ON CUSPIDAL SECTIONS OF ALGEBRAIC FUNDAMENTAL GROUPS

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Abstract — Rational points in the boundary of a hyperbolic curve over a field with sufficiently nontrivial Kummer theory are the source for an abundance of sections of the fundamental group exact sequence. We follow and refine Nakamura's approach towards these boundary sections. For example, we obtain a weak anabelian theorem for hyperbolic genus 0 curves over quite general fields including for example $\mathbb{Q}^{ab}$.

Contents

1. Introduction

The anabelian geometry of cusps governs the anabelian geometry of affine smooth curves of genus 0. By understanding cuspidal sections of algebraic fundamental groups well enough we extract anabelian results for curves of genus 0 over rather general base fields as an application.

We will work over fields k of characteristic 0 and with full pro-finite fundamental groups only. Tame or logarithmic as well as pro-ℓ versions of the material presented here involves only trivial modifications.

1.1. **Anabelian geometry for curves of genus** 0. The étale fundamental group of a geometrically connected variety U over a field k with algebraic closure k^{alg} sits in an extension, see [4] IX Thm 6.1,

$$1 \to \pi_1(\overline{U}) \to \pi_1(U) \to \mathrm{Gal}(k^{\mathrm{alg}}/k) \to 1, \tag{1.1}$$

where $\overline{U} = U \otimes_k k^{\mathrm{alg}}$, and $\mathrm{Gal}(k^{\mathrm{alg}}/k) = \pi_1(\mathrm{Spec}(k))$ is the absolute Galois group of k that we denote by Gal_k. The group $\pi_1(\overline{U})$ is called the **geometric fundamental group** of U/k. We abbreviate the extension (1.1) by $\pi_1(U/k)$. Due to neglecting base points, the extension $\pi_1(U/k)$ is only functorial in U/k if regarded as an extension of Gal_k with $\pi_1(\overline{U})$-conjugacy classes[1] of maps of extensions.

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[1]By definition, two maps of extensions $f, g : \pi_1(U_1/k) \to \pi_1(U_2/k)$ are $\pi_1(\overline{U}_2)$-conjugate if there is an element $\gamma \in \pi_1(\overline{U}_2)$ such that $f = \gamma(-)\gamma^{-1} \circ g$.

In a sequence of papers [14] [15] [16], Nakamura developed anabelian geometry of hyperbolic (i.e., the Euler characteristic is negative) smooth curves of genus 0 based on a detailed study of cuspidal sections (to be defined below in Section 1.2). Nakamura shows in [15] Theorem 1.1 that, for a finitely generated extension $k/\mathbb{Q}$, two hyperbolic genus 0 curves U/k and V/k are isomorphic over k if and only if the extensions $\pi_1(U/k)$ and $\pi_1(V/k)$ are isomorphic as extensions of Gal_k. The picture was later completed to include also a natural bijection

$$\mathrm{Isom}_k(U,V) = \mathrm{Isom}_{\mathrm{Gal}_k}\big(\pi_1(U/k), \pi_1(V/k)\big)$$

as a consequence of more general results on affine anabelian curves by Tamagawa [24]. Subsequent work by Mochizuki [9] [10] extended the scope further to base fields k contained in a finitely generated extension of a p-adic field $\mathbb{Q}_p$. The author [20] [21] included also base fields of positive characteristic[2], with the case of a finite base field already being covered by Tamagawa [24]. The main result of the present note with respect to anabelian geometry of genus 0 curves extends Nakamura's result to more general base fields.

Theorem A. *Let U and V be hyperbolic curves over k with smooth projective completions of genus 0. If k is algebraic over $\mathbb{Q}$ such that $k\mathbb{Q}^{\mathrm{ab}}$ is finite over $\mathbb{Q}^{\mathrm{ab}}$, then $U \cong V$ as k-curves if and only if there is an isomorphism $\pi_1(U/k) \cong \pi_1(V/k)$ of Gal_k-extensions.*

This is a special case of Theorem 63 proven in Section 9 with Theorem 30 providing the fact that over such base fields k every isomorphism of fundamental group extensions automatically preserves the anabelian weight filtration, see Section 5. The proof benefits from the terminology and structure of the anabelian local cohomology sequence, see Section 6, the notion of orientation, see Section 7, and an anabelian theory of units, see Section 8. Moreover, the proof relies on a precise analysis of the properties of the base field k that allow to characterise cuspidal sections and to define actual parameters in the base field, the double ratios, which serve as moduli parameters to distinguish different genus 0 curves.

Actually, extracting moduli parameters from a given $\pi_1(U/k)$ rather than showing that an isomorphism $\pi_1(U/k) \cong \pi_1(V/k)$ necessarily comes from an isomorphism $U \cong V$ as k-curves is a first step towards a better understanding of anabelian geometry of curves. While the understanding of morphisms shows that the functor π_1 is fully faithful, see [10], we would also like to know an a priori description of the essential image of π_1. The approximation that our approach here has to offer focuses on the moduli parameter values that we can extract from the extension, a priori elements in the pro-$\mathbb{N}$ completion $\widehat{k^*}$, see Definition 2. Obviously, as a necessary condition for an extension to lie in the essential image, these moduli parameters have to lie in the image of $k^* \to \widehat{k^*}$. In Section 10 we show how the j-invariant can be extracted from $\pi_1(E - \{e\}/k)$ where E/k is an elliptic curve with origin $e \in E$.

1.2. **Cuspidal sections.** A k-rational point $u \in U(k)$ yields by functoriality a section

$$s_u = \pi_1(u) \,:\, \mathrm{Gal}_k \to \pi_1(U)$$

of $\pi_1(U/k)$, with image the decomposition group of a point above u. Having neglected base points only the class of a section up to conjugation by elements from $\pi_1(\overline{U})$ is well defined. Let us denote by $\mathscr{S}_{\pi_1(U/k)}$ the set of conjugacy classes of sections of $\pi_1(U/k)$. The **section conjecture** of Grothendieck's anabelian geometry [5] speculates the following.

Section Conjecture (Grothendieck)**.** *The map $U(k) \to \mathscr{S}_{\pi_1(U/k)}$ which sends $u \mapsto s_u$ is bijective if U/k is a geometrically connected, smooth, projective curve of genus ≥ 2 over a finite extension $k/\mathbb{Q}$.*

The section conjecture is wide open in general, see [23] for a survey. The first affirmative examples can be found in [22] §7 and very few others have been found in the meantime. An approach using Tannakian formalism has been advocated by Esnault and Hai [3]. The injectivity part in the section conjecture is a consequence of the Mordell–Weil theorem and was known

[2] The presence of Frobenius requires some care with the correct statement.

to Grothendieck. A stronger pro-p injectivity result was obtained by Mochizuki [10] Theorem 19.1, see [23] §7.2 for a survey.

For affine smooth hyperbolic curves U/k there is an obvious failure of the section conjecture if there are k-rational points in the complement $Y = X \setminus U$ of U in its smooth projective completion X, since these points give rise to cuspidal sections to be defined now.

Let $y \in Y(k^{\mathrm{alg}})$ be a geometric point and $\mathscr{O}^{\mathrm{h}}_{X,y}$ the corresponding henselisation of the local ring. The fundamental group of the connected **scheme of nearby points** $U_y = U \times_X \mathrm{Spec}(\mathscr{O}^{\mathrm{h}}_{X,y})$ at y can be computed as the decomposition subgroup of a place above y and sits in an extension

$$1 \to \hat{\mathbb{Z}}(1) \to \pi_1(U_y) \to \mathrm{Gal}_{\kappa(y)} \to 1,$$

which we abbreviate by $\pi_1(U_y/\kappa(y))$, and where $\kappa(y) \subset k^{\mathrm{alg}}$ is the residue field of the closed point in Y underlying y. Then $U_y \to U$ yields a natural map of extensions

$$\pi_1(U_y/\kappa(y)) \to \pi_1(U/k),$$

the image of which coincides with the extension described by the inertia and decomposition subgroup at y, see Section 3.

Definition 1. A **cuspidal** section of $\pi_1(U/k)$ is a section that factors over $\pi_1(U_y)$ for a point $y \in Y$ with (necessarily) residue field k. The image of

$$\mathscr{S}_{\pi_1(U_y/k)} \to \mathscr{S}_{\pi_1(U/k)}$$

is called the **Paket (engl. packet)** based at the cusp y in [5].

Being cuspidal does not depend on the representative chosen in a given conjugacy class of sections. The correct version of the section conjecture for affine curves asks that all but cuspidal sections come from rational points. The condition on the genus gets replaced by asking the Euler-characteristic to be negative.

Grothendieck in [5] raised the question of how many inequivalent cuspidal sections there are per boundary point, and he also gave the correct answer: uncountably many. This question was first addressed by Esnault and Hai in [3] where Tannakian methods are applied. We give here a fairly elementary treatment[3] based on pro-finite group theory which works over fairly general base fields. The notion of a field with non-trivial Kummer theory naturally occurs, see Section 2. The main results here are Theorem 17 and Theorem 21 in Section 4 that we can summarize as follows.

Theorem B. *Let k be a field with nontrivial Kummer theory, and let U/k be an affine hyperbolic curve with a k-rational cusp y. Then the cardinality of the packet of cuspidal sections based at y is uncountable, more precisely has the same cardinality as $\widehat{k^*}$, if one of the following conditions holds.*

(i) *The injectivity part of the section conjecture holds for proper smooth curves of genus at least 2 over k.*
(ii) *For any abelian variety A/k the group of k-rational torsion points $A(k)_{\mathrm{tors}}$ contains no nontrivial divisible subgroup.*
(iii) *The $\hat{\mathbb{Z}}$-rank of $\widehat{k^*}$ is infinite.*

1.3. **A guide through the paper.** This paper deals with the anabelian geometry of cuspidal sections in the case of curves with an emphasis on imposing only mild assumptions on the base field. There are three main topics with the common thread of a detailed study of the anabelian geometry of cusps.

In Sections 2–4 we deal within the realm of pro-finite group theory with the abundance of cuspidal sections over fairly general base fields. Preliminaries for the discussion are a study of the notion of fields with nontrivial Kummer theory and a study of the local theory of cusps.

[3]The author thanks Tamás Szamuely for inquiring a *pro-finite* presentation.

In Sections 5–8 we pursue a general investigation on additional structure of a fundamental group of a curve defining anabelian versions of a weight filtration[4], an orientation and units. This part is very much influenced by work of Nakamura [14], [15], [16], [17].

As an application of the general investigation we obtain in Sections 9 and 10 anabelian results where the fundamental group determines the isomorphy type of a curve. As a particular feature, our approach, especially in Section 10, directly addresses how the moduli parameters in question are encoded in the arithmetic structure of the fundamental group extension.

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2. Fields with nontrivial Kummer theory

In this section we discuss the notion of a field with nontrivial Kummer theory. That notion naturally appears in the anabelian question about the cardinality of cuspidal packets, to be discussed in Section 4.

Definition 2. The **pro-$\mathbb{N}$ completion** of an abelian group A is

$$\widehat{A} = \varprojlim_{\mathbb{N}} A/nA.$$

By Kummer theory, we can identify $\widehat{k^*} = \mathrm{H}^1(k, \hat{\mathbb{Z}}(1))$ where the cohomology group is continuous cohomology of Gal_k, i.e., cohomology as a pro-finite group with continuous cochains, with values in the Tate module $\hat{\mathbb{Z}}(1) = \varprojlim_n \mu_n$.

Definition 3. We say that a field k has **nontrivial Kummer theory** if the group

$$\widehat{k^*} = \mathrm{H}^1(k, \hat{\mathbb{Z}}(1))$$

is infinite. Otherwise we say that k has **trivial Kummer theory**.

Example 4. (1) Among fields with trivial Kummer theory are real closed fields and the maximal solvable extension k^{solv} of an arbitrary field k.

(2) Finite extensions of $\mathbb{Q}$, local fields and more generally fields k that admit a discrete valuation $\nu : k^* \to \Gamma$ with nondivisible value group Γ have nontrivial Kummer theory. Indeed, the induced map $\hat{\nu} : \widehat{k^*} \to \widehat{\Gamma}$ has infinite image.

The notation for the torsion subgroup of an abelian group A is A_{tors}, while $A[n]$ denotes the n-torsion of A.

Lemma 5. *The group $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}(1))$ vanishes if and only if the map $\widehat{k^*_{\mathrm{tors}}} \to \widehat{k^*}$ is an isomorphism. Both properties hold if the projection $\widehat{k^*} \to k^*/(k^*)^n$ is an isomorphism for some $n \geq 1$.*

Proof: The multiplication by n sequence of $\mathbb{Q}/\mathbb{Z}(1) = \mathbb{G}_{\mathrm{m,tors}}$ and Kummer theory yield an exact sequence

$$1 \to k^*_{\mathrm{tors}}/(k^*_{\mathrm{tors}})^n \to k^*/(k^*)^n \to \mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}(1))[n] \to 1.$$

We conclude that $k^*_{\mathrm{tors}}/(k^*_{\mathrm{tors}})^n = k^*/(k^*)^n$ if and only if $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}(1))$ has no n-torsion. This proves the claimed equivalence.

Let $\widehat{k^*} \to k^*/(k^*)^n$ be an isomorphism. Then for every $a \in k^*$ and $m \in \mathbb{N}$ we have $a^n \in (k^*)^{n \cdot m!}$. Hence there is $\zeta \in \mu_n(k)$, a priori depending on m, such that $a\zeta \in (k^*)^{m!}$. Since $\mu_n(k)$ is finite, one ζ is good for all m, so $a \equiv \zeta^{-1}$ in $\widehat{k^*}$ and the always injective map $\widehat{k^*_{\mathrm{tors}}} \to \widehat{k^*}$ is also surjective. $\square$

Proposition 6. *Let k be a field. Then the following are equivalent.*

(a) *$\widehat{k^*}$ is finite, cyclic and generated by a root of unity,*

[4]The idea of an anabelian weight filtration is due to Nakamura [17] §2.1.

(b) $\widehat{k^*}$ *is countable,*

(c) $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}(1)) = 0$ *and* $k^*_{\mathrm{tors}} = \mathrm{H}^0(k, \mathbb{Q}/\mathbb{Z}(1))$ *is finite modulo its maximal divisible subgroup* $\mathrm{Div}(k^*_{\mathrm{tors}})$.

In particular, if k has nontrivial Kummer theory, then $\widehat{k^*}$ is uncountable.

Proof: Clearly (a) implies (b). If (b) holds, then for some $n \geq 1$ we have $\widehat{k^*} \cong k^*/(k^*)^n$, because otherwise infinitely many of the surjections

$$k^*/(k^*)^{(n+1)!} \twoheadrightarrow k^*/(k^*)^{n!}$$

have nontrivial kernel and the cardinality of $\widehat{k^*}$ is uncountable. From Lemma 5 we deduce that $\mathrm{H}^1(k, \mathbb{Q}/\mathbb{Z}(1)) = 0$ and with the n as above also the maps

$$\widehat{k^*_{\mathrm{tors}}} \twoheadrightarrow k^*_{\mathrm{tors}}/(k^*_{\mathrm{tors}})^n \hookrightarrow k^*/(k^*)^n \cong \widehat{k^*}$$

are isomorphisms. Hence

$$(k^*_{\mathrm{tors}})^n = \bigcap_{d \geq 1} (k^*_{\mathrm{tors}})^{nd} = \mathrm{Div}(k^*_{\mathrm{tors}})$$

and (c) follows, because $k^*_{\mathrm{tors}}/(k^*_{\mathrm{tors}})^n$ is finite.

