Galois sections and p-adic period maps

Jakob Stix

Institut für Mathematik
Goethe–Universität Frankfurt am Main

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joint work with
L. Alexander Betts (Harvard)

the letter

Grothendieck → Faltings 1

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Lieber Herr Falting,

Vielen Dank für Ihre rasche Antwort und Übersendung der Separate! Ih
Kommentar zur sog. "Theorie der Motive" ist von der üblichen Art, die woh
grossenteils der in der Mathematik stark eingewurzelten Tradition entspringt, nur denjenigen mathematischen Situationen und Zusammenhängen einer (eventuell langatmiger) Untersuchung und Aufmerksamkeit zuzuwenden,

$\vdots$

Nun einige Worte zum "Yoga" der anabelschen Geometrie. Es geht dabe
um "absolute" alg. Geometrie, nämlich über Grundkörpern (etwa) K, die endlich erzeugt über dem Primkörper sind. Die allgemeine Grundidee ist, dass für gewisse, sog. "anabelsche", Schemata X (von endlichem Typ) über K, die Geometrie von X vollständig durch die (profinite) Fundamentalgruppe $\pi_1(X,\xi)$ bestimmt ist (wo ξ ein "geometrischer Punkt" von X ist, etwa

what's anabelian

遠 = “far away from (abelian)”

geometry and arithmetic that is governed by the group theory of the étale fundamental group $\pi_1^{\text{ét}}(X, \bar{x})$

hyperbolic curves

Nakamura, Tamagawa, Mochizuki, S.

Theorem (Mochizuki 1997)

K sub-p-adic, X/K smooth curve with $\chi < 0$, geometrically connected, T/K smooth

$$\mathrm{Hom}_K^{\mathrm{dom}}(T, X) \xrightarrow{\sim} \mathrm{Hom}_{\pi_1(K)}^{\mathrm{open,out}}(\pi_1(T), \pi_1(X))$$

non-abelian $\mathrm{K}(\pi, 1)$-version of Tate-conjecture

rational points: anabelian perspective

Galois sections:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\bar{k}}, \bar{x}) \arrow[r] & \pi_1(X, \bar{x}) \arrow[r] & \pi_1(k) \arrow[r] \arrow[l, bend right, "s_a"'] \arrow[dl, "a_*"] & 1 \\
 & & \pi_1(X, \bar{a}) \arrow[u, "\gamma(-)\gamma^{-1}", "\cong"'] & &
\end{tikzcd}$$

pro-finite Kummer map:

$$X(k) \to \mathscr{S}_{X/k} := \{\pi_1(X_{\bar{k}})\text{-conj. classes } \pi_1(k) \xrightarrow{s} \pi_1(X)\}$$

Section Conjecture: bijective . . . k, X/k, . . . variants

injective: anabelian proof

$a \in X(k)$, set $U_a = X - \{a\}$

$$
\begin{tikzcd}
1 \arrow[r] & \pi_1(U_{a,\bar{k}}) \arrow[r] \arrow[d, "\cong"] & \pi_1(U_a) \arrow[r] \arrow[d] & \pi_1(k) \arrow[r] \arrow[d, "s_a"] & 1 \\
1 \arrow[r] & \pi_1(U_{a,\bar{k}}) \arrow[r] & \pi_1(X \times X \setminus \Delta) \arrow[r] & \pi_1(X) \arrow[r] & 1
\end{tikzcd}
$$

if $s_a = s_b$, then $\pi_1(U_a) \simeq \pi_1(U_b)$

$\Longrightarrow U_a \simeq U_b$ over id_X, hence $a = b$.

weak versus strong

- **weak SC**: $\mathscr{S}_{X/k} \neq \emptyset$, then $X(k) \neq \emptyset$
- **neighbourhood**: $Y \xrightarrow{\text{fét}} X$ & $t \in \mathscr{S}_{Y/k}$ with $t \mapsto s$: i.e. $s(\pi_1(k)) \subseteq \pi_1(Y) \subseteq \pi_1(X)$.
- k-form of pro-étale universal cover

$$X_s = \varprojlim_{(Y,t) \text{ ngbh of } s} Y$$

- s of the form $s_a \iff a \in \operatorname{im}\big(X_s(k) \to X(k)\big)$
- Mordell-Faltings: weak SC for all neighb. $\implies$ SC for X

$$X_s(k) = \varprojlim_{(Y,t) \text{ ngbh of } s} Y(k)$$

limit of compact non-empty sets is non-empty

real section conjecture

Theorem (Mochizuki, Huisman, Wojtkowiak, Wickelgren, Esnault–Wittenberg, Pál, Bresciani-Vistoli, ...)

