Mixed Volume Techniques for Embeddings of Laman Graphs

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Abstract

We use Bernstein's Theorem [1] to obtain combinatorial bounds for the number of embeddings of Laman graph frameworks modulo rigid motions. For this, we study the mixed volume of suitable systems of polynomial equations obtained from the edge length constraints. The bounds can easily be computed and for some classes of graphs, the bounds are tight.

Keywords: *Mixed volume, Laman graphs, minimally rigid graphs, Bernstein's Theorem*

1 Introduction

Let $G = (V, E)$ be a graph with $|E| = 2|V| - 3$ edges. If each subset of k vertices spans at most $2k - 3$ edges, we say that G has the *Laman property* and call it a *Laman graph* (see [18]). For generic edge lengths, Laman graphs are minimally rigid (see [6]), i.e. they become flexible if any edge is removed.

A *Henneberg sequence* for a graph G is a sequence $(G_i)_{3\leq i\leq n}$ of Laman graphs such that G_3 is a triangle, $G_n = G$ and each G_i is obtained by G_{i-1} via one of the following two types of steps: A *Henneberg I step* adds one new vertex v_{i+1} and two new edges, connecting v_{i+1} to two arbitrary vertices of G_i. A *Henneberg II step* adds one new vertex v_{i+1} and three new edges, connecting v_{i+1} to three vertices of G_i such that at least two of these vertices are connected via an edge e of G_i and this certain edge e is removed (see Figure 1). Any Laman graph G can be constructed via a Henneberg sequence and any graph constructed via a Henneberg sequence has the Laman property (see [25]). We call G a *Henneberg I graph* if it is constructable using only Henneberg I steps. Otherwise we call it *Henneberg II*.

In the following we look at *frameworks* which are tuples (G, L) where $G = (V, E)$ is a graph and $L = \{l_{i,j} : [v_i, v_j] \in E\}$ is a set of $|E|$ positive numbers interpreted as edge lengths. Given a framework we want to know how many embeddings, i.e. maps $\alpha : V \to \mathbb{R}^2$, exist such that the Euclidean distance between two points in the image is exactly $l_{i,j}$ for all $[v_i, v_j] \in E$. Since every rotation or translation of an embedding gives another one, we ask how many embeddings exist *modulo rigid motions*.

[*]Research supported through DFG grant TH1333/1-1

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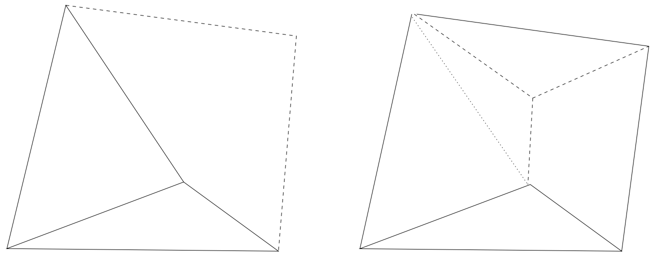

Figure 1: A Henneberg I and a Henneberg II step. New edges are dashed and the deleted edge is pointed.

Due to the minimal rigidity property, questions about embeddings of Laman graphs arise naturally in rigidity and linkage problems (see [23] or [14]). Graphs with less edges will have zero or infinitely many embeddings modulo rigid motions, and graphs with more edges do not have any embeddings for a generic choice of edge lengths.

Determining the maximal number of embeddings (modulo rigid motions) for a given Laman graph is an open problem. The best upper bounds are due to Borcea and Streinu (see [3] and [4]) who show that the number of embeddings is bounded by $\binom{2|V|-4}{|V|-2}$. Their bounds are based on degree results of determinantal varieties, but do not seem to fully exploit the specific combinatorial structure of Laman graphs.

Here, we present an alternative, combinatorial approach to bound the number of embeddings of a Laman graph based on Bernstein's Theorem for sparse polynomial systems. Since the systems of polynomial equations describing the Laman embeddings are sparse, the mixed volume of the Newton polytopes provides a simple combinatorial upper bound on the number of solutions. It is particularly interesting that for some classes of graphs, the mixed volume bound is tight (and in these cases improves the general bound in [3]).

To use algebraic tools for this problem we formulate the embedding problem as a system of polynomial equations. Each prescribed edge length translates into a polynomial equation. I.e. if $[v_i, v_j] \in E$ with length $l_{i,j}$, we require $(x_i - x_j)^2 + (y_i - y_j)^2 = l_{i,j}^2$ where (x_i, y_i) denote the coordinates of the embedding of the vertex v_i. Thus we obtain a system of $|E|$ quadratic equations whose solutions represent the embeddings of our framework. To get rid of translations and rotations we fix one point $(x_1, y_1) = (c_1, c_2)$ and the direction of the embedding of the edge $[v_1, v_2]$ by setting $y_2 = c_3$. (Here we assume without loss of generality that there is an edge between v_1 and v_2.) For practical reasons we choose $c_i \neq 0$ and as well $c_1 \neq l_{1,2}$. Hence we want to study the solutions to the following system.