It remains to show that (c) implies (a). Assuming (c), Lemma 5 shows

$$\widehat{k^*} = \widehat{k^*_{\mathrm{tors}}} = \left(\widehat{\frac{k^*_{\mathrm{tors}}}{\mathrm{Div}(k^*_{\mathrm{tors}})}}\right)$$

is finite and generated by roots of unity, hence cyclic and generated by one root of unity, so (a) holds. □

Corollary 7. *The field k has nontrivial Kummer theory if there is an $a \in k$ and $n \geq 1$ such that $k(\sqrt[n]{a})/k$ is not an abelian field extension.*

Proof: We argue by contradiction and assume $\widehat{k^*}$ is finite. If the group

$$\mathrm{Gal}(\bigcup_{n \geq 1} k(\sqrt[n]{a}, \mu_n)/k)$$

is not abelian, then not all $\sqrt[n]{a}$ are contained in the maximal cyclotomic extension $k(\mu_\infty)$. Thus a is not contained in the kernel of $\widehat{k^*} \to \widehat{k(\mu_\infty)^*}$, which contadicts (a) of Proposition 6. □

Proposition 8. *Let k be a field. The following are equivalent:*

(a) *The extension $k(\mu_{2^\infty})/k$ is infinite or μ_4 is contained in k.*

(b) *For any finite extension E/F of fields which are finite algebraic over k the natural map $\widehat{F^*} \to \widehat{E^*}$ is injective.*

Proof: In (b) we may assume that E/F is Galois. The inflation-restriction sequence then reads

$$0 \to \mathrm{H}^1\left(E/F, \mathrm{H}^0(E, \hat{\mathbb{Z}}(1))\right) \to \widehat{F^*} \to \widehat{E^*}.$$

We may restrict to the p-part. By the usual corestriction argument with the p-Sylow subgroup of $\mathrm{Gal}(E/F)$ we may further assume that $\mathrm{Gal}(E/F)$ is a p-group. By dévissage, since p-groups are nilpotent, we see that in (b) it is enough to ask $\mathrm{H}^1\left(E/F, \mathrm{H}^0(E, \mathbb{Z}_p(1))\right) = 0$ for all p-cyclic Galois extensions E/F.

The coefficients $\mathrm{H}^0\left(E, \mathbb{Z}_p(1)\right)$ either vanishes or μ_{p^∞} is contained in E. For p odd, either way the action of $\mathrm{Gal}(E/F)$ must be trivial as $\mathbb{Z}_p^*$ contains no p-torsion. Hence in this case

$$\mathrm{H}^1\left(E/F, \mathrm{H}^0(E, \mathbb{Z}_p(1))\right) = \mathrm{Hom}\left(\mathrm{Gal}(E/F), \mathrm{H}^0(E, \mathbb{Z}_p(1))\right) = 0.$$

For $p = 2$ we may argue analogously except in the case that $\mathrm{Gal}(E/F)$ acts nontrivially via complex conjugation, which means $\mu_4 \not\subset F$ and $\mu_{2^\infty} \subset E$. Therefore (a) implies (b).

But if (a) fails, then there is a quadratic extension E/F finite over k with $\mu_{2^\infty} \subset E$ and $E = F(i)$ such that

$$\ker\left(\widehat{F^*} \to \widehat{E^*}\right) = \mathrm{H}^1(\mathbb{Z}/2\mathbb{Z}, \hat{\mathbb{Z}}(1)) = \ker\left(\widehat{\mathbb{R}^*} \to \widehat{\mathbb{C}^*}\right) = \{\pm 1\}$$

contradicting (b). □

Corollary 9. *Let E/F be a finite extension. The map $\widehat{F^*} \to \widehat{E^*}$ is either injective or has kernel of order 2 generated by the class of -1.*

Proof: By Proposition 8 the kernel is contained in the corresponding kernel for $E = F(i)$, which at most equals the kernel for the extension $\mathbb{C}/\mathbb{R}$ as shown in the proof above. □

3. Cusps and inertia subgroups

In this section we introduce the notion of cusps of affine smooth curves and study the group theory of the inertia subgroup that a cusp gives rise to. The results are well known and included for convenience of the reader.

From now on U will be the complement of a reduced divisor Y in a geometrically connected, smooth, projective curve X/k. The Euler-characteristic is

$$\chi(U) = 2 - 2g - \deg(Y),$$

where g is the genus of X. We say that U is **hyperbolic** if $\chi(U) < 0$. In that case it turns out that inertia groups at different cusps are in some sense independent. The results can be viewed as analogues for the unramified quotient $\pi_1(U)$ of old results of F. K. Schmidt [19] for the decomposition groups of places.

3.1. **The universal pro-étale cover.** Recall that a **universal pro-étale cover** $\tilde{U} \to U$ of U is a pro-system of connected finite étale covers $U_i \to U$ such that for any finite étale cover $V \to U$ there is a map $U_i \to V$ above U for i sufficiently large. For a geometric point $\bar{u} \in U$, a choice of point $\xi \in \tilde{U}$ above $\bar{u}$ consists of a compatible choice of points $\xi_i \in U_i$ above $\bar{u}$. The pair $(\tilde{U}, \xi)$ then pro-represents the functor $F_{\bar{u}} =$ (fibre in $\bar{u}$) from the category of finite étale covers of U with values in finite sets by

$$\varinjlim_i \mathrm{Hom}(U_i, V) = F_{\bar{u}}(V)$$

$$\big(f : U_i \to V\big) \quad \mapsto \quad f(\xi_i),$$

see [4] exposé V. In particular, the fundamental group of U with base point $\bar{u}$, i.e.,

$$\pi_1(U, \bar{u}) = \mathrm{Aut}(F_{\bar{u}})$$

can be identified by the choice of ξ and the Yoneda embedding with the opposite group of $\mathrm{Aut}(\tilde{U}/U)$. Since $\overline{U} \to U$ is a pro-étale cover, we can lift $\tilde{U} \to U$ to a map $\tilde{U} \to \overline{U}$. Thus we fix a k^{alg}-structure on $\tilde{U}$.

3.2. **Cusps.** We now fix a universal pro-étale cover $\tilde{U} \to U$ and a lift $\tilde{U} \to \overline{U}$ together with an isomorphism of Gal_k-extensions of $\pi_1(U/k)$ with the opposite groups of

$$1 \to \mathrm{Aut}(\tilde{U}/\overline{U}) \to \mathrm{Aut}(\tilde{U}/U) \xrightarrow{\mathrm{pr}} \mathrm{Aut}(\mathrm{Spec}(k^{\mathrm{alg}})/\mathrm{Spec}(k)) \to 1.$$

Note that here $\mathrm{Aut}(-/-)$ means by definition automorphisms of pro-objects. But in fact, due to these pro-covers having a cofinal family of finite Galois covers, an element of $\mathrm{Aut}(-/-)$ is nothing but a compatible system of automorphisms for these intermediate finite Galois covers.

Let $\tilde{X}$ be the normalization of X in $\tilde{U} \to U$, i.e., $\tilde{X}$ is the pro-system (X_i) of normalizations X_i of X in $U_i \to U$. Let $\tilde{Y}$ be the reduced preimage of Y under $\tilde{X} \to X$, i.e., the pro-system of the $Y_i = X_i \setminus U_i$. The k^{alg}-structure on $\tilde{U}$ induces a k^{alg}-structure on $\tilde{X}$ and on $\tilde{Y}$. As normalization is unique we obtain an inclusion

$$\mathrm{Aut}(\tilde{U}/U) \subseteq \mathrm{Aut}(\tilde{X}/X)$$

by which we can let $\mathrm{Aut}(\tilde{U}/U)$ act on $\tilde{X}$ and in particular $\tilde{Y}$.

Definition 10. The set of **cusps** of U is

$$\mathrm{Cusps}(U) = \mathrm{Hom}_k(\mathrm{Spec}(k^{\mathrm{alg}}), Y)$$

as a set with Gal_k action. The set of **prolongations of cusps** to $\tilde{U}$ is the pro-finite set

$$\widetilde{\mathrm{Cusps}}(U) = \mathrm{Hom}_{k^{\mathrm{alg}}}(\mathrm{Spec}(k^{\mathrm{alg}}), \tilde{Y}),$$

with the continuous $\pi_1(U)$ action by $\gamma.\tilde{y} = \gamma^{-1} \circ \tilde{y} \circ \mathrm{pr}(\gamma)$ for $\gamma \in \pi_1(U)$ and $\tilde{y} \in \widetilde{\mathrm{Cusps}}(U)$.

The projection map $\widetilde{\mathrm{Cusps}}(U) \twoheadrightarrow \mathrm{Cusps}(U), \tilde{y} \mapsto y$ is equivariant and the quotient map for the induced $\pi_1(\overline{U})$ action.

3.3. **Inertia subgroups at cusps.** The **inertia group** $\mathrm{I}_{\tilde{y}/y}$ (resp. the **decomposition group** $\mathrm{D}_{\tilde{y}/y}$) of a cusp $\tilde{y} \in \widetilde{\mathrm{Cusps}}(U)$ is the stabilizer under the action of $\pi_1(\overline{U})$ (resp. $\pi_1(U)$). Hence for $\gamma \in \pi_1(U)$ we have $\mathrm{I}_{\gamma\tilde{y}/\gamma y} = \gamma\, \mathrm{I}_{\tilde{y}/y}\, \gamma^{-1}$ and $\mathrm{D}_{\gamma\tilde{y}/\gamma y} = \gamma\, \mathrm{D}_{\tilde{y}/y}\, \gamma^{-1}$. The choice of $\tilde{y} \in \widetilde{\mathrm{Cusps}}(U)$ determines a representative for the map of extensions $\pi_1(U_y/\kappa(y)) \to \pi_1(U/k)$ with image the extension $\mathscr{D}_{\tilde{y}/y}$ given by

$$1 \to \mathrm{I}_{\tilde{y}/y} \to \mathrm{D}_{\tilde{y}/y} \to \mathrm{Gal}_{\kappa(y)} \to 1.$$

Proposition 11. *The extension $\pi_1(U_y/\kappa(y))$ splits and there is a natural free transitive action by $\widehat{\kappa(y)^*}$ on $\mathscr{S}_{\pi_1(U_y/\kappa(y))}$.*

Proof: The roots $(t^{1/n})_{n\in\mathbb{N}}$ of a fixed parameter t at y define a tower of finite étale covers of U_y. The intersection of the corresponding family of open subgroups of $\pi_1(U_y)$ defines a splitting. The free transitive action of $\mathrm{H}^1(\kappa(y), \hat{\mathbb{Z}}(1)) = \widehat{\kappa(y)^*}$ on the set $\mathscr{S}_{\pi_1(U_y/\kappa(y))}$ is then standard, see for example [23] §1.2 Proposition 8. □

3.4. **Unramified local theory.** The maximal abelian quotient of $\pi_1(-)$ will be denoted by $\pi_1^{\mathrm{ab}}(-)$. From the topological computation of fundamental groups via GAGA we get the following exact sequence of Gal_k modules

$$\hat{\mathbb{Z}}(1) \xrightarrow{\Delta} \hat{\mathbb{Z}}(1) \otimes \mathbb{Z}[\mathrm{Cusps}(U)] \to \pi_1^{\mathrm{ab}}(\overline{U}) \to \pi_1^{\mathrm{ab}}(\overline{X}) \to 0, \tag{3.1}$$

where moreover $\hat{\mathbb{Z}}(1) \cdot y$ for a cusp $y \in \mathrm{Cusps}(U)$ maps to the image of $\mathrm{I}_{\tilde{y}/y}$ in $\pi_1^{\mathrm{ab}}(\overline{U})$ regardless of the prolongation $\tilde{y}$ of y, and Δ is the diagonal map. The group $\pi_1^{\mathrm{ab}}(\overline{X})$ is a free $\hat{\mathbb{Z}}$ module of rank $2g$.

Definition 12. A **neighbourhood** of a section $s \in \mathscr{S}_{\pi_1(X/k)}$ is a finite étale map $f : X' \to X$ with X'/k geometrically connected and a section $s' \in \mathscr{S}_{\pi_1(X'/k)}$ with $\pi_1(f) \circ s' = s$. Equivalently, a neighbourhood is an open subgroup H of $\pi_1(X)$ together with a representative of s with $s(\mathrm{Gal}_k) \subset H$ up to conjugation by $H \cap \pi_1(\overline{X})$, see [23] §4.2.

Lemma 13. *If $\chi(U) < 0$ and U is affine, then $\#\,\mathrm{Cusps}(V) \to \infty$ as V ranges over connected finite étale covers of U that, moreover, we may choose among the neighbourhoods of a section $s \in \mathscr{S}_{\pi_1(U/k)}$.*

Proof: By (3.1) the group $\mathrm{I}_{\tilde{y}/y}$ has infinite index in $\pi_1(\overline{U})$. □

Lemma 14. *If $\chi(U) \le 0$ then $\pi_1(U_y/\kappa(y)) \cong \mathscr{D}_{\tilde{y}/y}$ injects into $\pi_1(U/k)$ for each cusp y of U.*

Proof: The kernel of $\pi_1(U_y) \to \pi_1(U)$ is contained in the inertia subgroup $\hat{\mathbb{Z}}(1) \subset \mathscr{D}_{\tilde{y}/y}$. As $\hat{\mathbb{Z}}(1)$ is torsion free we may replace U by a finite étale cover and thus may assume $\#\,\mathrm{Cusps}(U) \ge 2$ by Lemma 13 in case $\chi(U) < 0$ and U is affine. Of course, if U is proper there is nothing to do and in the remaining case of $\chi(U) = 0$ we have U is a form of $\mathbb{G}_m$ and $\#\,\mathrm{Cusps}(U) = 2$ anyway. With at least two cusps the lemma follows at once from (3.1), because the inertia group then even injects into $\pi_1^{\mathrm{ab}}(\overline{U})$. □

Corollary 15. *If $\chi(U) \le 0$, then for any $\tilde{y} \in \widetilde{\mathrm{Cusps}}(U)$ the inertia group $\mathrm{I}_{\tilde{y}/y}$ is free pro-cyclic.*

Proof: This follows at once from the isomorphism $\hat{\mathbb{Z}}(1) \xrightarrow{\sim} \mathrm{I}_{\tilde{y}/y}$ implied by Lemma 14. $\square$

Proposition 16 (Local theory)**.** *Let $\chi(U) < 0$ be negative.*

(1) *Let $\tilde{y}_1, \tilde{y}_2 \in \widetilde{\mathrm{Cusps}}(U)$ be prolongations of cusps with nontrivial intersection $\mathrm{I}_{\tilde{y}_1/y_1} \cap \mathrm{I}_{\tilde{y}_2/y_2} \neq \mathbf{1}$. Then already $\tilde{y}_1 = \tilde{y}_2$.*
(2) *For any cusp y, the normalizer of $\mathrm{I}_{\tilde{y}/y}$ in $\pi_1(\overline{U})$ is $\mathrm{I}_{\tilde{y}/y}$, and the normalizer of $\mathrm{I}_{\tilde{y}/y}$ in $\pi_1(U)$ is $\mathrm{D}_{\tilde{y}/y}$.*
(3) *If an abelian subgroup $A \subset \pi_1(\overline{U})$ intersects $\mathrm{I}_{\tilde{y}/y}$ nontrivially for a cusp y, then A is contained in $\mathrm{I}_{\tilde{y}/y}$.*

Proof: (1) Since $\pi_1(\overline{U})$ is torsion free, see [24] Prop. 1.6, the intersection is either trivial or infinite. We argue by contradiction and may replace U by a finite étale cover to assume $y_1 \neq y_2$ and $\#\,\mathrm{Cusps}(U) \geq 3$. Then $\mathrm{I}_{\tilde{y}_1/y_1}$ and $\mathrm{I}_{\tilde{y}_2/y_2}$ intersect trivially in $\pi_1^{\mathrm{ab}}(\overline{U})$ by (3.1).

