system that is particularly suited for the regular simplex; these coordinates have also been used in [4, 31].

The equation $x_1 + \ldots + x_{n+1} = 1$ defines an $n$-dimensional affine subspace in $\mathbb{E}^{n+1}$. Now let the regular simplex in this $n$-dimensional subspace be given by the $n+1$ vertices $p_i = e_i$, where $e_i$ denotes the $i$-th standard unit vector, $1 \leq i \leq n+1$. We consider the tangency equation (3.2) for the point $p_{n+1}$,

$$v^2 u^2 - 2v^2 u_{n+1} + v^2 - v_{n+1}^2 - r^2 v^2 \;=\; 0\,.$$

Subtracting this equation from the equation for $p_i$, $1 \leq i \leq n$, yields

$$2v^2(u_i - u_{n+1}) \;=\; -(v_i^2 - v_{n+1}^2)\,, \quad 1 \leq i \leq n\,.$$

Moreover, the embedding into the hyperplane $\sum_{i=1}^{n+1} x_i = 1$ implies $\sum_{i=1}^{n+1} u_i = 1$. In order to solve these $n+1$ equations for $u$, let $M$ be the $(n+1) \times (n+1)$-matrix whose $i$-th row contains the vector $e_i^T - e_{n+1}^T$ and whose $n$-th row is $(1, 1, \ldots, 1)$. Since $M$ is invertible, we obtain

$$u = \frac{1}{2v^2} M^{-1} \begin{pmatrix} -(v_1^2 - v_{n+1}^2) \\ \vdots \\ -(v_n^2 - v_{n+1}^2) \\ 2v^2 \end{pmatrix} . \tag{4.5}$$

As before, substituting this expression into $u \cdot v = 0$ and setting $v^2 = 1$ in the denominator gives a cubic equation $g_1(v) = 0$. Hence, we obtain the following optimization problem. Here, the objective function $f$ stems from the condition for the vertex $p_{n+1}$, and the condition $\sum_{i=1}^{n+1} v_i = 0$ comes from the embedding.

$$\begin{array}{rrcl} & \min u^2 - 2u_{n+1} + 1 - v_{n+1}^2 & & \\ \text{s.t.} & g_1(v_1, \ldots, v_{n+1}) & = & 0\,, \\ & \sum\limits_{i=1}^{n+1} v_i & = & 0\,, \\ & v^2 & = & 1\,. \end{array} \tag{4.6}$$

First we record that the functions $f$ and $g_1$ are symmetric polynomials in the variables $v_1, \ldots, v_{n+1}$. In order to show this, let $\sigma_1, \ldots, \sigma_{n+1}$ be the elementary symmetric functions in $v_1, \ldots, v_{n+1}$,

$$\begin{array}{rcl} \sigma_1 & = & v_1 + \ldots + v_{n+1}\,, \\ & \vdots & \\ \sigma_k & = & \sum\limits_{1 \le i_1 < \ldots < i_k \le n+1} v_{i_1} v_{i_2} \cdots v_{i_k}\,, \\ & \vdots & \\ \sigma_{n+1} & = & v_1 v_2 \cdots v_{n+1} \end{array}$$

(see, e.g., [7, 28]). By providing explicit expressions for $f$ and $g_1$ as polynomials in the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_{n+1}$, the symmetry of $f$ and $g_1$ follows. More precisely, we obtain:

**Lemma 10.** *The quartic polynomial $f(v_1, \ldots, v_{n+1})$ and the cubic polynomial $g_1(v_1, \ldots, v_{n+1})$ are symmetric polynomials in the variables $v_1, \ldots, v_{n+1}$. In terms of the elementary symmetric functions, $f$ results in*

$$f = \frac{1}{4(n+1)} \left( n\sigma_1^4 - 4n\sigma_1^2\sigma_2 + 2(n-1)\sigma_2^2 - 4\sigma_1^2 + 8\sigma_2 + 4n \right) + \sigma_1\sigma_3 - \sigma_4\,,$$

*and the homogeneous polynomial $g_1$ results in*

$$g_1 = \frac{1}{2(n+1)} \left( -(n-2)\sigma_1^3 + 3(n-1)\sigma_1\sigma_2 \right) - \frac{3}{2}\sigma_3\,.$$