$$\left\{\begin{array}{l} x_1 - c_1 = 0 \\ y_1 - c_2 = 0 \\ x_2 - (l_{1,2} - c_1) = 0 \\ y_2 - c_3 = 0 \\ (x_i - x_j)^2 + (y_i - y_j)^2 - l_{i,j}^2 = 0 \quad \forall [v_i, v_j] \in E - \{[v_1, v_2]\} \end{array}\right\} \tag{1}$$

We will give bounds on the number of solutions in $\mathbb{C}$. To do this we will study the mixed volume of the Newton polytopes (i.e. the convex hulls of the monomial exponent vectors, see for example [22]) of the system (1).

2 Mixed volumes and mixed subdivisions

Let $P_1, \ldots, P_n$ be n polytopes in $\mathbb{R}^n$. We call $P+Q := \{p_i + q_j \mid p_i \in P,\ q_j \in Q\}$ the *Minkowski sum* of P and Q and denote the Euclidean volume in $\mathbb{R}^n$ by vol_n. For non-negative parameters $\lambda_1, \ldots, \lambda_n$, the function $\mathrm{vol}_n(\lambda_1 P_1 + \ldots + \lambda_n P_n)$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$ with non-negative coefficients (see [26]). The coefficient of the monomial $\lambda_1 \cdots \lambda_n$ is called the *mixed volume of* $P_1, \ldots, P_n$ and is denoted by $\mathrm{MV}_n(P_1, \ldots, P_n)$.

The mixed volume is independent of the order of its arguments and linear in each argument, i.e.

$$\begin{aligned} &\mathrm{MV}_n(P_1, \ldots, \alpha P_i + \beta P_i', \ldots, P_n) = \\ &\qquad \alpha\, \mathrm{MV}_n(P_1, \ldots, P_i, \ldots, P_n) + \beta\, \mathrm{MV}_n(P_1, \ldots, P_i', \ldots, P_n) \end{aligned} \tag{2}$$

and it generalizes the usual volume in the sense that

$$\mathrm{MV}_n(P, \ldots, P) = n!\, \mathrm{vol}_n(P) \tag{3}$$

holds (see [21]).

We state here two explicit formulas for this quantity (see [22] and [15]):

$$\begin{aligned} &\mathrm{MV}_n(P_1, \ldots, P_n) \\ &\quad = (-1)^n \sum_{(\alpha_1, \ldots, \alpha_n) \in \{0,1\}^n} (-1)^{\sum_i \alpha_i}\ \mathrm{vol}_n\left(\sum_i \alpha_i P_i\right) && (4) \\ &\quad = \sum_{\substack{Q \text{ mixed cell of a} \\ \text{mixed subdivision} \\ \text{of } (P_1, \ldots, P_n)}} \mathrm{vol}_n\,(Q) && (5) \end{aligned}$$

The first formula (4) is obtained by using inclusion and exclusion formulas to compute the coefficient of $\lambda_1 \cdots \lambda_n$ in $\mathrm{vol}_n(\lambda_1 P_1 + \ldots + \lambda_n P_n)$, see [21]. To understand the second formula (5) and for further considerations we have to introduce the reader to mixed subdivisions. For technical reasons we prefer here to define mixed subdivisions on point sets rather then on polytopes. This definition can then easily be extended to polytopes by considering their vertex sets as point sets.

Let $\mathcal{S} = (S^{(1)}, \ldots, S^{(m)})$ be a sequence of finite point sets in $\mathbb{R}^n$ that affinely spans the full space. A sequence $C = (C^{(1)}, \ldots, C^{(m)})$ of subsets $C^{(i)} \subseteq S^{(i)}$ is called a *cell* of $\mathcal{S}$. A *subdivision* of $\mathcal{S}$ is a collection $\Gamma = (C_1, \ldots, C_k)$ of cells such that

i) $\dim(\mathrm{conv}(C_i)) = n$ for all cells C_i,

ii) $\mathrm{conv}(C_i) \cap \mathrm{conv}(C_j)$ is a face of both convex hulls and

iii) $\bigcup_{i=1}^{k} \mathrm{conv}(C_i) = \mathrm{conv}(\mathcal{S})$

where $\text{conv}(A) := \text{conv}(A^{(1)} + \ldots + A^{(m)})$ for a sequence of point sets A. A subdivision is called *mixed* if additionally

iv) $\sum_{i=1}^{m} \dim(\text{conv}(C_j^{(i)})) = n$ for all cells C_j in Γ

and it is called *fine mixed* if additionally

v) $\sum_{i=1}^{m} (|C_j^{(i)}| - 1)) = n$ for all cells C_j in Γ

where $|A|$ denotes the number of points in a finite set $A \subset \mathbb{R}^n$. The *type* of a cell is defined as

$$\textit{type}(C) = (\dim(\text{conv}(C^{(1)})), \ldots, \dim(\text{conv}(C^{(m)})))$$

and cells of type $(1, 1, \ldots, 1)$ will be called *mixed cells*. These definitions extend naturally to sequences of polytopes by considering their vertices as the point sets above. In this case every mixed subdivision will be a fine mixed subdivision. If all cells of a subdivision are simplices we will call the subdivision a *triangulation*.