(2) If $\gamma \in \pi_1(U)$ normalises $\mathrm{I}_{\tilde{y}/y}$, then $\mathrm{I}_{\tilde{y}/y} \cap \mathrm{I}_{\gamma.\tilde{y}/\gamma.y} \neq \mathbf{1}$, hence $\gamma.\tilde{y} = \tilde{y}$ by (1) and thus $\gamma \in \mathrm{D}_{\tilde{y}/y}$. Note that $\mathrm{I}_{\tilde{y}/y} = \mathrm{D}_{\tilde{y}/y} \cap \pi_1(\overline{U})$.

(3) Let γ be a nontrivial element of $A \cap \mathrm{I}_{\tilde{y}/y}$ and $a \in A$ arbitrary. Then the intersection of $\mathrm{I}_{\tilde{y}/y}$ with $a\,\mathrm{I}_{\tilde{y}/y}\,a^{-1} = \mathrm{I}_{a.\tilde{y}/a.y}$ contains γ, hence $a.\tilde{y} = \tilde{y}$ by (1) and $a \in \mathrm{I}_{\tilde{y}/y}$. $\square$

4. Abundance of cuspidal sections

We give two proofs that quite generally the packets of cuspidal section are uncountable, see also Esnault and Hai in [3] Cor 6.9.

4.1. **The anabelian proof.**
We first present an anabelian proof that relies on an anabelian assumption.

Theorem 17. *Let us assume that the injectivity part of the section conjecture holds for proper curves of genus at least 2 over finite extensions of k. Then the following holds.*

(1) *A cuspidal section uniquely determines the cusp y for which it factors over $\pi_1(U_y)$.*
(2) *The natural map $\mathscr{S}_{\pi_1(U_y/k)} \to \mathscr{S}_{\pi_1(U/k)}$ for a k-rational cusp y of U is injective.*
(3) *The set of cuspidal sections is disjoint from the sections induced by rational points $u \in U(k)$.*

Proof: Let s be a section. Replacing U by a neighbourhood of s we may assume that X has genus ≥ 2.

(1) If the image of s is contained in $\mathrm{D}_{\tilde{y}_1/y_1}$ and $\mathrm{D}_{\tilde{y}_2/y_2}$ then s maps simultaneously to s_{y_1} and s_{y_2} under $\mathscr{S}_{\pi_1(U/k)} \to \mathscr{S}_{\pi_1(X/k)}$, hence $y_1 = y_2$ by assumption. The same argument proves (3).

(2) Let $s, t \in \mathscr{S}_{\pi_1(U_y/k)}$ become conjugate as sections of $\pi_1(U/k)$. Let $\gamma \in \pi_1(\overline{U})$ be such that $s = \gamma()\gamma^{-1} \circ t$. Then for suitable $\tilde{y}$ the image $s(\mathrm{Gal}_k)$ is contained in $\mathrm{D}_{\tilde{y}/y}$ and $\gamma(\mathrm{D}_{\tilde{y}/y})\gamma^{-1} = \mathrm{D}_{\gamma.\tilde{y}/y}$. Applying the same reasoning as in (1) to all finite étale covers of U that are neighbourhoods of s we also deduce $\gamma.\tilde{y} = \tilde{y}$. Hence $\gamma \in \mathrm{I}_{\tilde{y}/y}$ and so s and t are already conjugate as sections of $\pi_1(U_y/k)$. $\square$

Remark 18. (1) Let k be a field such that for any abelian variety A/k the group of k-rational poins $A(k)$ has no divisible elements. Then the assumption in Theorem 17 holds, see [23] §7.1 Prop 73.

(2) Part (2) of Theorem 17 actually follows from the weaker assumption that for all smooth projective hyperbolic curves X' over finite extensions k'/k and all $a \in X'(k')$ the centraliser of $s_a(\mathrm{Gal}_k)$ in $\pi_1(X')$ intersects only trivially with $\pi_1(\overline{X'})$. Namely, if $s(\mathrm{Gal}_k)$ is contained in both $\mathrm{D}_{\tilde{y}/y}$ and $\mathrm{D}_{\gamma.\tilde{y}/y}$ but $\tilde{y} \neq \gamma.\tilde{y}$, then there is a neighbourhood $U' \to U$ of s such that the image y' of $\tilde{y}$ is different from the image y'' of $\gamma.\tilde{y}$. After base change by a sufficiently ramified neighbourhood $V \to U$ of s, which in particular is (after possibly a finite extension k'/k) totally ramified above y, we may assume by Abhyankar's Lemma that $U' \times_U V \to V$ extends to a finite étale cover of the respective smooth completions. Replacing U by V, we may thus assume that $U' \to U$ extends to a finite étale cover $X' \to X$ and that s_y lifts simultaneously to $s_{y'}$ and $s_{y''}$

for $y' \neq y''$. This contradicts [23] §3.3 Prop 28 in light of our assumption on the centralizer of $s(\mathrm{Gal}_k)$.

(3) For $U = \mathbb{G}_m$ a cuspidal section fails to determine its cusp. In fact, all sections are cuspidal and belong to both packets, at 0 and ∞.

Corollary 19. *If the injectivity part of the section conjecture holds for proper curves of genus ≥ 2 over k and if k has a nontrivial Kummer theory then each k-rational cusp gives rise to an uncountable set of cuspidal sections.*

Proof: Immediately from Theorem 17 and Proposition 11. Indeed it suffices to assume injectivity of the section conjecture for curves over k, since we do apply this to neighbourhoods of the given cuspidal sections, and these are curves geometrically connected over k again. $\square$

Remark 20. As in Theorem 17 we can replace the assumption on the injectivity part of the section conjecture by the assumption on the centralizer of the image of geometric sections as in Remark 18 above.

4.2. **The abelian proof.** The abelianised fundamental group extension $\pi_1^{\mathrm{ab}}(U/k)$ is the push-out of the extension $\pi_1(U/k)$ by the quotient $\pi_1(\overline{U}) \twoheadrightarrow \pi_1^{\mathrm{ab}}(\overline{U})$.

Theorem 21. *Let k be a field with nontrivial Kummer theory such that one of the following two assumptions hold.*

(i) *For any abelian variety A/k the group of k-rational torsion points $A(k)_{\mathrm{tors}}$ contains no nontrivial divisible element.*
(ii) *The $\hat{\mathbb{Z}}$ rank of $\widehat{k^*}$ is infinite.*

The image of $\mathscr{S}_{\pi_1(U_y/k)} \to \mathscr{S}_{\pi_1(U/k)}$ is uncountable for every k-rational cusp y.

Remark 22. Assumption (ii) holds for $k = \mathbb{Q}^{\mathrm{ab}}$. For each prime p we have a valuation ν_p on $\mathbb{Q}^{\mathrm{ab}}$ with values in $1/(p-1)\mathbb{Z}[1/p]$. For different p these are mutually independent, whence for finite sets S of primes a surjective map

$$\widehat{\mathbb{Q}^{\mathrm{ab},*}} \to \prod_{p \in S} 1/(p\widehat{-1)\mathbb{Z}}[1/p].$$

Choosing a prime $\ell \notin S$ we conclude that the $\mathbb{Z}_\ell$ rank of $\widehat{\mathbb{Q}^{\mathrm{ab},*}} \otimes \mathbb{Z}_\ell$ is at least $\#S$, and (ii) follows.

Proof: Replacing U by a neighbourhood of a cuspidal section at y we may assume by Lemma 13 that U has at least 2 cusps. It suffices to prove that $\mathscr{S}_{\pi_1(U_y/k)} \to \mathscr{S}_{\pi_1^{\mathrm{ab}}(U/k)}$ has uncountable image. By the well-known description of the set of conjugacy classes of sections in the abelian case with an H^1-group, see for example [23] §1.2 Proposition 8, this translates into showing that the natural map

$$\widehat{k^*} = \mathrm{H}^1(k, I_{\tilde{y}/y}) \to \mathrm{H}^1(k, \pi_1^{\mathrm{ab}}(\overline{U}))$$

has uncountable image.

The k-rational cusp y can be used to split the diagonal map Δ in (3.1), so that Galois cohomology and the Shapiro lemma yields

$$\mathrm{H}^1(k, \hat{\mathbb{Z}}(1) \otimes \mathbb{Z}[\mathrm{Cusps}(U)]/\hat{\mathbb{Z}}(1)) = \mathrm{H}^1(k, \hat{\mathbb{Z}}(1) \otimes \mathbb{Z}[\mathrm{Cusps}(U)])/\,\mathrm{H}^1(k, \hat{\mathbb{Z}}(1))$$
$$= \left(\bigoplus_z \mathrm{H}^1(\kappa(z), \hat{\mathbb{Z}}(1))\right) /\, \mathrm{H}^1(k, \hat{\mathbb{Z}}(1)) = \big(\bigoplus_z \widehat{\kappa(z)^*}\big)/\widehat{k^*}$$

where z runs over a set of representatives of Gal_k orbits on $\mathrm{Cusps}(U)$. Now the exact sequence

$$0 \to \hat{\mathbb{Z}}(1) \otimes \mathbb{Z}[\mathrm{Cusps}(U)]/\hat{\mathbb{Z}}(1) \to \pi_1^{\mathrm{ab}}(\overline{U}) \to \pi_1^{\mathrm{ab}}(\overline{X}) \to 0$$

derived from (3.1), yields an exact sequence

$$\mathrm{H}^0(k, \pi_1^{\mathrm{ab}}(\overline{X})) \xrightarrow{\delta} \big(\bigoplus_z \widehat{\kappa(z)^*}\big)/\widehat{k^*} \to \mathrm{H}^1(k, \pi_1^{\mathrm{ab}}(\overline{U})).$$

By Corollary 9, the kernel of the map $\widehat{k^*} \to \big(\bigoplus_z \widehat{\kappa(z)^*}\big)/\widehat{k^*}$ onto the component of the cusp y is at most of order 2. The result follows because in case (i) the map δ is trivial (note that $\pi_1^{\mathrm{ab}}(\overline{X})$ is isomorphic to the Tate-module of the Jacobian of X) and in case (ii) it has image of finite rank, whereas $\widehat{k^*}$ has infinite rank. $\square$

Remark 23. Let y be a k-rational cusp of U. For a unit $f \in \Gamma(U, \mathbb{G}_m)$ with order of vanishing $n = v_y(f)$ in y the composite map

$$\pi_1(U_y/k) \to \pi_1(U/k) \xrightarrow{\pi_1(f)} \pi_1(\mathbb{G}_m/k)$$

induces the n^{th} power map

$$\widehat{k^*} = \mathscr{S}_{\pi_1(U_y/k)} \to \mathscr{S}_{\pi_1(\mathbb{G}_m/k)} = \widehat{k^*},$$

the kernel of which is generated by $\mu_n(k)$. It follows that, if there is a finite étale cover $U' \to U$, a cusp y' of U' above y and an $f \in \Gamma(U', \mathbb{G}_m)$ with $v_{y'}(f) \neq 0$, then the cuspidal packet at y is uncountable if k is a field with nontrivial Kummer theory. However, it seems to be an open question, whether a U' with such an f always exists, at least in the case when k is a number field, but see [13] Cor 5.7 for a negative example if U/k is a generic curve.

5. The anabelian weight filtration following Nakamura

We recall the theory of the anabelian weight filtration after Nakamura [15] §3, [17] §2.1. As before, U will be the complement of a reduced divisor Y in a geometrically connected, smooth, projective curve X/k.

Definition 24. The **anabelian weight filtration** of U is the subset $W_{-2}(U) \subset \pi_1(\overline{U})$ of all inertia elements $W_{-2}(U) = \bigcup \mathrm{I}_{\tilde{y}/y}$ where the union ranges over all $\tilde{y} \in \widetilde{\mathrm{Cusps}}(U)$.

The set $W_{-2}(U)$ is pro-finite, hence compact, and preserved under conjugation by $\pi_1(U)$. Strictly speaking, the terminology *anabelian* is only justified when $W_{-2}(U)$ can be described in terms of $\pi_1(U/k)$ alone.

Definition 25. A **weight preserving** map is a continuous group homomorphism $\varphi : \pi_1(U/k) \to \pi_1(V/k)$ of fundamental groups of hyperbolic curves such that $\varphi(W_{-2}(U)) \subseteq W_{-2}(V)$. This does not exclude that an inertia group of $\pi_1(U)$ may lie in the kernel of φ.

If $\varphi : \pi_1(U/k) \to \pi_1(V/k)$ is weight preserving, then Proposition 16 (3) implies that φ induces a partially defined π_1-equivariant map $\widetilde{\mathrm{Cusps}}(U) \to \widetilde{\mathrm{Cusps}}(V)$, which is defined precisely on those cusps $\tilde{y}/y$ for which $\varphi(\mathrm{I}_{\tilde{y}/y}) \neq \mathbf{1}$.

5.1. When the cyclotomic character is non-Tate.

Definition 26. Let ℓ be a prime number and let k be a field. Following [1] §3, we say that the ℓ-adic cyclotomic character $\mathrm{Gal}_k \to \mathbb{Z}_\ell^*$ is a **non-Tate character** if any Gal_k-map $\mathbb{Z}_\ell(1) \to \mathrm{T}_\ell A$ to the ℓ-adic Tate module of an abelian variety A/k vanishes.

Due to the theory of weights, for a finite extension $k/\mathbb{Q}$ and any prime number ℓ, the ℓ-adic cyclotomic character is non-Tate. This can be improved to the following criterion.

Lemma 27. *Let ℓ be a prime number and let $K_\ell/\mathbb{Q}^{\mathrm{ab}}$ be a maximal prime to ℓ extension, i.e., the fixed field of an ℓ-Sylow group of $\mathrm{Gal}_{\mathbb{Q}^{\mathrm{ab}}}$. Then, for an algebraic extension $k/\mathbb{Q}$ such that kK_ℓ/K_ℓ is finite, the ℓ-adic cyclotomic character is non-Tate.*

Proof: We argue by contradiction. Let $k_0 \subset k$ be a subfield that is finite over $\mathbb{Q}$, and let A/k_0 be an abelian variety such that there is a nontrivial Gal_k-map $\psi : \mathbb{Z}_\ell(1) \to \mathrm{T}_\ell A$. Since $\mathrm{GL}(\mathrm{T}_\ell A)$ has a pro-ℓ subgroup of finite index, we may assume by passing to a finite extension of k and k_0 that the action of Gal_{k_0} on $\mathrm{T}_\ell A$ is via a pro-ℓ group. Since kK_ℓ/K_ℓ is finite, a further finite extension of k_0 and k allows to assume that $k_0K_\ell = kK_\ell$ and moreover that Gal_{kK_ℓ} is an ℓ-Sylow subgroup of $\mathrm{Gal}_{k_0\mathbb{Q}^{\mathrm{ab}}}$. By pro-finite Sylow theory, we have then that

$$\mathrm{im}\big(\mathrm{Gal}_{\mathbb{Q}^{\mathrm{ab}}k_0} \to \mathrm{GL}(\mathrm{T}_\ell\, A)\big) = \mathrm{im}\big(\mathrm{Gal}_{kK_\ell} \to \mathrm{GL}(\mathrm{T}_\ell\, A)\big).$$

It follows that

$$\mathbb{Z}_\ell \cong \mathrm{H}^0(\mathbb{Q}^{\mathrm{ab}}k_0, \mathrm{im}(\psi)) \subseteq \mathrm{H}^0(\mathbb{Q}^{\mathrm{ab}}k_0, \mathrm{T}_\ell\, A) = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A(\mathbb{Q}^{\mathrm{ab}}k_0)),$$

which provides a contradiction, since $A(\mathbb{Q}^{\mathrm{ab}}k_0)$ is finite by Theorem 1 of Ribet's appendix to [6], and thus $\mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A(\mathbb{Q}^{\mathrm{ab}}k_0)) = 0$. $\square$

The following is a special case of [1] Lemma 3.1.