Since $\sigma_1 = 0$ and $\sum_{i=1}^{n+1} {v_i}^2 = \sigma_1^2 - 2\sigma_2$, we can also deduce the following formulation of our optimization problem:

**Corollary 11.** *Finding the critical values of the minimization problem* (4.6) *is equivalent to finding the critical values* $(v_1, \ldots, v_{n+1})^T$ *of the maximization problem*

$$\begin{array}{llcl} \max & \sigma_4 & & \\ \text{s.t.} & \sigma_1 & = & 0\,, \\ & \sigma_2 & = & -\dfrac{1}{2}\,, \\ & \sigma_3 & = & 0\,, \end{array} \tag{4.7}$$

*where* $\sigma_i$ *are the elementary symmetric functions in* $v_1, \ldots, v_{n+1}$.

**Theorem 12.** *The direction vector* $(v_1, \ldots, v_{n+1})^T$ *of any locally extreme circumscribing cylinder satisfies* $|\{v_1, \ldots, v_{n+1}\}| \leq 3$, *i.e., for each solution vector the components take at most three distinct values.*

*Proof.* For $n \leq 2$, the statement is trivial, so we can assume $n \geq 3$. Let $v$ be the direction vector of a locally extreme circumscribing cylinder with $v^2 = 1$. Using Corollary 11, let $f(v) := -\sigma_4(v)$, $g_1(v) := \sigma_3(v)$, $g_2(v) := \sigma_2(v) + 1/2$, and $g_3(v) := \sigma_1(v)$. As a necessary condition for a local extremum, for any pairwise different indices $a, b, c, d \in \{1, \ldots, n+1\}$ the determinant

$$\det \begin{pmatrix} -\frac{\partial f}{\partial v_a} & \frac{\partial g_1}{\partial v_a} & \frac{\partial g_2}{\partial v_a} & \frac{\partial g_3}{\partial v_a} \\ -\frac{\partial f}{\partial v_b} & \frac{\partial g_1}{\partial v_b} & \frac{\partial g_2}{\partial v_b} & \frac{\partial g_3}{\partial v_b} \\ -\frac{\partial f}{\partial v_c} & \frac{\partial g_1}{\partial v_c} & \frac{\partial g_2}{\partial v_c} & \frac{\partial g_3}{\partial v_c} \\ -\frac{\partial f}{\partial v_d} & \frac{\partial g_1}{\partial v_d} & \frac{\partial g_2}{\partial v_d} & \frac{\partial g_3}{\partial v_d} \end{pmatrix} \tag{4.8}$$

vanishes. Since $f$, $g_1$, $g_2$, and $g_3$ are symmetric functions in the variables $v_1, \ldots, v_{n+1}$, we can assume without loss of generality $a = 1$, $b = 2$, $c = 3$, and $d = 4$. Setting $\alpha_n := \sum_{i=5}^{n+1} v_i$ and $\beta_n = \sum_{i=5}^{n+1} v_i^2$, we can write

$$\begin{aligned} \frac{\partial g_3}{\partial v_i} &= 1\,, \\ \frac{\partial g_2}{\partial v_i} &= \sum_{\substack{j=1 \\ j \neq i}}^{4} v_j + \alpha_n\,, \\ \frac{\partial g_1}{\partial v_i} &= \sum_{\substack{1 \leq j < k \leq 4 \\ j,k \neq i}} v_j v_k + \alpha_n \sum_{\substack{j=1 \\ j \neq i}}^{4} v_j + \frac{1}{2}\left(\alpha_n^2 - \beta_n\right) \end{aligned}$$

$(1 \leq i \leq 4)$. Moreover, since $\sigma_3(v) = 0$, we can consider $\sigma_3 + \frac{\partial f}{\partial v_i}$ instead of $\frac{\partial f}{\partial v_i}$. This allows to express the resulting expression easily in terms of $\alpha_n$ and $\beta_n$. More precisely, we obtain

$$\sigma_3 + \frac{\partial f}{\partial v_i} = v_i \left( \sum_{\substack{1 \le j < k \le 4 \\ j,k \ne i}} v_j v_k + \alpha_n \sum_{\substack{j=1 \\ j \ne i}}^{4} v_j + \frac{1}{2}(\alpha_n^2 - \beta_n) \right) .$$