To construct mixed subdivisions we proceed as in [15]. Not every subdivision can be constructed in this way but since we will only need one arbitrary mixed subdivision we can use this simple construction. For each of the point sets $S^{(i)}$ from $\mathcal{S}$ we choose a linear lifting function $\mu_i : \mathbb{R}^n \to \mathbb{R}$ identified by an element of $\mathbb{R}^n$. By $\hat{A}$ we denote the lifted point sets $\{(q, \langle \mu_i, q \rangle) : q \in A\} \subset \mathbb{R}^{n+1}$.

The set of those facets of $\text{conv}(\hat{S}^{(1)} + \ldots + \hat{S}^{(m)})$ which have an inward pointing normal with a positive last coordinate is called the lower hull of the Minkowski sum. If we project down this lower hull back to $\mathbb{R}^n$ by forgetting the last coordinate we get a subdivision of $(S^{(1)}, \ldots, S^{(m)})$. We call such a subdivision *coherent* and will say it is *induced by* $\mu = (\mu_1, \ldots, \mu_m)$.

For the subdivision induced by μ to be a fine mixed subdivision it is sufficient that every vertex of the lower envelope can be expressed uniquely as a Minkowski sum (see [9]). Such a set of liftings will be called *(sufficiently) generic*.

3 Bernstein's Theorem

The core theorem that gives a connection between solutions to systems of polynomial equations and discrete geometry is the following.

Theorem 1 (Bernstein [1]) *Given polynomials $f_1, \ldots, f_n$ over $\mathbb{C}$ with finitely many common zeroes in $(\mathbb{C}^*)^n$ where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and let P_i denote the Newton polytope of f_i in $\mathbb{R}^n$. Then the number of common zeroes of the f_i in $(\mathbb{C}^*)^n$ is bounded above by the mixed volume $MV_n(P_1, \ldots, P_n)$. Moreover for generic choices of the coefficients in the f_i, the number of common solutions is exactly $MV_n(P_1, \ldots, P_n)$.*

Various attempts have been made to generalize these results to count all common roots in $\mathbb{C}^n$ (see for example [10], [16] and [19]). The easiest, but sometimes not the best bound is $\text{MV}_n(\text{conv}(P_1 \cup 0), \ldots, \text{conv}(P_n \cup 0))$ which is shown in [19]. Since the Newton polytopes of our system (1) all contain the point 0 as a vertex, the mixed volume of (1) will give us a bound on the number of solutions in $\mathbb{C}$.

The bound on the number of solutions of a polynomial system arising from Bernstein's Theorem is also often referred to as the *BKK bound* due to the work of Bernstein, Khovanskii and Kushnirenko. The BKK bound generalizes the Bézout bound (see [7] chapter 7) and for sparse polynomial systems it is often significantly better.

Bernstein also gives an explicit condition when a choice of coefficients is generic which we will state in the following. In [8] Canny and Rojas show that the BKK-bound is sharp under even weaker assumptions.

Let w be a non-zero vector and let $\partial_w P_i$ denote the face of P_i which is minimal with respect to the direction w. Also we set $\partial_w f_i = \sum_{\alpha \in \partial_w P_i} c_\alpha x^\alpha$ to be the face equation with respect to w.

Theorem 2 (Bernstein's Second Theorem [1]) *If $\forall\, w \neq 0$, the face system $\partial_w f_1 = 0, \ldots, \partial_w f_n = 0$ has no solution in $(\mathbb{C}^*)^n$, then the mixed volume of the Newton polytopes of the f_i gives the exact number of common zeros in $(\mathbb{C}^*)^n$ and all solutions are isolated. Otherwise it is a strict upper bound.*

Note that it is necessary for a direction w to be a witness of the degeneracy that it lies on the tropical prevariety (see [20]) of the polynomial equations $f_1, \ldots, f_n$.

In our case the system (1) allows to choose such a direction w. Namely if we choose $w = (0, 0, 0, 0, -1, -1, \ldots, -1)$ we get the face system

$$\left\{ \begin{array}{l} x_1 - c_1 = 0 \\ y_1 - c_2 = 0 \\ x_2 - (l_{1,2} - c_1) = 0 \\ y_2 - c_3 = 0 \\ x_i^2 + y_i^2 = 0 \quad \forall [v_1, v_i], [v_2, v_i] \in E \\ (x_i - x_j)^2 + (y_i - y_j)^2 = 0 \quad \forall [v_i, v_j] \in E \text{ with } i, j \neq 1, 2 \end{array} \right\}$$

which has $(x_1, y_1, \ldots, x_{|V|}, y_{|V|}) = (c_1, c_2, l_{1,2} - c_1, c_3, 1, i, 1, i, \ldots, 1, i)$ as a solution with non-zero complex entries. So the mixed volume of (1) will always be a strict upper bound on the number of graph embeddings.