Lemma 28. *Let ℓ be a prime number and let k'/k be a finite field extension. Then the ℓ-adic cyclotomic character is non-Tate for k if and only if it is non-Tate for k'.*

Proof: If there is an abelian variety A'/k' and a nontrivial $\mathrm{Gal}_{k'}$-map $\mathbb{Z}_\ell(1) \to \mathrm{T}_\ell\, A'$, then with the Weil-restriction of scalars $A = R_{k'/k}(A')$ we find a nontrivial Gal_k-map $\mathbb{Z}_\ell(1) \to \mathrm{T}_\ell\, A = \mathrm{ind}_{k'/k}(\mathrm{T}_\ell\, A')$ by Frobenius reciprocity. The other direction is even more obvious. $\square$

5.2. **Characterisation of inertia subgroups after Nakamura.** Local theory in anabelian geometry aims to show that any group homomorphism of Gal_k-extensions automatically preserves weights. For hyperbolic curves over fields such that the ℓ-adic cyclotomic character is non-Tate for some prime number ℓ, this was achieved by Nakamura as we will recall now.

Definition 29. A pro-cyclic closed subgroup $I \subset \pi_1(\overline{U})$ is **essentially cyclotomically normalized** if there is a subgroup N of the normalizer $N_{\pi_1(U)}(I)$ of I in $\pi_1(U)$ which projects to an open subgroup in Gal_k and the induced action of N on I is via the cyclotomic character. The pro-cyclic closed subgroup $I \subset \pi_1(\overline{U})$ is **cyclotomically normalized** if it is essentially cyclotomically normalized with $N = N_{\pi_1(U)}(I)$.

Theorem 30 (Nakamura)**.** *Let k be a field and let U/k be a hyperbolic curve. Let ℓ be a prime number such that the ℓ-adic cyclotomic character is non-Tate. Then the following are equivalent for a nontrivial closed pro-ℓ subgroup I in $\pi_1(\overline{U})$:*

(a) I *is contained in an inertia subgroup* $\mathrm{I}_{\tilde{y}/y}$,
(b) $\mathrm{I} \cong \mathbb{Z}_\ell$ *is cyclotomically normalized,*
(c) $\mathrm{I} \cong \mathbb{Z}_\ell$ *is essentially cyclotomically normalized.*

Proof: We assume (a). As $\mathrm{I}_{\tilde{y}/y}$ is pro-cyclic, we find $\mathrm{I} \cong \mathbb{Z}_\ell$. If $\gamma \in \pi_1(U)$ normalizes I then $\mathrm{I}_{\gamma.\tilde{y}/\gamma.y} = \gamma\, \mathrm{I}_{\tilde{y}/y}\, \gamma^{-1}$ intersects $\mathrm{I}_{\tilde{y}/y}$ nontrivially in I. Hence $\gamma.\tilde{y} = \tilde{y}$ and $\gamma \in \mathrm{D}_{\tilde{y}/y} = N_{\pi_1(U)}(\mathrm{I}_{\tilde{y}/y})$. As $\mathrm{I}_{\tilde{y}/y}$ is cyclotomically normalized the same follows for I and (b) follows.

That (b) implies (c) is trivial. It remains to deduce (a) from (c). We argue by contradiction. Then, by Proposition 16 (3), we have trivial intersection $I \cap W_{-2}(U) = 1$, and can choose a compact set $\emptyset \neq C \subset \mathrm{I}$ avoiding 1. The Hausdorff property yields an open normal subgroup N of $\pi_1(\overline{U})$ such that CN/N has empty intersection with $W_{-2}(U)N/N$ in $\pi_1(\overline{U})/N$.

The group generated by $W_{-2}(U) \cap N\,\mathrm{I}$ in $N\,\mathrm{I}/N$ is cyclic of order a power of ℓ hence the image of some $\mathrm{I}_{\tilde{y}/y} \cap N\,\mathrm{I}$ which therefore avoids CN/N. Thus the intermediate ℓ-cyclic cover $U'' \to U'$ corresponding to $N(\mathrm{I})^\ell \subset N\,\mathrm{I}$ is unramified over the corresponding cusps $X' - U'$. For some finite field extension we obtain a nontrivial Galois invariant map from $\mathrm{I} \cong \mathbb{Z}_\ell(1) \to \mathrm{T}_\ell\, A'$ where A' is the Albanese variety of X', a contradiction to Lemma 28. $\square$

We deduce an anabelian description of inertia subgroups, and hence the set $\widetilde{\mathrm{Cusps}}(U)$ and the anabelian weight filtration $W_{-2}(U)$.

Corollary 31. *Let k be a field such that the ℓ-adic cyclotomic character is non-Tate for a prime number ℓ, and let U/k be a hyperbolic curve. Then the set of inertia subgroups in $\pi_1(\overline{U})$ coincides with the set of maximal closed subgroups of $\pi_1(\overline{U})$ among those which are pro-cyclic with nontrivial ℓ-part and which are cyclotomically normalized.*

Proof: An inertia subgroup $\mathrm{I}_{\tilde{y}/y}$ is pro-cyclic, has nontrivial ℓ-part, and is cyclotomically normalized. Moreover, an inertia subgroup is a maximal such subgroup as otherwise $\mathrm{I}_{\tilde{y}/y}$ were properly contained in an abelian subgroup contradicting Proposition 16 (3).

Conversely, let I be a maximal pro-cyclic and cyclotomically normalized subgroup with non-trivial ℓ-part. Then, by Theorem 30, there is some $\mathrm{I}_{\tilde{y}/y}$ which contains $\mathrm{I} \otimes \mathbb{Z}_\ell$. Proposition 16 (3) shows that $\mathrm{I} \subseteq \mathrm{I}_{\tilde{y}/y}$ and by maximality in fact $\mathrm{I} = \mathrm{I}_{\tilde{y}/y}$. $\square$

5.3. Characterisation of cuspidal sections after Nakamura.

Corollary 32. *Let k be a field such that the ℓ-adic cyclotomic character is non-Tate for a prime number ℓ. A section $s \in \mathscr{S}_{\pi_1(U/k)}$ for a hyperbolic curve U/k is cuspidal if and only if the image $s(\mathrm{Gal}_k)$ cyclotomically normalizes a subgroup of $\pi_1(\overline{U})$ isomorphic to $\mathbb{Z}_\ell(1)$.*

Proof: A cuspidal section at the cusp y cyclotomically normalizes $\mathrm{I}_{\tilde{y}/y}$ and thus $\mathrm{I}_{\tilde{y}/y} \otimes \mathbb{Z}_\ell \cong \mathbb{Z}_\ell(1)$ for some choice of $\tilde{y}$ depending on various base points and paths. Conversely, let $s(\mathrm{Gal}_k)$ cyclotomically normalizes the subgroup $\mathrm{I} \cong \mathbb{Z}_\ell(1)$. By Theorem 30 there is an inertia subgroup $\mathrm{I}_{\tilde{y}/y}$ that contains I. For $\sigma \in \mathrm{Gal}_k$ we find $\mathbf{1} \neq \mathrm{I} \subseteq \mathrm{I}_{\tilde{y}/y} \cap \mathrm{I}_{s(\sigma).\tilde{y}/s(\sigma).y}$ so that $s(\sigma).\tilde{y} = \tilde{y}$ by Proposition 16 (1). Thus $s(\mathrm{Gal}_k) \subseteq \mathrm{D}_{\tilde{y}/y} = \pi_1(U_y)$ and s is cuspidal. $\square$

6. Anabelian local cohomology for curves

The purpose of this section is to give an anabelian definition of the sequence (3.1). We will introduce an anabelian orientation module that is isomorphic to $\hat{\mathbb{Z}}(1)$ but defined entirely in terms of the fundamental group of the curve in question.

The Pontrjagin dual of the localization sequence

$$\mathrm{H}^1(\overline{X}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(\overline{U}, \mathbb{Q}/\mathbb{Z}) \to \bigoplus_{y \in \mathrm{Cusps}(\bar{U})} \mathrm{H}^2_y(\overline{X}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(\overline{X}, \mathbb{Q}/\mathbb{Z})$$

for the pair $(\overline{X}, \overline{U})$ with coefficients in $\mathbb{Q}/\mathbb{Z}$ reads

$$\mathrm{H}^2(\overline{X}, \mathbb{Q}/\mathbb{Z})^\vee \xrightarrow{\Delta} \bigoplus_{y \in \mathrm{Cusps}(\bar{U})} \mathrm{H}^2_y(\overline{X}, \mathbb{Q}/\mathbb{Z})^\vee \to \pi_1^{\mathrm{ab}}(\overline{U}) \to \pi_1^{\mathrm{ab}}(\overline{X}). \tag{6.1}$$

The map Δ is even injective except if $X = U$.

Let U_y^{sh} be the scheme of **geometric nearby points** of y, i.e., the preimage of U in $X_y^{\mathrm{sh}} = \mathrm{Spec}(\mathscr{O}_{X,y}^{\mathrm{sh}})$, where $\mathscr{O}_{X,y}^{\mathrm{sh}}$ is the strict henselisation of the local ring at $y \in X$. By excision we have

$$\mathrm{H}^2_y(\overline{X}, \mathbb{Q}/\mathbb{Z})^\vee = \mathrm{H}^2_y(X_y^{\mathrm{sh}}, \mathbb{Q}/\mathbb{Z})^\vee$$

Naturality of (6.1) and summation over all $y \in \mathrm{Cusps}(\overline{U})$ leads to a commutative diagram

$$\begin{tikzcd}
\bigoplus_{y \in \mathrm{Cusps}(\bar{U})} \mathrm{H}^2_y(X_y^{\mathrm{sh}}, \mathbb{Q}/\mathbb{Z})^\vee \arrow[r, "\sim"] \arrow[d, equal] & \bigoplus_{y \in \mathrm{Cusps}(\bar{U})} \pi_1^{\mathrm{ab}}(U_y^{\mathrm{sh}}) \arrow[d] \\
\bigoplus_{y \in \mathrm{Cusps}(\bar{U})} \mathrm{H}^2_y(\overline{X}, \mathbb{Q}/\mathbb{Z})^\vee \arrow[r] & \pi_1^{\mathrm{ab}}(\overline{U}).
\end{tikzcd}$$

where the top horizontal map is an isomorphism since X_y^{sh} has no cohomology in degrees ≥ 1. This identifies the map

$$\mathrm{H}^2_y(\overline{X}, \mathbb{Q}/\mathbb{Z})^\vee \to \pi_1^{\mathrm{ab}}(\overline{U})$$

in the dual of the localization sequence with the natural map of an inertia group

$$\mathrm{I}_{\tilde{y}/y} = \pi_1^{\mathrm{ab}}(U_y^{\mathrm{sh}}) \to \pi_1^{\mathrm{ab}}(\overline{U})$$

for a choice of cusp $\tilde{y}$ above y reflected in the choice of base points. The image is independent of the choice of the prolongation $\tilde{y}$ and will be denoted by I_y.

For a hyperbolic curve U/k, apart from the map Δ, all constituents of (6.1) have an anabelian definition in terms of the fundamental group $\pi_1(U/k)$ together with its weight filtration alone. Here is the anabelian definition of the map Δ.

Definition 33. For X of genus 0 we define the **orientation module** $\mathrm{Or}_{\pi_1(U/k)}$ and Δ through the exactness of the sequence

$$0 \to \mathrm{Or}_{\pi_1(U/k)} \xrightarrow{\Delta} \bigoplus_{y \in \mathrm{Cusps}(\bar{U})} \mathrm{I}_y \to \pi_1^{\mathrm{ab}}(\overline{U}).$$

If X has positive genus, then $\overline{X}$ is an étale $K(\pi,1)$, cf. [21] Appendix A, which in particular means that the natural map

$$\mathrm{H}^2(\pi_1(\overline{X}), \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(\overline{X}, \mathbb{Q}/\mathbb{Z})$$

is an isomorphism. We define the **orientation module** as

$$\mathrm{Or}_{\pi_1(U/k)} = \mathrm{H}^2(\pi_1(\overline{X}), \mathbb{Q}/\mathbb{Z})^\vee = \mathrm{H}^2(\overline{X}, \mathbb{Q}/\mathbb{Z})^\vee.$$

The y component of the map Δ is defined by functoriality from the case of the pair $(X, X - \{y\})$ and as the Pontrjagin dual to a map

$$\delta_y : \mathrm{Hom}(\mathrm{I}_y, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^2(\pi_1(\overline{X}), \mathbb{Q}/\mathbb{Z})$$

as follows. There is a maximal intermediate quotient

$$\pi_1(\overline{X \setminus \{y\}}) \twoheadrightarrow E \xrightarrow{\alpha} \pi_1(\overline{X})$$

with $\ker(\alpha)$ central in E. Then canonically $\ker(\alpha) = I_y$ and pushing

$$1 \to I_y \to E \to \pi_1(\overline{X}) \to 1 \tag{6.2}$$

by $\chi \in \mathrm{Hom}(\mathrm{I}_y, \mathbb{Q}/\mathbb{Z})$ defines $\delta_y(\chi) \in \mathrm{H}^2(\pi_1(\overline{X}), \mathbb{Q}/\mathbb{Z})$ as a central extension.

Definition 34. The **anabelian local cohomology sequence** for the pair (X, U) of a hyperbolic curve U and its smooth completion X is the sequence

$$\mathrm{Or}_{\pi_1(U/k)} \xrightarrow{\Delta} \bigoplus_{y \in \mathrm{Cusps}(\bar{U})} \mathrm{I}_y \to \pi_1^{\mathrm{ab}}(\overline{U}) \to \pi_1^{\mathrm{ab}}(\overline{X}) \to 0, \tag{6.3}$$

as constructed above. The map Δ is even injective except if $X = U$, and the groups $\mathrm{Or}_{\pi_1(U/k)}$ and I_y are all isomorphic to $\hat{\mathbb{Z}}(1)$ as Galois modules.

It remains to compare the anabelian definition of Δ with the geometric definition.

Theorem 35. *Let U be a hyperbolic curve with smooth completion X. The anabelian local cohomology sequence (6.3)*

$$\mathrm{Or}_{\pi_1(U/k)} \xrightarrow{\Delta} \bigoplus_{y \in \mathrm{Cusps}(\bar{U})} \mathrm{I}_y \to \pi_1^{\mathrm{ab}}(\overline{U}) \to \pi_1^{\mathrm{ab}}(\overline{X}) \to 0,$$

is naturally isomorphic to the local cohomology sequence in étale cohomology (6.1) up to a sign for the map Δ.