Thus we can consider the determinant (4.8) as a polynomial in $v_1, v_2, v_3, v_4, \alpha_n, \beta_n$. Evaluating this $4 \times 4$-determinant $\Delta$ shows that it is independent of $\alpha_n$, $\beta_n$ and that it factors as

$$\Delta = (v_1 - v_2)(v_1 - v_3)(v_1 - v_4)(v_2 - v_3)(v_2 - v_4)(v_3 - v_4) .$$

Hence, $|\{v_1, v_2, v_3, v_4\}| \le 3$, and this holds true for any quadruple $(a, b, c, d)$ of indices. □

Using this result, we illustrate the occurrence of the Stirling numbers in Lemma 9 for the case of a regular simplex. There are $\left\{ {n+1 \atop 3} \right\}$ ways to partition the set $V := \{v_1, \ldots, v_{n+1}\}$ into three nonempty subsets $V_1$, $V_2$, $V_3$. We assume that $v_i \in V_i$, $1 \le i \le 3$, and that all variables within the same set take the same value. Setting $k := |V_1|$ and $l := |V_2|$, the formulation in Corollary 11 yields the system of equations

$$\begin{aligned} kv_1 + lv_2 + (n+1-k-l)v_3 &= 0 , \\ kv_1^2 + lv_2^2 + (n+1-k-l)v_3^2 &= 1 , \\ \sum_{\substack{0 \le i_1 < i_2 < i_3 \le 3 \\ i_1+i_2+i_3=3}} \binom{k}{i_1}\binom{l}{i_2}\binom{n+1-k-l}{i_3} v_1^{i_1} v_2^{i_2} v_3^{i_3} &= 0 . \end{aligned} \tag{4.9}$$

If one of the indices $k$, $l$, or $n+1-k-l$ is zero then this system consists of three equations in two variables, so we do not expect any solutions. For every choice of $k$, $l$ corresponding to a partition into nonempty subsets, we obtain a system of equations with Bézout number 6. Thus, whenever the values of $v_1$, $v_2$, and $v_3$ in the solutions to (4.9) are distinct, then this reflects the bound in Lemma 9.

In particular, in the case $n = 4$ we obtain the following 150 solutions.

**$k = 1$, $l = 1$:** The six solutions for $(v_1, v_2, v_3)^T$ of the system (4.9) are

$$\left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right)^T, \quad \left( \frac{1}{20}\sqrt{110 - 30i\sqrt{15}}, \frac{1}{20}\sqrt{110 + 30i\sqrt{15}}, -\frac{1}{10}\sqrt{15} \right)^T,$$

and the solutions obtained by permuting the first two components of the first solution and by changing the signs and/or permuting the first two components in the second solution.

For the program (4.7) in the variables $(v_1, \ldots, v_5)^T$, this gives $\binom{5}{2}\binom{2}{1} = 20$ critical positions of the form (i.e., up to variable permutations)

$$\left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0, 0 \right)^T,$$

20 complex solutions of the form

$$\left( -\frac{1}{20}\sqrt{110 - 30i\sqrt{15}}, -\frac{1}{20}\sqrt{110 + 30i\sqrt{15}}, \frac{1}{10}\sqrt{15}, \frac{1}{10}\sqrt{15}, \frac{1}{10}\sqrt{15} \right)^T,$$

and 20 complex solutions of the form

$$\left(\frac{1}{20}\sqrt{110-30i\sqrt{15}}, \frac{1}{20}\sqrt{110+30i\sqrt{15}}, -\frac{1}{10}\sqrt{15}, -\frac{1}{10}\sqrt{15}, -\frac{1}{10}\sqrt{15}\right)^T .$$

**$k = 1$, $l = 2$:** Here, we obtain 30 solutions of the form

$$\left(0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^T ,$$

30 solutions of the form

$$\left(\frac{1}{5}\sqrt{10}, \frac{1}{4}\sqrt{2}-\frac{1}{20}\sqrt{10}, \frac{1}{4}\sqrt{2}-\frac{1}{20}\sqrt{10}, -\frac{1}{4}\sqrt{2}-\frac{1}{20}\sqrt{10}, -\frac{1}{4}\sqrt{2}-\frac{1}{20}\sqrt{10}\right)^T ,$$

and 30 solutions of the form

$$\left(-\frac{1}{5}\sqrt{10}, \frac{1}{4}\sqrt{2}+\frac{1}{20}\sqrt{10}, \frac{1}{4}\sqrt{2}+\frac{1}{20}\sqrt{10}, -\frac{1}{4}\sqrt{2}+\frac{1}{20}\sqrt{10}, -\frac{1}{4}\sqrt{2}+\frac{1}{20}\sqrt{10}\right)^T .$$

The global minimum is attained for the vector $\left(0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)^T$, and the objective value of the global optimum is 49/80. Hence, the radius of the smallest circumscribing cylinder for a regular simplex in $\mathbb{E}^4$ with edge length $\sqrt{2}$ is $\sqrt{49/80} = 7\sqrt{5}/20 \approx 0.7826$.