To remove this degeneracy Ioannis Emiris[1] proposed a simple substitution. The idea is to introduce new variables s_i with new equations $s_i = x_i^2 + y_i^2$ for $i = 1, \ldots, |V|$. This way the quadratic terms in the equations given by the edge lengths disappear. So we should deal with the following system of equations.

$$\left\{ \begin{array}{l} x_1 - c_1 = 0 \\ y_1 - c_2 = 0 \\ x_2 - (l_{1,2} - c_1) = 0 \\ y_2 - c_3 = 0 \\ s_i + s_j - 2x_i x_j - 2y_i y_j - l_{i,j}^2 = 0 \quad \forall [v_i, v_j] \in E - \{[v_1, v_2]\} \\ s_i - x_i^2 - y_i^2 = 0 \quad \forall i = 1, \ldots, |V| \end{array} \right\} \tag{6}$$

4 New technical tools to simplify mixed volume calculation

In the special case of Henneberg I graphs our system (1) will be in a shape that allows to separate the mixed volume calculation into smaller pieces. Our main

[1] Personal communication at EuroCG 2008, Nancy

tool to do this is the following Lemma. It is most often used in the special case when all polytopes involved have integer vertices. In this case there is a shorter proof using Bernstein's Theorem. However we would like to state it here in the general case and give a purely geometric proof for it.

Lemma 3 *Let $P_1, \ldots, P_k$ be polytopes in $\mathbb{R}^{m+k}$ and $Q_1, \ldots, Q_m$ be polytopes in $\mathbb{R}^m \subset \mathbb{R}^{m+k}$. Then*

$$\mathrm{MV}_{m+k}(Q_1, ..., Q_m, P_1, ..., P_k) = \mathrm{MV}_m(Q_1, ..., Q_m) * \mathrm{MV}_k(\pi(P_1), ..., \pi(P_k)) \tag{7}$$

where $\pi : \mathbb{R}^{m+k} \to \mathbb{R}^k$ denotes the projection on the last k coordinates.

An equivalent decomposition result was already mentioned in [5] in which the authors refer to [12] for the proof which unfortunately we were unable to check and therefore we give a full proof ourselves.

Proof. First we will show the Lemma in the *semimixed case* where $Q_1 = \ldots = Q_m = Q$ and $P_1 = \ldots = P_k = P$, then we will show that both sides of the desired equation define a symmetric multilinear function and then we will use combinatorial identities for symmetric multilinear functions to show the full result.

By (3) we have to show first that

$$\mathrm{MV}_{m+k}(Q, \ldots, Q, P, \ldots, P) = m!\, k!\ \mathrm{vol}_m(Q) * \mathrm{vol}_k(\pi(P)) \tag{8}$$

where Q is taken m times and P is taken k times. But this formula for semimixed systems is a special case of Lemma 4.9 in [11] or also of Theorem 1 in [2].

Let $\mathcal{P}^m$ (resp. $\mathcal{P}^{m+k}$) be the set of all m-dimensional (resp. $m+k$-dimensional) polytopes and define two functions g_1 and g_2 on $\mathcal{P}^m \times \ldots \times \mathcal{P}^m \times \mathcal{P}^{m+k} \times \ldots \times \mathcal{P}^{m+k}$ via

$$\begin{aligned} g_1(Q_1, \ldots, Q_m, P_1, \ldots, P_k) &:= \mathrm{MV}_{m+k}(Q_1, \ldots, Q_m, P_1, \ldots, P_k) \\ g_2(Q_1, \ldots, Q_m, P_1, \ldots, P_k) &:= \mathrm{MV}_m(Q_1, ..., Q_m) * \mathrm{MV}_k(\pi(P_1), ..., \pi(P_k)) \ . \end{aligned}$$

It is easy to see that g_1 and g_2 are invariant under changing the order of the Q_i and also changing the order of the P_j. Furthermore it follows from (2) that both functions are linear in each argument. Let $f : A \times \ldots \times A \to B$ be a symmetric multilinear function, where A and B are semigroups. By expanding the right hand side it can be seen that

$$f(a_1, ..., a_n) = \frac{1}{n!} \sum_{1 \le i_1 < \ldots < i_q \le n} (-1)^{n-q} f(a_{i_1} + ... + a_{i_q}, ..., a_{i_1} + ... + a_{i_q}) \ . \tag{9}$$