Proof: We only need to compare the anabelian definition of Δ with the geometric definition. For that purpose we restrict to the n-torsion part and twist by μ_n. In the following diagram

$$\begin{tikzcd}
\mathrm{H}^0\left(U_y^{\mathrm{sh}}, \mathbb{G}_m\right) \arrow[r, two heads, "\delta_{\mathrm{loc}}"] \arrow[d, "\delta_{\mathrm{Kum}}"] \arrow[dr, phantom, "\boxed{-1}"] & \mathrm{H}^1_y\left(X_y^{\mathrm{sh}}, \mathbb{G}_m\right) \arrow[d, "\delta_{\mathrm{Kum}}"] & \mathrm{H}^1_y\left(\overline{X}, \mathbb{G}_m\right) \arrow[l, "\mathrm{exc}"', "\cong"] \arrow[r] \arrow[d, "\delta_{\mathrm{Kum}}"] & \mathrm{H}^1\left(\overline{X}, \mathbb{G}_m\right) \arrow[d, "\delta_{\mathrm{Kum}}"] \\
\mathrm{H}^1\left(U_y^{\mathrm{sh}}, \mu_n\right) \arrow[r, "\delta_{\mathrm{loc}}", "\cong"'] & \mathrm{H}^2_y\left(X_y^{\mathrm{sh}}, \mu_n\right) & \mathrm{H}^2_y\left(\overline{X}, \mu_n\right) \arrow[l, "\mathrm{exc}"', "\cong"] \arrow[r, "\Delta_y^{\vee}"] & \mathrm{H}^2\left(\overline{X}, \mu_n\right) \\
\mathrm{H}^1\left(\pi_1(U_y^{\mathrm{sh}}), \mu_n\right) \arrow[u, "\mathrm{comp}", "\cong"'] \arrow[r, equal] & \mathrm{Hom}(I_y, \mu_n) \arrow[u, phantom, "\boxed{-1}"] \arrow[rr, "\delta_y = \mathrm{push}\ E"] & & \mathrm{H}^2\left(\pi_1(\overline{X}), \mu_n\right) \arrow[u, "\mathrm{comp}", "\cong"']
\end{tikzcd}$$

the upper left facet commutes up to sign by [2] cycle 2.1.3. We compute the maps going around the diagram along the outside border and establish that the bottom facet also commutes only up to a sign, which yields precisely the claim of the theorem.

Let f be a parameter of X at y. The mod n tame character in $\mathrm{Hom}(I_y, \mu_n)$

$$I_y \to \mu_n \quad \sigma \mapsto \sigma(\sqrt[n]{f})/\sqrt[n]{f}$$

maps under the comparison map to the μ_n torsor

$$[\sqrt[n]{f}] \in \mathrm{H}^1\left(U_y^{\mathrm{sh}}, \mu_n\right)$$

of n^{th} roots of f, which equals $\delta_{\mathrm{Kum}}(f)$ by [2] cycle 1.1.1. By [2] cycle 1.1.4, we may interpret

$$\delta_{\mathrm{loc}}(f) \in \mathrm{H}^1_y\left(X_y^{\mathrm{sh}}, \mathbb{G}_m\right)$$

as the trivial line bundle on X_y^{sh} plus the trivialisation given by f over U_y^{sh} which maps under excision and the natural map to the class of the line bundle $\mathcal{O}(y) \in \mathrm{H}^1\left(\overline{X}, \mathbb{G}_m\right)$ associated to the divisor y. By the very definition of [2] cycle 2.1.2, the class $\delta_{\mathrm{Kum}}(\mathcal{O}(y)) = \mathrm{c}_1(\mathcal{O}(y))$ is the cycle class cl_y of y. By [11] Lemma 4.5, the corresponding central extension in $\mathrm{H}^2(\pi_1(\overline{X}), \mu_n)$ is given by the mod n push of the homotopy sequence

$$1 \to \hat{\mathbb{Z}}(1) \to \pi_1(\mathbb{L}^\circ) \to \pi_1(\overline{X}) \to 1 \tag{6.4}$$

of the complement $\mathbb{L}^\circ$ of the zero section in

$$\mathbb{L} = \underline{\mathrm{Spec}}_{\bar{X}}\left(\mathrm{Sym}^\bullet \mathcal{O}(-y)\right)$$

which is the geometric line bundle whose sections naturally coincide with sections of $\mathcal{O}(y)$.

The section 1 above $X - \{y\}$ defines a map

$$\overline{X - \{y\}} \to \mathbb{L}^\circ \to \overline{X}$$

which leads to a map of extensions

$$\begin{tikzcd}
1 \arrow[r] & I_y \arrow[r] \arrow[d, "\chi"] & E \arrow[r] \arrow[d] & \pi_1(\overline{X}) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \hat{\mathbb{Z}}(1) \arrow[r] & \pi_1(\mathbb{L}^\circ) \arrow[r] & \pi_1(\overline{X}) \arrow[r] & 1
\end{tikzcd}$$

and all that remains is to identify the character χ as the tame character. The trivialisation of $\mathbb{L}$ over X_y^{sh} via f leads to a diagram

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[dr, equal] & \mathbb{L}^\circ|_{X_y^{\mathrm{sh}}} \arrow[r, "\cong"] \arrow[d, "\mathrm{pr}_1"] & \mathbb{G}_m \times X_y^{\mathrm{sh}} \\
& \mathbb{G}_m & U_y^{\mathrm{sh}} \arrow[l, "f"'] \arrow[ul, "1"'] \arrow[u, "f{,}\mathrm{incl}"']
\end{tikzcd}$$

which on fundamental groups gives

$$\begin{tikzcd}
\hat{\mathbb{Z}}(1) \arrow[r, "\cong"] \arrow[dr, equal] & \pi_1(\mathbb{L}^\circ|_{X_y^{\mathrm{sh}}}) \arrow[r, "\cong"] \arrow[d, "\pi_1(\mathrm{pr}_1)"] & \pi_1(\mathbb{G}_m \times X_y^{\mathrm{sh}}) \\
& \hat{\mathbb{Z}}(1) & \pi_1(U_y^{\mathrm{sh}}) = I_y \arrow[l, "\pi_1(f)"'] \arrow[ul, "\chi"'] \arrow[u, "\pi_1(f,\mathrm{incl})"']
\end{tikzcd}$$

and reveals that $\chi = \pi_1(f)$ which is exactly the tame character. □

7. Orientation and degree

As it matters here, we stress that the Tate twist $\hat{\mathbb{Z}}(1)$ has an anabelian definition as the geometric fundamental group $\pi_1(\overline{\mathbb{G}}_{\mathrm{m}})$ of $\mathbb{G}_m$ with the Gal_k-action induced from $\pi_1(\mathbb{G}_m/k)$.

Definition 36. An **orientation** on $\pi_1(U/k)$ consists of an isomorphism τ of the orientation module $\mathrm{Or}_{\pi_1(U/k)}$ with $\hat{\mathbb{Z}}(1)$. Note that every isomorphism τ is automatically an isomorphism of Gal_k-modules.

Definition 37. A **local orientation** at each cusp $\tilde{y}/y$ is a family of isomorphisms $\tau_{\tilde{y}/y} : \mathrm{I}_{\tilde{y}/y} \to \hat{\mathbb{Z}}(1)$ such that the following holds.

(i) Equivariance with respect to $\pi_1(U)$ and the cyclotomic character χ^{cyl} of Gal_k, namely, for $\gamma \in \pi_1(U)$ and a cusp $\tilde{y}/y$ the following commutes

$$\begin{tikzcd}
\mathrm{I}_{\tilde{y}/y} \arrow[r, "\gamma()\gamma^{-1}"] \arrow[d, "\tau_{\tilde{y}/y}"] & \mathrm{I}_{\gamma.\tilde{y}/\gamma.y} \arrow[d, "\tau_{\gamma.\tilde{y}/\gamma.y}"] \\
\hat{\mathbb{Z}}(1) \arrow[r, "\chi^{\mathrm{cyl}}(\gamma)\cdot"] & \hat{\mathbb{Z}}(1).
\end{tikzcd}$$

(ii) The kernel of the map $\bigoplus_{y \in \mathrm{Cusps}(U)} \mathrm{I}_y \to \pi_1^{\mathrm{ab}}(\overline{U})$ from (6.3) is the diagonally embedded $\hat{\mathbb{Z}}(1)$ under the identification by the local orientations, which is well defined by (i).

Definition 38. The **standard orientation** on $\pi_1(U/k)$ is given locally by the tame characters and globally by the evaluation at the fundamental class

$$\mathrm{Or}_{\pi_1(U/k)} = \mathrm{H}^2(\overline{X}, \mathbb{Q}/\mathbb{Z})^\vee = \mathrm{Hom}\left(\mathrm{H}^2(\overline{X}, \hat{\mathbb{Z}}(1)), \hat{\mathbb{Z}}(1)\right) \xrightarrow{\cong} \hat{\mathbb{Z}}(1),$$

whichever is applicable for $\pi_1(U/k)$.

Remark 39. (1) The anabelian local cohomology sequence (6.3) and its comparison with (6.1) shows that an orientation on $\pi_1(U/k)$ induces canonically **local orientations** at each cusp $\tilde{y}/y$.

(2) Since inertia subgroups at different prolongations of cusps intersect only trivially, see Proposition 16, local orientations fuse to form a $\pi_1(U)$ equivariant map

$$\tau : W_{-2}(U) \to \hat{\mathbb{Z}}(1)$$

that satisfies a global consistency condition, namely condition (ii) of Definition 37.

(3) The core of the argument in Section 6 showed that tame character and evaluation at the fundamental class are indeed compatible in cases both definitions make sense.

(4) An arbitrary orientation on $\pi_1(U/k)$ will be a multiple of the standard orientation by a factor $\varepsilon \in \hat{\mathbb{Z}}^*$. We call the resulting orientation '*ε times the standard orientation*' or simply the ε orientation.

A weight preserving map of Gal_k-extensions of hyperbolic curves induces a natural map between the respective orientation modules.

Definition 40. The **degree** of a map $\varphi : \pi_1(U/k) \to \pi_1(V/k)$ which is weight preserving between the fundamental groups of hyperbolic curves U/k and V/k equipped with an orientation is the element $\deg(\varphi) \in \hat{\mathbb{Z}}$, such that multiplication by $\deg(\varphi)$ agrees with the map

$$\hat{\mathbb{Z}}(1) = \mathrm{Or}_{\pi_1(U/k)} \to \mathrm{Or}_{\pi_1(V/k)} = \hat{\mathbb{Z}}(1)$$

under the identification induced by the chosen orientations.

The **local degree** or **ramification index** of φ at $\tilde{y} \in \widetilde{\mathrm{Cusps}}(U)$ over $\tilde{z} \in \widetilde{\mathrm{Cusps}}(V)$ is the unique $\varepsilon_\varphi(\tilde{y}/\tilde{z}) \in \hat{\mathbb{Z}}$, so that $\varphi|_{\mathrm{I}_{\tilde{y}/y}} : \mathrm{I}_{\tilde{y}/y} \to \mathrm{I}_{\tilde{z}/z}$ equals multiplication by $\varepsilon_\varphi(\tilde{y}/\tilde{z})$ after identifying the inertia groups with $\hat{\mathbb{Z}}(1)$ according to the chosen local orientations.

As the restriction to $W_{-2}(U)$ of φ is equivariant with respect to conjugation, the local degree only depends on the cusps $y \in \mathrm{Cusps}(U)$ and $\varphi(y) := z \in \mathrm{Cusps}(V)$ and we write $\varepsilon_\varphi(y/z) = \varepsilon_\varphi(\tilde{y}/\tilde{z})$.

The following absorbs an argument exploited by Pop in the context of birational anabelian geometry.

Proposition 41. *Let $\varphi : \pi_1(U/k) \to \pi_1(V/k)$ be a weight preserving map between oriented fundamental groups. Then for any cusp $z \in \mathrm{Cusps}(V)$ the **fundamental equation***

$$\deg(\varphi) = \sum_{y \mapsto z} \varepsilon_\varphi(y/z)$$

holds in $\hat{\mathbb{Z}}$, where y ranges over cusps of U with $\varphi(y) = z$.

Proof: Using orientations, we extract from the map of anabelian local cohomology sequences for $\pi_1(U/k)$ and $\pi_1(V/k)$ induced by φ the following commutative diagram

$$\begin{tikzcd}
\hat{\mathbb{Z}}(1) \arrow[r, "\Delta"] \arrow[d, "\cdot \deg(\varphi)"] & \bigoplus_{y \in \mathrm{Cusps}(U)} \hat{\mathbb{Z}}(1) \arrow[d, "\cdot \varepsilon_\varphi(y/z)"] \\
\hat{\mathbb{Z}}(1) \arrow[r, "\Delta"] & \bigoplus_{z \in \mathrm{Cusps}(V)} \hat{\mathbb{Z}}(1).
\end{tikzcd}$$

The proposition follows since Δ is the diagonal embedding. $\square$

If we restrict the discussion to isomorphisms $\varphi : \pi_1(U/k) \to \pi_1(V/k)$ preserving the weight filtration, then $\varphi : \mathrm{Cusps}(U) \to \mathrm{Cusps}(V)$ is bijective and the fundamental equation shows that $\deg(\varphi) = \varepsilon_\varphi(y/z) \in \hat{\mathbb{Z}}^*$ for all cusps y and $z = \varphi(y)$.

Definition 42. An **orientation preserving** isomorphism is an isomorphism φ as above with $\deg(\varphi) = 1$.

8. Anabelian theory of units

The anabelian approach to units on $\pi_1(U/k)$ is by maps to $\pi_1(\mathbb{G}_m/k)$. The extension $\pi_1(\mathbb{G}_m/k)$ is a group object in the category of Gal_k-extensions with multiplication given by

$$\pi_1(\mathrm{mult}) : \pi_1(\mathbb{G}_m) \times_{\mathrm{Gal}_k} \pi_1(\mathbb{G}_m) = \pi_1(\mathbb{G}_m \times_k \mathbb{G}_m) \to \pi_1(\mathbb{G}_m)$$

and unit $s_1 : \mathrm{Gal}_k \to \pi_1(\mathbb{G}_m)$ induced by the rational point $1 \in \mathbb{G}_m(k)$.

Definition 43. The group of **pro-units** on $\pi_1(U/k)$ is the group

$$\hat{\mathcal{O}}^*\big(\pi_1(U/k)\big) := \mathrm{Hom}\,\big(\pi_1(U/k), \pi_1(\mathbb{G}_m/k)\big),$$

with the group structure inherited from the group object $\pi_1(\mathbb{G}_m/k)$. The **constant pro-units** are those which factor through Gal_k.

Definition 44. The group of **units** on an oriented $\pi_1(U/k)$ is the subgroup $\mathcal{O}^*\big(\pi_1(U/k)\big) \subseteq \hat{\mathcal{O}}^*\big(\pi_1(U/k)\big)$ consisting of those maps that for each cusp $y \in \mathrm{Cusps}(U)$ induce integral maps $\mathrm{I}_y = \hat{\mathbb{Z}}(1) \to \hat{\mathbb{Z}}(1)$, i.e., the multiplication by some $\varepsilon_y \in \mathbb{Z} \subset \hat{\mathbb{Z}}$.