## Appendix: An error in the results of Weissbach

In the course of our investigations, we discovered a subtle but severe mistake in the paper [32] on the explicit determination of the outer $(n-1)$-radius of a regular simplex in $\mathbb{E}^n$. Since this error completely invalidates the proof given there[2], we give a description of that flaw, including some computer-algebraic calculations illustrating it.

In that paper, the computation of the outer $(n-1)$-radius of a regular simplex (with edge length $\sqrt{2}$) is reduced to the analysis of the following optimization problem.

$$\begin{array}{rrcl} & \min \sum_{i=1}^{n+1} u_i^4 & & \\ \text{s.t.} & \sum_{i=1}^{n+1} u_i^2 & = & 1 , \\ & \sum_{i=1}^{n+1} u_i & = & 0 . \end{array} \tag{4.10}$$

[2]In a personal communication this has been confirmed by B. Weißbach.

For any local optimum $(u_1, \ldots, u_{n+1})^T$ there exist Lagrange multipliers $\lambda_1,\ \lambda_2 \in \mathbb{R}$ such that

$$\begin{aligned} 4u_i^3 + 2\lambda_1 u_i + \lambda_2 &= 0\,, \qquad 1 \le i \le n+1\,, \\ \sum_{i=1}^{n+1} u_i^2 &= 1\,, \\ \sum_{i=1}^{n+1} u_i &= 0\,. \end{aligned} \tag{4.11}$$

Erroneously, in [32] it is argued that symmetry arguments imply that $\lambda_2 = 0$ in any solution. The following calculation in the computer algebra system Singular [15] shows that for $n = 3$ this system has 26 solutions (counting multiplicity) over $\mathbb{C}$.

```
ring R = 0, (u1,u2,u3,u4,la1,la2), (dp);

ideal I =
  4*u1^3 + 2*la1*u1 + la2,
  4*u2^3 + 2*la1*u2 + la2,
  4*u3^3 + 2*la1*u3 + la2,
  4*u4^3 + 2*la1*u4 + la2,
  u1^2 + u2^2 + u3^2 + u4^2 - 1,
  u1 + u2 + u3 + u4;

degree(std(I));
```

This program first defines a polynomial ring in the variables $u_1, \ldots, u_4, \lambda_1, \lambda_2$ over a field of characteristic zero. We then use the `degree` command to compute the dimension and the degree of the ideal defined by our equations. The output of that command is

```
// codimension = 6
// dimension   = 0
// degree      = 26
```

Hence, there are finitely many solutions (since the dimension of the ideal is zero), and the degree of the ideal (the sum of the multiplicities of the solutions) is 26.

18 of these solutions refer to the case $\lambda_2 = 0$ (and those were the ones computed in [32]). Namely, if $\lambda_2 = 0$ then the first row of (4.11) simplifies to

$$u_i(2u_i^2 + \lambda_1) \;=\; 0\,, \qquad 1 \le i \le n+1\,.$$

If we are only interested in the real solutions to this system, then setting $\lambda_1 = -2\lambda^2$ for some $\lambda \ge 0$ gives

$$u_i(u_i^2 - \lambda^2) \;=\; 0\,, \qquad 1 \le i \le n+1\,.$$

Since the vector $(u_1, \ldots, u_{n+1})^T = (0, \ldots, 0)^T$ does not satisfy the second row in (4.11), the solutions with $\lambda_2 = 0$ are

$$\begin{aligned} u_i &= \lambda, & i &\in \{i_1, \ldots, i_h\}, \\ u_i &= -\lambda, & i &\in \{i_{h+1}, \ldots, i_{2h}\}, \\ u_i &= 0, & i &\in \{1, \ldots, n+1\} \setminus \{i_1, \ldots, i_{2h}\} \end{aligned}$$

for some $h \geq 1$, some set $\{i_1, \ldots, i_{2h}\}$ of pairwise different indices, and $\lambda = (2h)^{-1/2}$. In the case $n = 3$, there are 12 possibilities to choose the indices and the signs for $|h| = 1$ and 6 possibilities to choose the indices and the signs for $|h| = 2$, giving 18 solutions to (4.11).