$X/\mathbb{R}$ smooth hyperbolic curve: $\pi_0(X(\mathbb{R})) \xrightarrow{\sim} \mathscr{S}_{X/\mathbb{R}}$

Theorem (Tim Holzschuh, 2024)

$X/\mathbb{R}$ quasi-projective, geom connected, $X(\mathbb{C})$ simply connected

$$\pi_0(X(\mathbb{R})) \xrightarrow{\sim} \pi_0\ \mathrm{maps}_{B\,\mathsf{Gal}_{\mathbb{R}}}\big(B\,\mathsf{Gal}_{\mathbb{R}}, \Pi_\infty^{\text{ét}}(X)\big)$$

birational section conjecture

- sections of $\pi_1(k(X)) \to \pi_1(k)$
- cuspidal packets: image in decomposition subgroup
- **birational SC**: only cuspidal sections

Theorem (Koenigsmann, 2003)

$k/\mathbb{Q}_p$ finite, X/k smooth geom. connected curve:
$\Longrightarrow$ birational SC holds.

sketch: $M_s := \overline{k(X)}^{s(\pi_1(k))}$ elementary equivalent to k
generic point in $X(M_s)$, hence $X(k)$ not empty

t-birational sections

base change: $L/K \rightsquigarrow \mathscr{S}_{X/K} \to \mathscr{S}_{X_L/L}$, $s \mapsto s_L$ because of pull back square

$$\begin{array}{ccc} \pi_1(X_L) & \longrightarrow & \pi_1(L) \\ \downarrow & & \downarrow \\ \pi_1(X) & \longrightarrow & \pi_1(K) \end{array}$$

t-**birational**: $s \in \mathscr{S}_{X/k}$ such that $s_{k(t)}$ lifts to birational section

Theorem (Bresciani 2021)

$k/\mathbb{Q}$ finitely generated, X/k smooth hyperbolic curve
$s \in \mathscr{S}_{X/k}$ from rational point or cuspidal
$\iff$ s is t-birational

adèle map: selmer sections

- $F/\mathbb{Q}$ finite, X/F smooth projective hyperbolic curve
- (as in) Koenigsmann & real SC: **Selmer sections**

$$\begin{tikzcd} X(F) \arrow[r, phantom, "\subseteq"] & \mathscr{S}^{\mathrm{bir}}_{X/F} \arrow[r, phantom, "\subseteq"] & \mathscr{S}^{\mathrm{Sel}}_{X/F} \arrow[r, phantom, "\subseteq"] \arrow[d, "\underline{a}(s)=(\mathrm{loc}_v(s))_v"] & \mathscr{S}_{X/F} \arrow[d, "\mathrm{loc}"] \\ & & X(\mathbb{A}_F)_\bullet \arrow[r, phantom, "\subseteq"] & \prod_v \mathscr{S}_{X_{F_v}/F_v} \end{tikzcd}$$

- Theorem (Porowski 2023): loc injective
- Theorem (Stoll, Harari-S.): image of $\underline{a}$ equals $X(\mathbb{A}_F)_\bullet^{\mathrm{f-cov}}$
- SC for Selmer sections
  $\iff X(F) = X(\mathbb{A}_F)_\bullet^{\mathrm{f-cov}}$ (Conjecture of Stoll 2007)
- Theorem (S., 2013): F imaginary quadratic or totally real, and $s \in \mathscr{S}^{\mathrm{bir}}_{X/F}$ *adelic*, then s comes from a point

main result — joint with L. Alexander Betts

Theorem (Betts-S., 2022)

$F/\mathbb{Q}$ finite, no CM-subfield, X/F smooth projective, $g \geq 2$. Then for all finite places v of F

$$\mathrm{loc}_v \colon \mathscr{S}^{\mathrm{Sel}}_{X/F} \longrightarrow X(F_v)$$

has finite image.