Proposition 45. *The automorphism group of $\pi_1(\mathbb{G}_m/k)$ is isomorphic to the semidirect product $\widehat{k^*} \rtimes \hat{\mathbb{Z}}^*$ with respect to the natural action of $\hat{\mathbb{Z}}^*$ on pro-$\mathbb{N}$ completions. The group of automorphisms which act as the identity on $\pi_1(\overline{\mathbb{G}}_\mathrm{m}) = \hat{\mathbb{Z}}(1)$ is $\widehat{k^*}$.*

Before we start the proof we recall that we identify $\pi_1(\overline{\mathbb{G}}_\mathrm{m}) = \hat{\mathbb{Z}}(1)$ and that the Tate-module $\hat{\mathbb{Z}}(1)$ is defined as $\varprojlim_n \mu_n$. We therefore write composition in $\hat{\mathbb{Z}}(1)$ multiplicatively. Additive notation would suggest having chosen an isomorphism $\hat{\mathbb{Z}}(1) \cong \hat{\mathbb{Z}}$ of the underlying pro-finite groups, i.e., having chosen a compatible system of roots of unity, a choice that we avoid to make.

Proof: The group $\pi_1(\mathbb{G}_m)$ is a semi direct product with respect to the section s_1 associated to $1 \in \mathbb{G}_m(k)$. We write elements of $\pi_1(\mathbb{G}_m)$ as $a\sigma = as_1(\sigma)$ with $a \in \hat{\mathbb{Z}}(1)$ and $\sigma \in \mathrm{Gal}_k$.

A unit $\varepsilon \in \hat{\mathbb{Z}}^*$ gives an automorphism $\varepsilon : a\sigma \mapsto a^\varepsilon\sigma$. An element $\alpha \in \widehat{k^*}$ corresponds by Kummer theory to the cocycle

$$\chi_\alpha : \sigma \mapsto \chi_\alpha(\sigma) = \big(\sigma(\sqrt[n]{\alpha})/\sqrt[n]{\alpha}\big)_{n\in\mathbb{N}} \in \hat{\mathbb{Z}}(1).$$

The automorphism **scaling with** α is defined as $\alpha : a\sigma \mapsto a\chi_\alpha(\sigma)\sigma$. We have $\varepsilon\alpha\varepsilon^{-1} : a\sigma \mapsto \big(a^{\varepsilon^{-1}}\chi_\alpha(\sigma)\big)^\varepsilon\sigma = a\chi_\alpha(\sigma)^\varepsilon\sigma$ which equals the automorphism 'scaling by α^ε'. Thus we have constructed a subgroup

$$\widehat{k^*} \rtimes \hat{\mathbb{Z}}^* \subseteq \mathrm{Aut}(\pi_1(\mathbb{G}_m/k))$$

Let $\varphi \in \mathrm{Aut}(\pi_1(\mathbb{G}_m/k))$ be an arbitrary automorphism. Then φ agrees with some $\varepsilon \in \hat{\mathbb{Z}}^*$ on $\pi_1(\overline{\mathbb{G}}_\mathrm{m})$ and after composing with ε^{-1} we may assume that φ differs from the identity by a map $\sigma \mapsto \varphi(\sigma)\sigma^{-1}$

$$\varphi/\mathrm{id} \;:\; \pi_1(\mathbb{G}_m) \to \mathrm{Gal}_k \to \hat{\mathbb{Z}}(1) = \pi_1(\overline{\mathbb{G}}_\mathrm{m}) \subset \pi_1(\mathbb{G}_m),$$

more precisely a cocycle of which, due to the definition of maps of extensions, only the class in $\mathrm{H}^1(k, \hat{\mathbb{Z}}(1)) = \widehat{k^*}$ matters. This completes the proof of the proposition. $\square$

Proposition 46. *Let x be a parameter for $\mathbb{G}_m$. For $\varepsilon = \pm 1$ and $\alpha \in k^*$ the geometric automorphism $x \mapsto \alpha x^\varepsilon$ of $\mathbb{G}_m$ induces the automorphism $(\alpha, \varepsilon) \in \widehat{k^*} \rtimes \hat{\mathbb{Z}}^*$ of $\pi_1(\mathbb{G}_m/k)$.*

Proof: The map $x \mapsto x^\varepsilon$ preserves the section s_1 and the effect on $\hat{\mathbb{Z}}(1)$ can be computed from cohomology, thus multiplies with ε. We find again the automorphism denoted ε in the proof above.

The scaling map $x \mapsto \alpha x$ acts trivially on étale cohomology of $\mathbb{G}_m$ by homotopy invariance, hence the associated map on $\pi_1(\mathbb{G}_m/k)$ has $\varepsilon = 1$. It remains to observe how the section s_1 is shifted. But this is done using the Kummer character $\chi_\alpha(-)$ which mediates an isomorphism of fibre functors from fibres above 1 to fibres above α, hence 'scaling by α' acts on sections as the geometric map $x \mapsto \alpha x$ does. $\square$

Proposition 47. *There is a natural isomorphism*

$$\hat{\mathcal{O}}^*\big(\pi_1(U/k)\big) \xrightarrow{\sim} \mathrm{H}^1\left(\pi_1(U), \hat{\mathbb{Z}}(1)\right)$$

with the following properties.

1. *The image of inflation $\mathrm{H}^1\left(k, \hat{\mathbb{Z}}(1)\right) \hookrightarrow \mathrm{H}^1\left(\pi_1(U), \hat{\mathbb{Z}}(1)\right)$ agrees with the group of constant pro-units.*
2. *The map $\kappa : \mathcal{O}^*(U) \to \hat{\mathcal{O}}^*\big(\pi_1(U/k)\big)$ given by $f \mapsto \kappa_f := \pi_1(f)$ agrees with the boundary map δ_{Kum} of Kummer theory on U.*

Proof: A unit is an equivalence class of sections of the extension $\pi_1(\mathbb{G}_m/k)$ pulled back via $\pi_1(U) \to \mathrm{Gal}_k$. Thus the existence of the isomorphism and part (1) follow from the identification of differences of sections with 1-cocycles.

Let $\chi_f : \pi_1(U) \to \hat{\mathbb{Z}}(1)$ be the Kummer character

$$\gamma \mapsto \chi_f(\gamma) = (\gamma(\sqrt[n]{f})/\sqrt[n]{f})_{n\in\mathbb{N}}$$

associated to $f \in \mathcal{O}^*(U)$, and let $\mathrm{pr} : \pi_1(U) \to \mathrm{Gal}_k$ be the projection. For (2) and $\gamma \in \pi_1(U)$ we compute $\kappa_f(\gamma)$ as the effect of γ on the Galois extension $k^{\mathrm{alg}}(U)[\sqrt[n]{f}]/k(U)$ for all $n \geq 1$:

$$\kappa_f(\gamma) = \chi_f(\gamma) \cdot \mathrm{pr}(\gamma) \in \hat{\mathbb{Z}}(1) \rtimes \mathrm{Gal}_k = \pi_1(\mathbb{G}_m),$$

which under the identification of (1) becomes $\delta_{\mathrm{Kum}}(f) = \chi_f$. □

Let $W_{-2}\big(\pi_1^{\mathrm{ab}}(\overline{U})\big) \subset \pi_1^{\mathrm{ab}}(\overline{U})$ be the subgroup generated by inertia. The group $\mathrm{Div}_Y^0(X)$ of divisors of degree 0 on X with support in Y has a map

$$\tau \ : \ \mathrm{Div}_Y^0(X) \to \mathrm{Hom}_k\left(W_{-2}(\pi_1^{\mathrm{ab}}(\overline{U})), \hat{\mathbb{Z}}(1)\right)$$

which assigns to a divisor the corresponding linear combination of tame characters, and which induces an isomorphism

$$\mathrm{Div}_Y^0(X) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \xrightarrow{\cong} \mathrm{Hom}_k\left(W_{-2}(\pi_1^{\mathrm{ab}}(\overline{U})), \hat{\mathbb{Z}}(1)\right).$$

The Jacobian of the curve X is Pic_X^0 and the Albanese variety of X is denoted by Alb_X. Of course, it is well known that for proper curves Pic_X^0 is isomorphic to Alb_X. Distinguishing Albanese variety and Jacobian helps to keep track of the covariance of the objects and allows for easier potential generalizations to the case beyond curves.

For two Gal_k-modules M, M' we denote by $\mathscr{H}om(M, M')$ the inner Hom, i.e., the group of homomorphisms as a Gal_k-module.

We define a homomorphism

$$\gamma \ : \ \mathrm{Pic}_X^0(k) \to \mathrm{Ext}_k^1\left(\pi_1^{\mathrm{ab}}(\overline{X}), \hat{\mathbb{Z}}(1)\right)$$

as the composite of the boundary map of Kummer theory for Pic_X^0, the map $\pi_1(\overline{\mathrm{Pic}^0}_X) \to \mathscr{H}om(\pi_1(\overline{\mathrm{Alb}}_X), \hat{\mathbb{Z}}(1))$ of Gal_k-modules given by the Weil pairing, and the edge map in the Ext-spectral sequence:

$$\begin{tikzcd}
\mathrm{Pic}_X^0(k) \arrow[r, "\delta_{\mathrm{Kum}}"] \arrow[d, dotted, "\gamma"] & \mathrm{H}^1\left(k, \pi_1(\overline{\mathrm{Pic}}_X^0)\right) \arrow[d, "\mathrm{Weil}"', "\cong"] \\
\mathrm{Ext}_k^1\left(\pi_1^{\mathrm{ab}}(\overline{X}), \hat{\mathbb{Z}}(1)\right) & \mathrm{H}^1\left(k, \mathscr{H}om(\pi_1(\overline{\mathrm{Alb}}_X), \hat{\mathbb{Z}}(1))\right). \arrow[l, "\cong"', "\mathrm{edge}"]
\end{tikzcd}$$

Proposition 48. *The maps κ, τ and γ fit into the following map of exact sequences*

$$\begin{tikzcd}
 & 1 \arrow[d] \\
1 \arrow[r] \arrow[d] & \left(T\,\mathrm{Pic}_X^0\right)(k) \arrow[d] \\
\mathcal{O}^*(U)/k^* \arrow[r, "\kappa"] \arrow[d, "\mathrm{div}"] & \hat{\mathcal{O}}^*\big(\pi_1(U/k)\big)/\widehat{k^*} \arrow[d, "\mathrm{res}"] \\
\mathrm{Div}_Y^0(X) \arrow[r, "\tau"] \arrow[d] & \mathrm{Hom}_k\left(W_{-2}(\pi_1^{\mathrm{ab}}(\overline{U})), \hat{\mathbb{Z}}(1)\right) \arrow[d, "\delta"] \\
\mathrm{Pic}_X^0(k) \arrow[r, "\gamma"] & \mathrm{Ext}_k^1\left(\pi_1^{\mathrm{ab}}(\overline{X}), \hat{\mathbb{Z}}(1)\right).
\end{tikzcd}$$

Proof: The right column is exact because it can be identified with the long exact Ext-sequence of Gal_k modules for

(8.1) $$0 \to W_{-2}\big(\pi_1^{\mathrm{ab}}(\overline{U})\big) \to \pi_1^{\mathrm{ab}}(\overline{U}) \to \pi_1^{\mathrm{ab}}(\overline{X}) \to 0.$$

The identification is based on the following. First, the Weil-pairing yields
$$\mathrm{Hom}_k\left(\pi_1^{\mathrm{ab}}(\overline{X}), \hat{\mathbb{Z}}(1)\right) = \left(T\,\mathrm{Pic}^0_X\right)(k).$$
Secondly, the edge map
$$\mathrm{H}^2\left(k, \hat{\mathbb{Z}}(1)\right) \to \mathrm{H}^2\left(\pi_1(U), \hat{\mathbb{Z}}(1)\right)$$
in the Hochschild–Serre spectral sequence for $\pi_1(U) \to \mathrm{Gal}_k$ and the module $\hat{\mathbb{Z}}(1)$ is injective, because the projection $\pi_1(U) \to \mathrm{Gal}_k$ is split over an open subgroup and $\mathrm{H}^2(k, \hat{\mathbb{Z}}(1)) = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathrm{Br}(k))$ contains no torsion. Therefore the differential $d_2^{0,1}$ vanishes and the exact sequence of low degree terms reads by Proposition 47
$$0 \to \widehat{k^*} \to \hat{\mathcal{O}}^*\left(\pi_1(U/k)\right) \to \mathrm{H}^0\left(k, \mathrm{H}^1(\pi_1(\overline{U}), \hat{\mathbb{Z}}(1))\right) \xrightarrow{d_2^{0,1}} 0$$
which determines
$$\hat{\mathcal{O}}^*\left(\pi_1(U/k)\right)/\widehat{k^*} \xrightarrow{\sim} \mathrm{H}^0\left(k, \mathrm{H}^1(\pi_1(\overline{U}), \hat{\mathbb{Z}}(1))\right) = \mathrm{Hom}_k\left(\pi_1^{\mathrm{ab}}(\overline{U}), \hat{\mathbb{Z}}(1)\right).$$

Now we show the commutativity of the two squares. We have
$$\mathrm{res} \circ \kappa = \tau \circ \mathrm{div}$$
since for a function $f \in \mathcal{O}^*(U)$ the restriction $\kappa_f|_{\mathrm{I}_y}$ to inertia at y equals $\nu_y(f)$ times the tame character, where ν_y is the valuation associated to $y \in X$.

It remains to show commutativity of the bottom square. For a divisor $D \in \mathrm{Div}^0_Y(X)$ the extension $E_D = \delta(\tau(D))$ is obtained by pushing the extension of Gal_k modules (8.1) with $\tau(D)$. The line bundle associated to D comes from a unique degree 0 line bundle on the Albanese variety $A = \mathrm{Alb}_X$ of X. The complement $\mathbb{L}^0(D)$ of the 0-section in the corresponding geometric line bundle $\mathbb{L}(D) \to A$ has an abelian geometric fundamental group $\pi_1(\overline{\mathbb{L}^0(D)})$ as its first Chern class vanishes and therefore the extension
$$0 \to \hat{\mathbb{Z}}(1) \to \pi_1(\overline{\mathbb{L}^0(D)}) \to \pi_1(\overline{A}) \to 0$$
splits representing
$$0 = \mathrm{c}_1(\mathbb{L}(D)) \in \mathrm{H}^2(\overline{A}, \hat{\mathbb{Z}}(1)) = \mathrm{H}^2(\pi_1(\overline{A}), \hat{\mathbb{Z}}(1))$$
Hence, from the canonical trivialisation of $\mathbb{L}(D)$ above U we obtain a map of extensions of Gal_k modules
$$\begin{tikzcd}
0 \arrow[r] & W_{-2}\left(\pi_1^{\mathrm{ab}}(\overline{U})\right) \arrow[r] \arrow[d, "\tau(D)"] & \pi_1^{\mathrm{ab}}(\overline{U}) \arrow[r] \arrow[d] & \pi_1^{\mathrm{ab}}(\overline{X}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \hat{\mathbb{Z}}(1) \arrow[r] & \pi_1(\overline{\mathbb{L}^0(D)}) \arrow[r] & \pi_1(\overline{A}) \arrow[r] & 0,
\end{tikzcd}$$
from which we conclude that E_D equals $\pi_1(\overline{\mathbb{L}^0(D)})$ as an extension of Gal_k modules. This completes the computation of $\delta(\tau(D))$.

We now compute $\gamma(D)$, where we abuse notation to denote also the class of D in Pic^0_X by D. Let $i_D : A \to A \times B$ be the inclusion induced by D as a k-rational point of $B = \mathrm{Pic}^0_X$. The line bundle $\mathbb{L}(D)$ is the restriction of the geometric Poincaré bundle $\mathbb{P}_{A\times B}$ along i_D. We thus find the extension E_D as the image E of
$$\pi_1(i_D) : \pi_1(\overline{\mathbb{L}^0(D)}) \hookrightarrow \pi_1(\overline{\mathbb{P}^0_{A\times B}})$$
and can do the corresponding cocycle calculations in $\pi_1(\mathbb{P}^0_{A\times B})$.