The functions

$$\begin{aligned} \tilde{g}_i^{(P_1, \ldots, P_k)}(Q_1, \ldots, Q_m) &:= g_i(Q_1, \ldots, Q_m, P_1, \ldots, P_k) \text{ and} \\ \bar{g}_i^{(Q)}(P_1, \ldots, P_k) &:= g_i(Q, \ldots, Q, P_1, \ldots, P_k) \text{ for } i = 1, 2 \end{aligned}$$

satisfy these conditions. Hence we have for $i = 1, 2$ that

$$\begin{aligned}
& g_i(Q_1, \dots, Q_m, P_1, \dots, P_k) \\
& \quad = \tilde{g}_i^{(P_1,\dots,P_k)}(Q_1, \dots, Q_m) \\
& \quad = \frac{1}{m!} \sum_{1 \le i_1 < \dots < i_q \le m} (-1)^{m-q} \; \tilde{g}_i^{(P_1,\dots,P_k)}(Q_{i_1} + ... + Q_{i_q}, ..., Q_{i_1} + ... + Q_{i_q}) \\
& \quad = \frac{1}{m!} \sum_{1 \le i_1 < \dots < i_q \le m} (-1)^{m-q} \; \bar{g}_i^{(Q_{i_1}+\dots+Q_{i_q})}(P_1, \dots, P_k) \; .
\end{aligned}$$

Since we can expand $\bar{g}_i^{(Q_{i_1}+\dots+Q_{i_q})}(P_1, \dots, P_k)$ by using (9) as well, we see that both functions g_1 and g_2 are fully determined by their images of tuples of polytopes where $Q_1 = \dots = Q_m = Q$ and $P_1 = \dots = P_k = P$. This proves the Lemma. $\square$

Another technical tool which will be needed in a subsequent proof is the following Lemma. This goes back to an idea of Emiris and Canny [9] to use linear programming and the formula (5) to compute the mixed volume. The proof is based on the duality theorem for linear programming.

Lemma 4 *Given polytopes $P_1, \dots, P_n \subset \mathbb{R}^n$ and lifting vectors $\mu_1, \dots, \mu_n \in \mathbb{R}^n_{\ge 0}$. Denote the vertices of P_i by $v_1^{(i)}, \dots, v_{r_i}^{(i)}$ and choose one edge $e_i = [v_{k_i}^{(i)}, v_{l_i}^{(i)}]$ from each P_i. Then $C := (e_1, \dots, e_n)$ is a mixed cell of the mixed subdivision induced by the liftings μ_i if and only if*

i) The edge matrix $E := V_a - V_b$ is non-singular (where $V_a := (v_{k_1}^{(1)}, \dots, v_{k_n}^{(n)})$ and $V_b := (v_{l_1}^{(1)}, \dots, v_{l_n}^{(n)})$) and

ii) For all polytopes P_i and all vertices $v_s^{(i)}$ of P_i which are not in e_i we have:

$$\left(\mathrm{diag}\left(\mu^T E\right)^T E^{-1} - \mu_i^T\right) \cdot \left(v_{l_i}^{(i)} - v_s^{(i)}\right) \ge 0 \tag{10}$$

where $\mu := (\mu_1, \dots, \mu_n)$ and where $\mathrm{diag}(V)$ denotes the vector of the diagonal entries of V.

Before we begin with the proof we start with some preliminary considerations about linear programming and how it can be applied here. In [9] it is shown that the test, if a cell lies on the lower envelope of the lifted Minkowski sum can be formulated as a linear program. Let $\hat{m}_i \in \mathbb{R}^{n+1}$ denote the midpoint of the lifted edge $\hat{e}_i$ of $\hat{P}_i$ such that $\hat{m} = \hat{m}_1 + \dots + \hat{m}_k$ is an interior point of the Minkowski sum $\hat{e}_1 + \dots + \hat{e}_k$. Consider the linear program,

$$\begin{aligned}
\text{maximize } s \quad & \in \quad \mathbb{R}_{\ge 0} \\
\text{s.t. } \hat{m} \quad & - \quad (0, \dots, 0, s) \in \hat{P}_1 + \dots + \hat{P}_k \; .
\end{aligned} \tag{11}$$

If we denote the vertices of P_i by $v_1^{(i)}, \dots, v_{r_i}^{(i)}$ this can be written as

$$\begin{aligned}
\text{maximize} \quad & s \in \mathbb{R}_{\geq 0} \\
\text{s.t.} \quad & \hat{m} - (0, \ldots, 0, s) = \sum_{i=1}^{n} \sum_{j=1}^{r_i} \lambda_j^{(i)} \hat{v}_j^{(i)} \\
& \sum_{j=1}^{r_i} \lambda_j^{(i)} = 1 \quad \forall\, i = 1, ..., n \\
& \lambda_j^{(i)} \geq 0 \quad \forall\, i, j \quad .
\end{aligned}$$

s measures the distance of $\hat{m}$ to the lower envelope of the Minkowski sum. Hence $\hat{m}$ lies on the lower envelope of $\hat{P}_1 + \ldots + \hat{P}_k$ if and only if the optimal value of (11) is zero.