- adapt p-adic method of Lawrence-Venkatesh to show finiteness statement for Galois sections
- extra work to get result for all places v

LV-locus

- abelian by finite family (Kodaira-Parshin):
  $f : Y \xrightarrow{\text{pol. AV}} X' \xrightarrow{\text{fét}} X$
- **symplectic pair**: $A = f_*\mathbb{Q}_p$ acts on $V = \mathrm{R}^1 f_*\mathbb{Q}_p$ with symplectic pairing $\Lambda^2 V \to \mathbb{Q}_p(-1)$
- for every $F \subseteq K$ and $x \in X(K)$ fibre in geometric point above x yields

$$(A_x, V_x) \text{ symplectic pair in } \mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(K))$$

Definition

$X(F_v)^{\mathrm{LV}}_{Y/X,S}$: locus where (A_x, V_x) is restriction of symplectic pair in $\mathrm{Rep}_{\mathbb{Q}_p}(\pi_1(F))$, with A unramified outside S, with V pure, integral, weight 1 outside S, and V de Rham at places above p with HT-weights in $\{0, 1\}$.

strategy

1. for all abelian-by-finite families: factorization for S large enough

$$\begin{array}{ccccc} X(F) & \subseteq & \mathscr{S}^{\mathrm{Sel}}_{X/F} & \subseteq & X(\mathbb{A}_F)_\bullet \\ & & \downarrow & & \downarrow \\ & & X(F_v)^{\mathrm{LV}}_{Y/X,S} & \subseteq & X(F_v) \end{array}$$

2. for suitable Y/X show finiteness of $X(F_v)^{\mathrm{LV}}_{Y/X,S}$ along suitably modified method of Lawrence-Venkatesh

part 1

$\mathscr{S}^{\mathrm{Sel}}_{X/F} \to X(F_v)^{\mathrm{LV}}_{Y/X,S}$:

- restriction (A_x, V_x) in $x \in X(K)$ only requires $x_* : \pi_1(K) \to \pi_1(X)$
- for $s \in \mathscr{S}_{X/F}$ get (A_s, V_s) by restriction along $s : \pi_1(F) \to \pi_1(X)$, conjugate s yields isomorphic pair
- if s Selmer, then local properties of (A_s, V_s) same as restriction to local component of adèle $\underline{a}(s)$
- thus $x_v = \mathrm{loc}_v(s)$ belongs to LV-locus (A_{x_v}, V_{x_v}) being the restriction of (A_s, V_s)

part 2: recall [LV]

Fix abelian by finite $Y/X'/X$

$$\begin{tikzcd}
X(F_v) \arrow[d, "x \mapsto (A_x, V_x)"] & \supset & U_{x_0}(F_v) \arrow[r, "\Phi_{x_0}"] & \mathcal{H}_{x_0}(F_v) \arrow[d, "(M^0, M^1(-))"] \\
\mathrm{SP}\big(\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\pi_1(F_v))\big) \arrow[rrr, hook, "\mathscr{D}_{\mathrm{pH}} = (\mathscr{D}_{\mathrm{pst}}, \mathscr{D}_{\mathrm{dR}}, c)"] & & & \mathrm{SP}\big(\mathrm{MF}(\varphi, N, \pi_1(F_v))\big)
\end{tikzcd}$$

- Φ_{x_0}: period map: parallel transport of Hodge filtration along Gauß-Manin
- U_{x_0}: small admissible neighbourhood in $X^{\mathrm{an}}_{F_v}$
- $\mathcal{H}_{x_0}$: Lagrangian Grassmannian parametrising Lagrangian $\mathrm{H}^0_{\mathrm{dR}}(Y_{x_0}/F_v)$-submodules of $\mathrm{H}^1_{\mathrm{dR}}(Y_{x_0}/F_v)$
- Fontaine: $\mathscr{D}_{pH}$ is fully faithful
- $M^1(W) := \mathscr{D}_{\mathrm{pst}}(\mathrm{H}^1_{\text{ét}}(Y_{x_0, \bar{F}_v}, \mathbb{Q}_p))$ with filtration $W \subseteq \mathrm{H}^1_{\mathrm{dR}}(Y_{x_0}/F_v) = \mathscr{D}_{\mathrm{dR}}(\mathrm{H}^1_{\text{ét}}(Y_{x_0, \bar{F}_v}, \mathbb{Q}_p))$

strategy:

- $Y/X'/X$ full monodromy: $\mathbb{C}/v$-adic analytic switch

diagram commutes

$$
\begin{tikzcd}
X(F_v) \arrow[d, "x \mapsto (A_x, V_x)"] & \supset & U_{x_0}(F_v) \arrow[r, "\Phi_{x_0}"] & \mathcal{H}_{x_0}(F_v) \arrow[d, "(M^0, M^1(-))"] \\
\mathrm{SP}\big(\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\pi_1(F_v))\big) \arrow[rrr, hook, "\mathscr{D}_{\mathrm{pH}} = (\mathscr{D}_{\mathrm{pst}}, \mathscr{D}_{\mathrm{dR}}, c)"] & & & \mathrm{SP}\big(\mathrm{MF}(\varphi, N, \pi_1(F_v))\big)
\end{tikzcd}
$$

- may have bad reduction at v: no crystalline central fibre
- need parallel transport purely p-adic analytically

$$T^{\nabla}_{x,x_0} : \mathscr{D}_{\mathrm{dR}}(\mathrm{H}^1_{\text{ét}}(Y_{x,\bar{F}_v}, \mathbb{Q}_p)) \xrightarrow{\sim} \mathscr{D}_{\mathrm{dR}}(\mathrm{H}^1_{\text{ét}}(Y_{x_0,\bar{F}_v}, \mathbb{Q}_p))$$

induced by

$$T^{\nabla}_{x,x_0} : \mathscr{D}_{\mathrm{pst}}(\mathrm{H}^1_{\text{ét}}(Y_{x,\bar{F}_v}, \mathbb{Q}_p)) \xrightarrow{\sim} \mathscr{D}_{\mathrm{pst}}(\mathrm{H}^1_{\text{ét}}(Y_{x_0,\bar{F}_v}, \mathbb{Q}_p))$$

- (relative p-adic Hodge à la Scholze, Shimizu, plus ε)

further issues

$$\begin{tikzcd}
X(F_v) \arrow[d, "x \mapsto (A_x, V_x)"] & \supset & U_{x_0}(F_v) \arrow[r, "\Phi_{x_0}"] & \mathcal{H}_{x_0}(F_v) \arrow[d, "(M^0, M^1(-))"] \\
\mathrm{SP}\big(\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\pi_1(F_v))\big) \arrow[rrr, hook, "\mathscr{D}_{\mathrm{pH}} = (\mathscr{D}_{\mathrm{pst}}, \mathscr{D}_{\mathrm{dR}}, c)"] & & & \mathrm{SP}\big(\mathrm{MF}(\varphi, N, \pi_1(F_v))\big)
\end{tikzcd}$$

- do not have F-rational point in $U_{x_0}(F_v)$: missing center of v-adic period map to be compared with complex period map (minor issue)
- representations V_s not known to be semisimple: here the assumption on no CM-subfield enters as in [LV]

trichotomy

$\Sigma =$ geometric fibre of X'/X over x, set with $\pi_1(F_v)$-action:
$A_x = \prod_{\psi\in\Sigma}\mathbb{Q}_p$ and $V = \bigoplus_{\psi\in\Sigma} V_\psi$ and $G_\psi =$ stabilizer
at least one of the following occurs:

(a) there is a $\psi \in \Sigma$ such that $|\pi_1(F_v).\psi| \geq 4$ and V_ψ has no non-zero isotropic G_ψ-subrepresentation; or

(b) there is a $\psi \in \Sigma$ such that $|\pi_1(F_v).\psi| \geq 4$ and V_ψ has a non-zero isotropic G_ψ-subrepresentation W whose average Hodge–Tate weight is $\geq 1/2$; or

(c) the number of $\psi \in \Sigma$ satisfying $|\pi_1(F_v).\psi| < 4$ is $\geq \frac{1}{d+1}\dim_{\mathbb{Q}_p}(A)$. Here $d = \dim(Y/X')$.