The group $\pi_1(\mathbb{P}^0_{A\times B})$ is an extension of Gal_k by $\pi_1(\overline{\mathbb{P}^0_{A\times B}})$ which is a central extension of $\mathrm{T}\,A \times \mathrm{T}\,B$ by $\hat{\mathbb{Z}}(1)$. The factors $\mathrm{T}\,A$ and $\mathrm{T}\,B$ are Lagrangian subspaces for the symplectic commutator pairing
$$[\ ,\ ] : \bigwedge^2(\mathrm{T}\,A \times \mathrm{T}\,B) \to \hat{\mathbb{Z}}(1)$$
associated to this central extension. The induced pairing of $\mathrm{T}\,A$ with $\mathrm{T}\,B$ is nothing but the Weil-pairing induced by the Poincaré bundle $\mathbb{P}_{A\times B}$.

Note that the image E of $\pi_1(\overline{\mathbb{L}^0(D)})$ does not depend on D, whereas the Gal_k module structure does depend on D being induced by the conjugation action from the image of the full étale fundamental group $\pi_1(\mathbb{L}^0(D))$, or, what amounts to the same, as the conjugation action via

$$s_{0,D} = \pi_1(i_D) \circ s_0 : \mathrm{Gal}_k \to \pi_1(\mathbb{P}^0_{A\times B}).$$

Here s_0 belongs to $0 \in A(k)$ and $s_{0,D}$ to $(0,D) \in A \times B(k)$.

For $D = 0$ the extension $\pi_1(\mathbb{L}^0(0))$ splits as an extension of Gal_k modules via $s_{0,0}$. Let $f : \mathrm{T}\,A \to E$ be a splitting which is $s_{0,0}(\mathrm{Gal}_k)$ equivariant. By [23] §7.1 Prop 70 the difference cocycle

$$\sigma \mapsto \Delta_\sigma := s_{0,D}(\sigma)s_{0,0}(\sigma)^{-1} = s_D(\sigma)s_0(\sigma)^{-1}$$

represents $\delta_{\mathrm{Kum}}(D)$. The class

$$df = \sigma \mapsto df_\sigma = \big(x \mapsto \sigma f(\sigma^{-1}(x))) - f(x)\big)$$

of E_D in $\mathrm{H}^1\left(k, \mathscr{H}om(\mathrm{T}\,A, \hat{\mathbb{Z}}(1))\right)$ computes therefore as

$$\begin{aligned} df_\sigma(x) &= s_{0,D}(\sigma) f(s_{0,D}(\sigma)^{-1} x \; s_{0,D}(\sigma)) s_{0,D}(\sigma)^{-1} f(x)^{-1} \\ &= \Delta_\sigma f(x) \Delta_\sigma^{-1} f(x)^{-1} \end{aligned}$$

where the Δ_σ (resp. the $f(x)$) lift elements from the second (resp. first) component of $\mathrm{T}\,A \times \mathrm{T}\,B$ to $\pi_1(\overline{\mathbb{P}^0_{A\times B}})$. Consequently, the cocycles Δ_σ and df_σ agree under the Weil pairing. This completes the proof. □

Corollary 49. *Assume that $\mathrm{Pic}^0_X(k)$ does not contain a nontrivial divisible subgroup. Let the orientation fixed on $\pi_1(U/k)$ be ε times the standard orientation for some $\varepsilon \in \hat{\mathbb{Z}}^*$. Then*

$$\mathcal{O}^*(\pi_1(U/k)) = \kappa(\mathcal{O}^*(U))^\varepsilon \cdot \widehat{k^*}$$

holds in $\hat{\mathcal{O}}^*(\pi_1(U/k))$. In particular all units on $\pi_1(U/k)$ correspond to geometric units of U up to an automorphism of $\pi_1(\mathbb{G}_m/k)$.

Proof: We have $(T\,\mathrm{Pic}^0_X)(k) = \mathrm{Hom}\left(\mathbb{Q}/\mathbb{Z}, \mathrm{Pic}^0_X(k)\right) = 0$. Moreover, the map γ in Proposition 48 is injective by the assumption on $\mathrm{Pic}^0_X(k)$. The corollary now follows from the diagram in Proposition 48. Indeed, we have $\widehat{k^*} \subseteq \mathcal{O}^*(\pi_1(U/k))$ and

$$\mathcal{O}^*(\pi_1(U/k))/\widehat{k^*} = \mathrm{res}^{-1}(\varepsilon \cdot \tau(\mathrm{Div}^0_Y(X)) = \kappa(\mathcal{O}^*(U))^\varepsilon$$

by definition and a diagram chase exploiting the injectivity of γ. □

Remark 50. The assumption of Corollary 49 are fulfilled trivially if X has genus 0 or by the Mordell–Weil theorem if k is a finitely generated extension of $\mathbb{Q}$.

A unit $f : \pi_1(U/k) \to \pi_1(\mathbb{G}_m/k)$ can be evaluated in a section s of $\pi_1(U/k)$ as follows. The composition $f \circ s$ yields a section of $\pi_1(\mathbb{G}_m/k)$, hence by comparing with the section s_1 an element

$$f(s) \in \widehat{k^*} = \mathrm{H}^1(k, \hat{\mathbb{Z}}(1))$$

which we call the value of f in s. Composing f with an automorphism $\alpha\varepsilon$ of $\pi_1(\mathbb{G}_m/k)$ changes the value to $\big(\alpha\varepsilon.f\big)(s) = \alpha f(s)^\varepsilon$.

9. Anabelian geometry of genus 0 curves - revisited

Definition 51. A **cuspidal ratio over** k on the oriented fundamental group $\pi_1(U/k)$ is the value

$$\lambda = \lambda_{f;y,y'} = f(s_y)/f(s_{y'}) \in \widehat{k^*},$$

where $s_y, s_{y'}$ are cuspidal sections at k-rational cusps $y, y' \in \mathrm{Cusps}(U)$ and $f : \pi_1(U/k) \to \pi_1(\mathbb{G}_m/k)$ is a unit such that $\ker(f)$ contains the inertia subgroups $\mathrm{I}_y, \mathrm{I}_{y'}$ at y and y'.

A **cuspidal ratio** of $\pi_1(U/k)$ is a cuspidal ratio over some finite extension k'/k for the base change $\pi_1(U \otimes k'/k')$ with respect to the induced orientation.

The cuspidal ratio $\lambda_{f;y,y'}$ neither depends on the cuspidal section chosen within the corresponding cuspidal packets, nor changes its value when f is composed with a scaling automorphism of $\pi_1(\mathbb{G}_m/k)$.

Proposition 52. *Assume that X has genus 0 or k is a finitely generated extension of $\mathbb{Q}$. If the orientation on $\pi_1(U/k)$ is ε times the standard orientation for some $\varepsilon \in \hat{\mathbb{Z}}^*$ then $\lambda_{f;y,y'}$ is the ε-power of an element in the image of the natural map $k^* \to \widehat{k^*}$.*

Proof: Immediate from Corollary 49. □

From now on we will mainly be preoccupied with curves U which are complements of a divisor Y in a smooth, projective curve X of genus 0. The fundamental group $\pi_1(U/k)$ shall be endowed with its anabelian weight filtration and an orientation.

Definition 53. We say that the fundamental group $\pi_1(U/k)$ has **type** $(0,n)$ if the inertia groups generate the geometric fundamental group $\pi_1(\overline{U})$, all cusps of U are k-rational, and we are given a bijective ordering $y : \{1,\ldots,n\} \to \mathrm{Cusps}(U)$ sending i to y_i.

We remind the reader that the **double ratio** $\mathrm{DV}(a_1,a_2;a_3,a_4)$ **of a 4-tuple** (a_i) in k without repetition is defined by the formula

$$\mathrm{DV}(a_1,a_2;a_3,a_4) = \frac{a_1-a_3}{a_1-a_4} : \frac{a_2-a_3}{a_2-a_4}.$$

Definition 54. The **double ratio of an oriented fundamental group** $\pi_1(U/k)$ of type $(0,4)$ with ordering $y : \{1,2,3,4\} \to \mathrm{Cusps}(U)$ is defined as the cuspidal ratio

$$\mathrm{DV}(\pi_1(U/k),y) = \mathrm{DV}(y_1,y_2;y_3,y_4) = \lambda_{j;y_1,y_2}$$

where j is a unit on $\pi_1(U/k)$ such that $\mathrm{res}(j) = \varepsilon\cdot\tau(y_3 - y_4)$ with the notation as in Proposition 48 and where ε is determined so that $\pi_1(U/k)$ is endowed with ε times the standard orientation.

Remark 55. (1) The double ratio of $\pi_1(U/k)$ is the ε power of an element of the image of $k^* \to \widehat{k^*}$ and independent of the unit j chosen.

(2) Let U_λ be the complement in $\mathbb{P}^1_k$ of $Y = \{\lambda,1,0,\infty\}$ and equip $\pi_1(U_\lambda)$ with an ordering of the cusps accordingly and ε times the standard orientation. The resulting fundamental group of type $(0,4)$ has double ratio

$$\mathrm{DV}(\pi_1(U_\lambda/k),y) = \lambda^\varepsilon = \mathrm{DV}(\lambda,1;0,\infty)^\varepsilon,$$

where this follows from Corollary 49 as j may be chosen to be $\varepsilon\cdot\pi_1(t)$ where t is the coordinate on $\mathbb{P}^1_k$ and then $j(s_\lambda) = \lambda^\varepsilon$ and $j(s_1) = 1$.

If $\varepsilon = 1$ and so $\pi_1(U_\lambda/k)$ is equipped with the standard orientation then the following two double ratios add up to 1:

$$\mathrm{DV}(\lambda,1;0,\infty) + \mathrm{DV}(\lambda,0;1,\infty) = \lambda + (1-\lambda) = 1.$$

Definition 56. A **geometric subquotient** of a fundamental group $\pi_1(U/k)$ is the quotient of an open subgroup by the normal subgroup generated by a collection of inertia subgroups.

A **geometric subquotient of type** $(0,n)$ is a geometric subquotient together with a choice of a structure of a fundamental goup of type $(0,n)$ on that subquotient.

A geometric subquotient of $\pi_1(U/k)$ inherits naturally an orientation from an orientation on $\pi_1(U/k)$. The factor ε of the orientation remains the same.

Let U be the complement of a divisor Y in a smooth, projective curve X/k of genus 0. In this context we define the following list of hypothesis on the curve U and the field k.

(A) For any pair of prime numbers $p \neq q$ there is a valuation ν on k which is nontrivial on $F = k \cap \mathbb{Q}^{\mathrm{alg}}$ and with value group $\Gamma \subset \mathbb{Q}$ that is neither divisible by p nor q.

(B) There is a discrete valuation ν on k with value group $\mathbb{Z}$ such that U does not have good reduction over the valuation ring of ν, in the sense that in any smooth model of X some cusps of U coalesce.

(C) The field k is a function field over k_0 and U is not defined over an algebraic extension of k_0.

(D) The map $k_1^* \to \widehat{k_1^*}$ is injective for every finite extension k_1/k.

Note that (A) and (D) only depend on the base field k, but not on the curve U.

The standard orientation is subject to the following anabelian characterization.

Theorem 57 (Characterization of the standard orientation)**.** *Let U be the complement of a divisor Y in a smooth, projective curve X/k of genus 0. Let $\pi_1(U/k)$ be equiped with ε times the standard orientation. We will assume that property (A) or property (C) or properties (B) and (D) as above hold.*

Then the standard orientation, so $\varepsilon = 1$, is the only orientation such that the following two conditions hold

(i) *For all geometric subquotients of type $(0,4)$ with field of constants a finite extension k_1/k, the double ratio is in the image of $k_1^* \to \widehat{k_1^*}$, and more precisely*

(ii) *a suitable choice of preimages in k_1^* of $\mathrm{DV}(y_1, y_2; y_3, y_4)$ and $\mathrm{DV}(y_1, y_3; y_2, y_4)$ add up to 1.*

Proof: We prove first that $\varepsilon \in \{\pm 1\}$. By Belyi's theorem, the fundamental group of the curve $U_\lambda = \mathbb{P}^1 - \{\lambda, 1, 0, \infty\}$ over $k(\lambda)$ occurs for all $\lambda \in \mathbb{Q}^{\mathrm{alg}} - \{0,1\}$ as a geometric quotient of type $(0,4)$ of $\pi_1(U/k)$, at least if we allow some finite extension k_1/k. We obtain $\mu \in k_1^*$ with $\lambda^\varepsilon = \mu$ in $\widehat{k_1^*}$.

The assumption (A) of the suitable valuations on k transfers to the finite extension k_1. Choose a valuation ν suitable for p and q, choose $\lambda \in \mathbb{Q}^{\mathrm{alg}} - \{0,1\}$ with $\gamma = \nu(\lambda) \neq 0$ and set $\delta = \nu(\mu)$. The valuation extends to a map $\widehat{\nu} : \widehat{k_1^*} \to \widehat{\Gamma} \to \mathbb{Z}_p \times \mathbb{Z}_q$. The relation $\lambda^\varepsilon = \mu$ becomes $\varepsilon \cdot \gamma = \delta$ in $\mathbb{Z}_p \times \mathbb{Z}_q$, where moreover γ and δ come from the diagonally embedded $\mathbb{Q} \cap (\mathbb{Z}_p \times \mathbb{Z}_q)$. Thus, since $\gamma \neq 0$, the p and q component of ε are rational and agree. Applying this argument for all pairs p, q we find that $\varepsilon \in \hat{\mathbb{Z}}^*$ is in the diagonally embedded $\mathbb{Q} \cap \hat{\mathbb{Z}}^* = \{\pm 1\}$.

Under the assumptions (B) or (C), we find, upon restriction to a finite field extension k_1/k that makes the cusps of U rational, even a quotient of type $(0,4)$ with double ratio not a unit with respect to an extension of the valuation ν to k_1. The same but simpler argument with $\widehat{\nu} : \widehat{k_1^*} \to \hat{\mathbb{Z}}$ as above shows that $\varepsilon \in \{\pm 1\}$.

For the remaining part we assume that $\varepsilon = -1$ and argue by contradiction. For one of the geometric subquotients of type $(0,4)$ considered above, let λ be the double ratio in the geometric sense. Then by assumption we find a, b in the group $(k_1^*)_{\mathrm{div}}$ of divisible elements of k_1^* such that

$$\frac{a}{\lambda} + \frac{b}{1-\lambda} = 1. \tag{9.1}$$

Under assumption (D) we have $a = b = 1$ and thus $\lambda = \zeta_6^{\pm 1}$ is a primitive 6^{th} root of unity, which we can avoid by the ubiquity of subquotients of type $(0,4)$ provided by Belyi's theorem.

Property (C) implies that a and b belong to the constants $k_0 \subset k$ forcing λ to be a constant. But the assumption that U is not defined over the constants enables us to find a geometric subquotient of type $(0,4)$ with nonconstant double ratio, hence a contradiction.