We call a linear program in *standard form* if it is stated as follows

$$\begin{aligned}
\text{maximize} \quad & c^t x \\
\text{s.t.} \quad & Ax = b \\
& x_i \geq 0 \quad \forall i = 1, \ldots, n
\end{aligned}$$

where $c, x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{n \times m}$. Given a feasible point $\bar{x} \geq 0$ satisfying $A.\bar{x} = b$ we want to check whether $\bar{x}$ is an optimal solution. If $\bar{x}$ is a vertex of the polyhedron defined by the constraints and is not degenerate in the sense defined below, we can use linear programming duality to test for optimality. To $\bar{x}$ corresponds a (not necessarily unique) choice B of columns of A such that the submatrix consisting of these columns A_B satisfies $A_B^{-1}.b = \bar{x}$. ($\bar{x}$ is non-degenerate if the inverse of A_B exists.) Let A_N be the submatrix of A consisting of the remaining columns and define c_B and c_N in the same way. Then $\bar{x}$ is a feasible point of the dual program and therefore optimal (see [13]) if and only if

$$c_N^t - c_B^t \cdot A_B^{-1} \cdot A_N \leq 0 \quad \text{(componentwise)} \; . \tag{12}$$

Our linear program (11) can be written in standard form using the following notation.

$$\begin{aligned}
c^t &= (\mathbf{0}^t_{r_1+\ldots+r_n}, 1) \;\in \mathbb{R}^{r_1+\ldots+r_n+1} \\
x^t &= (\lambda_1^{(1)}, \ldots, \lambda_{r_1}^{(1)}, \ldots\ldots, \lambda_1^{(n)}, \ldots, \lambda_{r_n}^{(n)}, s) \;\in \mathbb{R}^{r_1+\ldots+r_n+1} \\
b^t &= (\hat{m}, \mathbf{1}_n^t) \;\in \mathbb{R}^{2n+1}
\end{aligned}$$

$$A = \begin{pmatrix}
v_1^{(1)} & \ldots & v_{r_1}^{(1)} & \ldots & \ldots & v_1^{(n)} & \ldots & v_{r_n}^{(n)} & \mathbf{0}_n \\
\langle \mu_1, v_1^{(1)} \rangle & \ldots & \langle \mu_1, v_{r_1}^{(1)} \rangle & \ldots & \ldots & \langle \mu_n, v_1^{(n)} \rangle & \ldots & \langle \mu_n, v_{r_n}^{(n)} \rangle & 1 \\
 & \mathbf{1}^t_{r_1} & & \mathbf{0}^t_{r_2} & \ldots & & \mathbf{0}^t_{r_n} & & 0 \\
 & \mathbf{0}^t_{r_1} & & \mathbf{1}^t_{r_2} & \ldots & & \mathbf{0}^t_{r_n} & & 0 \\
 & \vdots & & & \ddots & & \vdots & & \vdots \\
 & \mathbf{0}^t_{r_1} & & \mathbf{0}^t_{r_2} & \ldots & & \mathbf{1}^t_{r_n} & & 0
\end{pmatrix}$$

Here we denote by $\mathbf{0}_n$ and $\mathbf{1}_n$ the column vectors consisting only of 0's and 1's respectively.

In this notation our point $\hat{m}$ from (11) corresponds to $\bar{x} = (\lambda_1^{(1)}, \ldots\ldots, \lambda_{r_n}^{(n)}, s)$ where $s = 0$ and $\lambda_j^{(i)} = \frac{1}{2}$ if the edge $\hat{e}_i$ contains the vertex $\hat{v}_j^{(i)}$ and $\lambda_j^{(i)} = 0$ otherwise.

To prove Lemma 4 we will now assume that this $\bar{x}$ is optimal and deduce conditions on the lifting vectors μ_i by using the inequality (12).

Proof. (of Lemma 4) Note that C is full-dimensional and hence has a non-zero volume if and only if E is non-singular. In the following we will only consider this case. To simplify the notation we write $\mu(V)$ for $\operatorname{diag}(\mu^t.V)^t$.

We know that C is a mixed cell if and only if the following $\bar{x}$ is the optimal solution to the linear program defined above.

$$\bar{x} = (\lambda_{1,1}, \ldots, \lambda_{n,r_n}, s) \text{ where } s = 0 \text{ and } \lambda_{i,j} = \begin{cases} \frac{1}{2}, & j \in \{k_i, l_i\} \\ 0, & \text{else} \end{cases} .$$

For the submatrix of A corresponding to $\bar{x}$ we have

$$A_B = \begin{pmatrix} V_a & V_b & \mathbf{0}_n \\ \mu(V_a) & \mu(V_b) & 1 \\ \mathrm{Id}_n & \mathrm{Id}_n & \mathbf{0}_n \end{pmatrix} \quad \text{and} \quad A_N = \begin{pmatrix} v_s^{(i)} \\ \mu_r.v_s^{(i)} \\ \xi_i \end{pmatrix}_{\substack{1 \le i \le n \\ 1 \le s \le r_i \\ s \ne k_i, l_i}}$$

where ξ_i denotes the i^{th} unit vector. The inverse of A_B is

$$A_B^{-1} = \begin{pmatrix} E^{-1} & \mathbf{0}_n & -E^{-1}.V_b \\ -E^{-1} & \mathbf{0}_n & E^{-1}.V_a \\ -\mu(E).E^{-1} & 1 & \mu(E).E^{-1}.V_b - \mu(V_b) \end{pmatrix} .$$