However, there are 8 additional solutions, which in fact are also real! Namely, these solutions are

$$\begin{aligned} (u_1, \ldots, u_4)^T &= \frac{1}{2\sqrt{3}}(1, -3, 1, 1)^T, \quad \lambda_1 = -\frac{7}{6}, \quad \lambda_2 = \frac{1}{\sqrt{3}}, \\ (u_1, \ldots, u_4)^T &= \frac{1}{2\sqrt{3}}(-1, 3, -1, -1)^T, \quad \lambda_1 = -\frac{7}{6}, \quad \lambda_2 = -\frac{1}{\sqrt{3}}, \end{aligned}$$

as well as the six distinct solutions obtained from these two by permuting the variables $u_1, \ldots, u_4$. The additional solutions invalidate the subsequent arguments in [32].

The omissions get even worse in the higher-dimensional case. E.g., for $n = 4$, besides the $\binom{5}{2}\binom{2}{1} + \binom{5}{4}\binom{4}{2} = 20 + 30 = 50$ solutions described in [32], we obtain the following solutions:

$$\begin{aligned} (u_1, \ldots, u_5)^T &= \frac{1}{\sqrt{30}}(-2, -2, -2, 3, 3)^T, \quad \lambda_1 = -\frac{7}{15}, \quad \lambda_2 = -\frac{2}{75}\sqrt{30}, \\ (u_1, \ldots, u_5)^T &= \frac{1}{\sqrt{30}}(2, 2, 2, -3, -3)^T, \quad \lambda_1 = -\frac{7}{15}, \quad \lambda_2 = \frac{2}{75}\sqrt{30}, \\ (u_1, \ldots, u_5)^T &= \frac{1}{2\sqrt{5}}(1, -4, 1, 1, 1)^T, \quad \lambda_1 = -\frac{13}{10}, \quad \lambda_2 = \frac{6}{25}\sqrt{5}, \\ (u_1, \ldots, u_5)^T &= \frac{1}{2\sqrt{5}}(-1, 4, -1, -1, -1)^T, \quad \lambda_1 = -\frac{13}{10}, \quad \lambda_2 = -\frac{6}{25}\sqrt{5}, \end{aligned}$$

as well as those solutions obtained by permuting the variables. Altogether, we have $10 + 10 + 5 + 5 = 30$ solutions with $\lambda_2 \neq 0$, and thus a total number of 80 solutions.

Finally, we remark that the paper [31], which computes the outer $(n-1)$-radius of a regular simplex in *odd* dimension $n$, is correct (cf. also [4]).

## References

[1] P.K. Agarwal, B. Aronov, and M Sharir. Line transversals of balls and smallest enclosing cylinders in three dimensions. *Discrete Comput. Geom.*, 21:373–388, 1999.

[2] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper.* Springer-Verlag, Berlin, 1934.

[3] R. Brandenberg. *Radii of Convex Bodies.* Ph.D. thesis, Dept. of Mathematics, Technische Universität München, 2002. `http://tumb1.biblio.tu-muenchen.de/publ/diss/ma/2002/brandenberg.html`.

[4] R. Brandenberg. *Radii of regular polytopes.* Preprint, 2003. `math.GM/0308121`.

[5] R. Brandenberg and T. Theobald. Radii minimal projections of simplices and constrained optimization of symmetric polynomials. Preprint, 2003. `math.MG/0311017`.

[6] T.M. Chan. Approximating the diameter, width, smallest enclosing cylinder, and minimum-width annulus. *Internat. J. of Comp. Geom. and Applications 12:67–85, 2002.*

[7] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra.* UTM, Springer-Verlag, New York, 1996, second edition.

[8] D. Cox, J. Little, and D. O'Shea. *Using Algebraic Geometry.* Graduate Texts in Mathematics, vol. 185, Springer-Verlag, New York, 1998.

[9] H.S.M. Coxeter, Introduction to Geometry. John Wiley & Sons, 1961.