Since $c_N = (0, \ldots, 0)$ the criterion (12) tells us that $\bar{x}$ is optimal if

$$(0, \ldots, 0, 1) \cdot A_B^{-1} \cdot A_N \ge 0 \quad \text{componentwise} .$$

But a single entry of the vector on the left can be explicitly computed as

$$-\left(\mu(E) \cdot E^{-1}\right) \cdot v_s^{(i)} + \mu_r \cdot v_s^{(i)} + \left(\mu(E) \cdot E^{-1} \cdot V_b - \mu(V_b)\right) \cdot \xi_i$$

which equals the left hand side of (10). □

Note that (10) is linear in the μ_j. Hence given a choice of edges we can explicitly calculate $\sum_{i=1}^{n} r_i$ normal vectors defining a cone in $\mathbb{R}^{n^2}$. The interior of this cone consists of all liftings $(\mu_1^t, \ldots, \mu_n^t)$ which induce a mixed subdivision that contains our chosen cell as a mixed cell.

5 Henneberg I graphs

For this simple class of Laman graphs the mixed volume bound is tight as we will demonstrate below. Our proof exploits the inductive structure of Henneberg I graphs which is why it cannot be used for Henneberg II graphs.

Theorem 5 *A Henneberg I step at most doubles the number of embeddings of the framework and there is always a choice of edge lengths such that the number of embeddings is doubled.*

Proof. In a Henneberg I step we add one vertex $v_{|V|+1}$ and two edges $[v_r, v_{|V|+1}]$, $[v_q, v_{|V|+1}]$ with lengths $l_{r,|V|+1}$ and $l_{q,|V|+1}$. So our system of equations (6) gets three new equations, namely

$$s_{|V|+1} - x_{|V|+1}^2 - y_{|V|+1}^2 = 0 \quad (13)$$

$$s_r + s_{|V|+1} - 2x_r x_{|V|+1} - 2y_r y_{|V|+1} - l_{r,|V|+1}^2 = 0 \quad (14)$$

$$s_q + s_{|V|+1} - 2x_q x_{|V|+1} - 2y_q y_{|V|+1} - l_{q,|V|+1}^2 = 0. \quad (15)$$

In our new system of equations these three are the only polynomials involving $x_{|V|+1}$, $y_{|V|+1}$ and $s_{|V|+1}$, so we can use Lemma 3 to calculate the mixed volume separately. The projections of the Newton polytopes of equations (13), (14) and (15) to the coordinates $x_{|V|+1}$, $y_{|V|+1}$ and $s_{|V|+1}$ are

$$\mathrm{conv}\left\{\begin{pmatrix}2 & 0 & 0\end{pmatrix}^T, \begin{pmatrix}0 & 2 & 0\end{pmatrix}^T, \begin{pmatrix}0 & 0 & 1\end{pmatrix}^T\right\}$$

and twice

$$\mathrm{conv}\left\{\begin{pmatrix}1 & 0 & 0\end{pmatrix}^T, \begin{pmatrix}0 & 1 & 0\end{pmatrix}^T, \begin{pmatrix}0 & 0 & 1\end{pmatrix}^T, \begin{pmatrix}0 & 0 & 0\end{pmatrix}^T\right\} .$$

The mixed volumes of these equals 2. So by Lemma 3 the mixed volume of the new system is twice the mixed volume of the system before the Henneberg I step.

To get two new embeddings for each previous one we choose our new edge lengths to be almost equal to each other and much larger then all previous edge lengths (larger then the sum of all previous is certainly enough). This leads to the desired new embeddings. □

Each Henneberg sequence starts with a triangle which has obviously at most 2 embeddings up to rigid motions (we count reflections separately). Hence using our Theorem inductively we get the following corollary.

Corollary 6 *The number of embeddings of Henneberg I graphs is less than or equal* $2^{|V|-2}$ *and this bound is sharp.*

6 Laman graphs on 6 vertices

For Laman graphs on 6 vertices, the general bound in [3] on the number of embeddings is 70. From the Henneberg constructions and simple combinatorial considerations, it follows that the only Henneberg II Laman graphs on 6 vertices are the Desargues graph and $K_{3,3}$ (see figure 2). For the Desargues graph, an

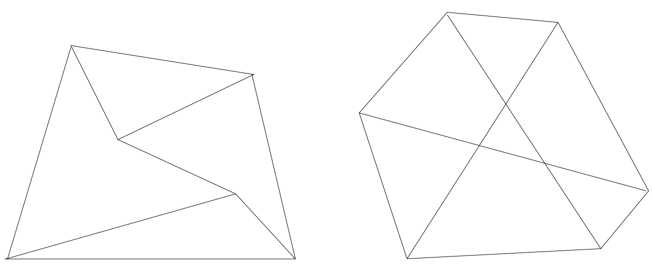

Figure 2: Left: Desargues graph. Right: $K_{3,3}$.

explicit analysis is given in [3] which shows that the correct number is only 24, and that there is a choice of edge lengths giving 24 different embeddings. For the $K_{3,3}$, Manfred Husty found a construction with 32 embeddings [17].

When we set up the system (6) our mixed volume approach yields a bound of 32 for both graph classes on 6 vertices[2]. So in the case of 6 vertices our

[2] We used the PHCpack by Jan Verschelde for our mixed volume calculations, see [24].

bound is tight. By glueing several copies of $K_{3,3}$ together and using Lemma 3 to calculate the mixed volume we get an infinite class of graphs where our bound is tight as well.

7 General case

For the classes discussed above (Henneberg I, graphs on six vertices) as well as some other special cases, our bound on the number of embeddings improves the known general bounds. For the general case, our mixed volume approach for the system (1) without the substitutions suggested by Ioannis Emiris provides a simple, but very weak bound. However, it may be of independent interest, that for this class of problems, it is possible to determine the mixed volume exactly.

Theorem 7 *The mixed volume of our initial system (1) is exactly* $4^{|V|-2}$.

Proof. The mixed volume of (1) is at most the product of the degrees of the polynomial equations because it is less than or equal to the Bézout bound (see [22]). To show that the mixed volume is at least this number we will use Lemma 4 to give a lifting that induces a mixed cell of volume $4^{|V|-2}$.

The first 4 equations of (1) give rise to a single edge as a Newton polytope which is part of any mixed cell. Now we claim that we can order the Newton polytopes P_i in such a way that, for $i \geq 5$, P_i contains the edge $[0, 2\xi_i]$ where ξ_i denotes the i^{th} unit vector. To see this, note first that every equation in (1) has a non vanishing constant term and therefore its Newton polytope contains the point 0. To see that P_i contains $2\xi_i$ we have to show that we can order the polynomials of our system (1) such that for $i > 5$ the i^{th} polynomial contains the term $x^2_{\frac{i+1}{2}}$ for i odd and $y^2_{\frac{i}{2}}$ for i even. To see this it is enough to show that there is a labeling of the edges of our graph with a direction such that each vertex except the fixed edge $[v_1, v_2]$ has exactly two incoming edges. We

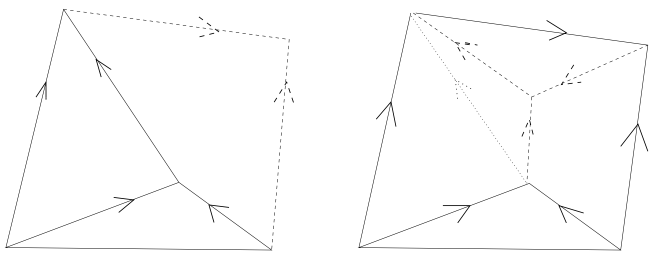

Figure 3: A Henneberg I and a Henneberg II step with directed edges.

use the Henneberg constructions to see that this is possible. How to choose the directions in the Henneberg steps is sketched in figure 3. In a Henneberg I step we let the two new edges point to the new vertex. While in a Henneberg II step we remember the direction of the deleted edge $\overrightarrow{[v_r, v_s]}$ and let the new edge, which connects the new vertex to v_s, point to v_s. The other two new edges point to the new vertex.

Now using Lemma 4 we describe a lifting that induces a subdivision that has $(\{0, \xi_1\}, \ldots, \{0, \xi_4\}, \{0, 2\xi_5\}, \ldots, \{0, 2\xi_{2|V|}\})$ as a mixed cell. In the notation of Lemma 4 our chosen edges give rise to the edge matrix $E = \begin{pmatrix} \mathbb{E}_4 & \mathbf{0} \\ \mathbf{0} & 2\mathbb{E}_{2|V|-4} \end{pmatrix}$. Substituting this into the second condition (10) we get that for each Newton polytope P_i all vertices $v_s^{(i)}$ of P_i which are not 0 or $2\xi_i$ have to satisfy

$$\left((\mu_1^{(1)}, \ldots, \mu_{2|V|}^{(2|V|)}) - \mu^{(i)} \right) \cdot v_s^{(i)} \leq 0\,,$$

where we denote by $\mu^{(j)} \in \mathbb{Q}^{2|V|}$ the lifting vector for P_j. Since all the entries of each $v_s^{(i)}$ are non-negative this can easily be done by choosing the vectors $\mu^{(j)}$ such that their j^{th} entry is relatively small and all other entries are relatively large. $\square$

Corollary 8 *The number of embeddings of a Laman graph framework with generic edge lengths is strictly less then* $4^{|V|-2}$.

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