[10] O. Devillers, B. Mourrain, F.P. Preparata, and Ph. Trébuchet. On circular cylinders through four or five points in space. *Discrete Comput. Geom.* 29:83–104, 2002.

[11] G. Dos Reis, B. Mourrain, F. Rouillier, and Ph. Trébuchet. SYNAPS: An environment for symbolic and numeric computation. *Proc. International Congress of Mathematical Software 2002*, Beijing, 239-249, World Scientific, 2002.

[12] H.G. Eggleston. Notes on Minkowski geometry (I): Relations between the circumradius, diameter, inradius, and minimal width of a convex set. *J. London Math. Soc.* 33:76–81 (1958).

[13] I. Emiris and J. Canny. Efficient incremental algorithms for the sparse resultant and the mixed volume. *J. Symb. Comp.* 20:117–149, 1995.

[14] R.L. Graham, D.E. Knuth, and O. Patashnik. *Concrete mathematics.* Addison-Wesley, 1989.

[15] G.-M. Greuel, G. Pfister, and H. Schönemann. SINGULAR 2.0. A computer algebra system for polynomial computations. Centre for Computer Algebra, University of Kaiserslautern, 2001. `http://www.singular.uni-kl.de` .

[16] P. Gritzmann and V. Klee. Inner and outer $j$-radii of convex bodies in finite-dimensional normed spaces. *Discrete Comput. Geom.* 7:255-280, 1992.

[17] P. Gritzmann and V. Klee. Computational complexity of inner and outer $j$-radii of polytopes in finite-dimensional normed spaces. *Math. Program.* 59A:163-213, 1993.

[18] P. Gritzmann and V. Klee. Computational convexity. In *Handbook of Discrete and Computational Geometry* (J.E. Goodman, J. O'Rourke, eds.), 491–515, CRC Press, Boca Raton, 1997.

[19] S. Har-Peled and K. Varadarajan. Projective clustering in high dimensions using core sets. Proc. ACM Symposium on Computational Geometry '02 (Barcelona), 312–318, 2002.

[20] D. Hilbert and S. Cohn-Vossen. *Anschauliche Geometrie.* Springer-Verlag, Berlin, 1932. Translation: *Geometry and the Imagination*, Chelsea Publ., New York, 1952.

[21] Y.S. Kupitz and H. Martini. Equifacial tetrahedra and a famous location problem. *Math. Gazette*, 83:464–467, 1999.

[22] I.G. Macdonald, J. Pach, and T. Theobald. Common tangents to four unit balls in $\mathbb{R}^3$. *Discrete Comput. Geom.* 26:1–17, 2001.

[23] H. Schaal. Ein geometrisches Problem der metrischen Getriebesynthese. In *Sitzungsber., Abt. II, Österr. Akad. Wiss.* 194:39–53, 1985.

[24] E. Schömer, J. Sellen, M. Teichmann, and C. Yap. Smallest enclosing cylinders. *Algorithmica*, 27:170–186, 2000.

[25] R.P. Stanley. *Enumerative Combinatorics.* Cambridge University Press, 1997.

[26] F. Sottile and T. Theobald. Lines tangent to $2n-2$ spheres in $\mathbb{R}^n$. *Trans. Amer. Math. Soc.* 354:4815–4829, 2002.

[27] B. Sturmfels. *Solving Systems of Polynomial Equations.* CBMS series, vol. 97, AMS, Providence, 2002.

[28] B. Sturmfels. *Algorithms in Invariant Theory.* RISC Series in Symbolic Computation, Springer-Verlag, Wien, 1993.

[29] T. Theobald. Visibility computations: From discrete algorithms to real algebraic geometry. In S. Basu and L. Gonzalez-Vega (eds.), *Algorithmic and Quantitative Real Algebraic Geometry in Mathematics and Computer Science*, AMS DIMACS series, vol. 60, 207–219, 2003.

[30] J. Verschelde. PHCpack: A general-purpose solver for polynomial systems by homotopy continuation. *ACM Trans. Math. Software 25:251–276, 1999.*

[31] B. Weißbach. Über die senkrechten Projektionen regulärer Simplexe. *Beitr. Algebra Geom.* 15:35–41, 1983.

[32] B. Weißbach. Über Umkugeln von Projektionen regulärer Simplexe. *Beitr. Algebra Geom.* 16:127–137, 1983.

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