Finally, let us assume property (A). Because the value group of a valuation ν as in (A) is not divisible, we find $\nu(a) = \nu(b) = 0$. If we pick $\lambda \in \mathbb{Q}^{\mathrm{alg}} - \{0,1\}$ with $\nu(\lambda) > 0$, then $\nu(1-\lambda) = 0$ and

$$\nu\Big(\frac{a}{\lambda} + \frac{b}{1-\lambda}\Big) < 0$$

in contradiction to (9.1). $\square$

Theorem 58 (Preserving orientation)**.** *Let U and V be hyperbolic curves with smooth, projective completions of genus 0. We will assume that property (A), or property (C), or properties (B) and (D) from Theorem 57 hold.*

Then a weight preserving isomorphism $\varphi : \pi_1(U/k) \to \pi_1(V/k)$ automatically preserves the standard orientation.

Proof: As the isomorphism φ preserves the anabelian weight filtration it also preserves the notion of a geometric subqotient of type $(0,4)$. Let $\lambda \in k^*$ be the double ratio of a geometric subqotient of type $(0,4)$ of $\pi_1(U/k)$ and $\mu \in k^*$ the double ratio of its companion for $\pi_1(V/k)$. Then the equation $\lambda^{\deg(\varphi)} = \mu$ holds in $\widehat{k^*}$. As the same λ's as in Theorem 57 are available we may argue as in the proof of Theorem 57 and conclude that $\deg(\varphi) = 1$. □

Remark 59. (1) A version of Theorem 58 for proper hyperbolic curves over p-adic local fields was obtained in [9] Lemma 9.1, and also in [12] Lemma 2.5(ii).

(2) Condition (A) is met by algebraic extensions $k/\mathbb{Q}$ with $k\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}^{\mathrm{ab}}$ finite, in particular by $\mathbb{Q}^{\mathrm{ab}}$, cf. Remark 22, or by finitely generated extensions of $\mathbb{Q}_p$.

(3) If k is a function field over k_0 which is algebraically closed and $U = U_0 \otimes_{k_0} k$, then $\pi_1(U) = \pi_1(\overline{U}) \times \mathrm{Gal}_k$ and the Galois action has nothing to say about the orientation chosen.

Lemma 60. *Let k be an algebraic extension of $\mathbb{Q}$ such that $k\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}^{\mathrm{ab}}$ finite. Then the group $(k^*)_{\mathrm{div}}$ of divisible elements in k^* is contained in its torsion group of roots of unity $\mu_\infty(k)$.*

Proof: Let first F be a finite extension of $\mathbb{Q}$. An element $a \in (F^*)_{\mathrm{div}}$ has trivial valuation everywhere and thus is a unit. The same applies to $\sqrt[n]{a}$ and thus $a \in (\mathfrak{o}_F^*)_{\mathrm{div}}$ where $\mathfrak{o}_F^*$ is the group of units of the ring of integers in F. By Dirichlet's unit theorem $(\mathfrak{o}_F^*)_{\mathrm{div}} = 1$ hence $(F^*)_{\mathrm{div}} = 1$.

We now argue by contradiction and assume $a \in (k^*)_{\mathrm{div}}$ is not a root of unity. Choose a subfield $F \subseteq k$ that is finite over $\mathbb{Q}$ containing a and such that $F\mathbb{Q}^{\mathrm{ab}} = k\mathbb{Q}^{\mathrm{ab}}$. Then there is a prime number $p \geq 3$ such that $\sqrt[p]{a} \notin F$ and consequently the field

$$F_\infty = \bigcup_{n\geq 1} F(\sqrt[p^n]{a}) \subset k\mathbb{Q}^{\mathrm{ab}} = F\mathbb{Q}^{\mathrm{ab}}$$

is an abelian $\mathbb{Z}_p$ extension of F. By the group theory of

$$\mathrm{Gal}(F\mathbb{Q}^{\mathrm{ab}}/F) \subseteq \hat{\mathbb{Z}}^*,$$

the field F_∞ is already contained in $\bigcup_{n\geq 1} F(\mu_{p^n})$.

Let $a = \alpha^{p^n}$ in $F(\mu_{p^{n+m}})$. We apply [18] Lemma (5.7), which gives the injectivity of

$$F^*/(F^*)^{p^N} \hookrightarrow F(\mu_{p^N})^*/\big(F(\mu_{p^N})^*\big)^{p^N}$$

for $N = n+m$. We find $\alpha_0 \in F$ with $a^{p^m} = \alpha^{p^{n+m}} = \alpha_0^{p^{n+m}}$. So $a\zeta$ is divisible by p^n in F^* for a suitable root of unity $\zeta \in F$. As $\mu_\infty(F)$ is finite, one ζ is sufficient for all n and with this choice we find $a\zeta \in \bigcap_n (F^*)^{p^n} \subset \mu_\infty(F)$, hence a was torsion to begin with. □

Corollary 61. *We have $(\mathbb{Q}^{\mathrm{ab},*})_{\mathrm{div}} = \mu_\infty$.* □

Lemma 62. *Let $\lambda = \mathrm{DV}(\zeta_1, \zeta_2; \zeta_3, \zeta_4)$ be a double ratio of roots of unity ζ_i. Then for any p the p-adic absolute value of λ satisfies $|\lambda|_p \leq 4$. If moreover $\lambda \in \mathbb{Q}$, then λ is among the following values:*

$$2, -1, 1/2; \quad 3, 1/3, -2, -1/2, 2/3, 3/2; \quad 4, 1/4, -3, -1/3, 4/3, 3/4.$$

Proof: For ζ a primitive n^{th}-root of unity, the value $\zeta - 1$ is a unit except if $n = p^m$ and then it is a uniformizer in $\mathbb{Q}_p(\mu_{p^m})$. We conclude that $1 \geq |\zeta - 1|_p \geq p^{-1/p^{m-1}(p-1)} \geq 1/2$, hence

$$|\lambda|_p = |\mathrm{DV}(\zeta_1, \zeta_2; \zeta_3, \zeta_4)|_p$$

$$= \left| \frac{\zeta_1 - \zeta_3}{\zeta_1 - \zeta_4} : \frac{\zeta_2 - \zeta_3}{\zeta_2 - \zeta_4} \right|_p = \left| \frac{1 - \zeta_3\zeta_1^{-1}}{1 - \zeta_4\zeta_1^{-1}} : \frac{1 - \zeta_3\zeta_2^{-1}}{1 - \zeta_4\zeta_2^{-1}} \right|_p \leq 4$$

For $\mathbb{Q}$-rational double ratios of roots of unity we observe that we have a natural S_3 action on the set of values including $\lambda \mapsto 1/\lambda$, and that by the estimate above the only prime factors that may occur are 2 and 3 with only 2 possibly occuring twice. Imposing this on the full S_3 orbit already pins down λ into the given list, whose values by the way are all attained for suitable roots of unity. □

Finally, we can prove a weak anabelian statement for hyperbolic curves with completion of genus 0.

Theorem 63. *Let U and V be hyperbolic curves with smooth, projective completion of genus 0. We will assume that property (C), or properties (B) and (D) from Theorem 57 hold, or property (A'):*

(A') *The field k is algebraic over $\mathbb{Q}$ with $k\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}^{\mathrm{ab}}$ finite.*

Then $U \cong V$ are isomorphic as k-curves if and only if there is an isomorphism $\pi_1(U/k) \xrightarrow{\sim} \pi_1(V/k)$ that respects the anabelian weight filtration.

Remark 64. Theorem 63 applies in particular to hyperbolic curves with completion of genus 0 over $\mathbb{Q}^{\mathrm{ab}}$ or to nonconstant hyperbolic curves with completion of genus 0 over a function field. The latter should be useful in applications to birational anabelian geometry for function fields over algebraically closed fields of charactersitic 0.

Proof of Theorem 63: By Galois descent as established by Nakamura in [15] Theorem 6.1, we may replace k by a finite extension and therefore assume that all cusps are k-rational.

Recall from Remark 59 (2) that (A') implies (A). Therefore, by Theorem 58, any isomorphism $\pi_1(U/k) \xrightarrow{\sim} \pi_1(V/k)$ automatically preserves the standard orientation. We may therefore in a compatible way compute double ratios.

Let $\varphi : \pi_1(U/k) \xrightarrow{\sim} \pi_1(V/k)$ be a weight preserving isomorphism. Then as in the remark after Definition 25 we obtain an induced bijection

$$\varphi : \mathrm{Cusps}(U) \xrightarrow{\sim} \mathrm{Cusps}(V).$$

If U is of type $(0,3)$, then V is of type $(0,3)$ and there is nothing to prove. Let $y_1, y_2, y_3, y_4 \in \mathrm{Cusps}(U)$ be a 4-tuple with $DV(y_1, y_2; y_3, y_4) = \lambda$, and set $\mu = DV(\varphi(y_1), \varphi(y_2); \varphi(y_3), \varphi(y_4))$ as double ratios of the respective quotient fundamental groups of curves of type $(0,4)$. A priori λ, μ are elements of $\widehat{k^*}$, but there are representatives in k^* such that we have open embeddings $U \subseteq U_\lambda$ and $V \subseteq U_\mu$, with notation as above, preserving the chosen cusps. As the kernel of $\pi_1(U) \twoheadrightarrow \pi_1(U_\lambda)$ is generated by the inertia groups at all the other cusps, and similarly for the kernel of $\pi_1(V) \twoheadrightarrow \pi_1(U_\mu)$, then the map φ induces a weight preserving isomorphism

$$\varphi : \pi_1(U_\lambda/k) \xrightarrow{\sim} \pi_1(U_\mu/k),$$

which on cusps maps $0 \mapsto 0$, $1 \mapsto 1$, $\infty \mapsto \infty$, and $\lambda \mapsto \mu$. If we can treat the case of curves of type $(0,4)$, then we deduce $\lambda = \mu$. We fix the cusps y_2, y_3, and y_4, and let $y = y_1$ range over all remaining cusps of U with λ_y the corresponding double ratio. Then we find

$$U \cong \bigcap_{y \in \mathrm{Cusps} \setminus \{y_2, y_3, y_4\}} U_{\lambda_y} \subset \mathbb{P}^1_k,$$

and since the same holds for V, we deduce that $U \cong V$ as k-curves.

It remains to discuss the case of genus 0 with 4 cusps where moreover both curves carry compatible structures as a fundamental group of type $(0,4)$, say $U \cong U_\lambda$ and $V \cong U_\mu$ with notations as above. We can then compare various double ratios and find $\lambda = \mu \cdot a$ and $1 - \lambda = (1 - \mu) \cdot b$ with $a, b \in (k^*)_{\mathrm{div}}$. It follows that $\lambda = \mu$ or $a \neq b$ and

$$\lambda = \frac{a - ab}{a - b} = \mathrm{DV}(a, \infty; ab, b) \qquad \mu = \frac{b - 1}{b - a} = \mathrm{DV}(b, \infty; 1, a), \tag{9.2}$$

and so we have to contradict that $\lambda \neq \mu$ are double ratios of divisible elements of k^*. Under condition (D) this is absurd.

Under condition (C) the divisible elements are constants and thus also λ and μ are constants, a contradiction to condition (C).

Under condition (A') we replace U and V by suitable finite étale covers $U' \to U$ of genus 0 and the corresponding $V' \to V$ with again all cusps rational after a finite extension k'/k such that at least one of the double ratios of quotients of $\pi_1(U'/k')$ of type $(0,4)$ is not a double ratio of divisible elements. This is possible by using Belyi's theorem and choosing a suitable $\lambda \in \mathbb{Q}^{\mathrm{alg}} - \{0,1\}$ as in the proof of Theorem 57 and by Lemma 62 and Lemma 60. $\square$

10. Anabelian geometry of $\mathscr{M}_{0,4}$ and the j-invariant

The algebraic stack $\mathscr{M}_{0,4}$, actually a scheme as we will recall shortly, parameterizes pairs

$$(X/S; \underline{y}) \in \mathscr{M}_{0,4}(S)$$

where X/S is a smooth, projective S-curve of genus 0 with 4 marked points

$$\underline{y} = (y_1, y_2, y_3, y_4)$$

in $X(S)$ with pairwise disjoint images. The double ratio of a marked curve $(X/S, \underline{y})$ is

$$\mathrm{DV}(\underline{y}) = \frac{y_1 - y_3}{y_1 - y_4} : \frac{y_2 - y_3}{y_2 - y_4} \in \mathcal{O}^*(S),$$

where the values y_i are taken with respect to and are independent of any choice of parameter on $X \cong \mathbb{P}^1$ locally on S. The double ratio map provides an isomorphism

$$\mathrm{DV} : \mathscr{M}_{0,4} \xrightarrow{\sim} \mathbb{P}^1 - \{0,1,\infty\},$$

since by [7] Thm 2.7 we have $\mathscr{M}_{0,3} = \mathrm{Spec}(\mathbb{Z})$ and by [7] Cor 2.6 the stack $\mathscr{M}_{0,4}$ is the complement of the marked points $0,1,\infty$ in the universal curve $\mathbb{P}^1_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z})$ for $\mathscr{M}_{0,3}$. In fact, it is not hard to show directly that every $(X/S, \underline{y}) \in \mathscr{M}_{0,4}(S)$ is uniquely isomorphic as a marked curve to the pull back via $\mathrm{DV}(\underline{y}) : S \to \mathbb{P}^1 - \{0,1,\infty\}$ of the universal curve

$$\mathrm{pr}_2 : \mathscr{C}_{0,4} = \mathbb{P}^1 \times (\mathbb{P}^1 - \{0,1,\infty\}) \to \mathbb{P}^1 - \{0,1,\infty\}$$

that is equipped with the marked points $(\Delta, 1, 0, \infty)$ where Δ is the diagonal map. Indeed, let $f : X \to S$ be the projection map and let by abuse of notation denote $1, 0, \infty$ the image divisors of y_2, y_3, y_4 on X. Then cohomology and base change shows that evaluation at $0, 1$ provides an isomorphism

$$f_*\mathcal{O}_X(\infty) \xrightarrow{\sim} \mathcal{O}_S \oplus \mathcal{O}_S.$$

Let $x_0, x_1 \in \mathrm{H}^0(X, \mathcal{O}_X(\infty))$ be the sections mapped under the above isomorphism to $(1,1), (0,1) \in \mathrm{H}^0(S, \mathcal{O}_S \oplus \mathcal{O}_S)$. Then x_0, x_1 locally generate $\mathcal{O}_X(\infty)$ and the induced map $X \to \mathbb{P}^1_S$ is a fibrewise isomorphism, hence by flatness an isomorphism that moreover maps y_1, y_2, y_3, y_4 to $\lambda, 1, 0, \infty$. The uniqueness is clear by the exact 3-transitivity of the PGL_2 action on $\mathbb{P}^1$.

10.1. On the section conjecture for $\mathscr{M}_{0,4}$.

The map SC from rational points to conjugacy classes of sections of the section conjecture, see Section 1, for $\mathscr{M}_{0,4}$ factors as follows.

$$\begin{tikzcd}
\mathscr{M}_{0,4}(k) \arrow[rr, "\mathrm{SC}"] \arrow[dr, "\pi_1"'] \arrow[dd, "\cong"', "\mathrm{DV}"] & & \mathscr{S}_{\pi_1(\mathscr{M}_{0,4}/k)} \arrow[dd, "\cong"', "\mathrm{DV}"] \\
& \left\{\begin{array}{c}\text{fundamental groups}\\ \pi_1(U/k) \text{ of type } (0,4)\\ \text{with anabelian}\\ \text{weight structure}\end{array}\right\} \arrow[dr, "\Theta"] & \\
\big(\mathbb{P}^1 - \{0,1,\infty\}\big)(k) \arrow[rr, "\mathrm{SC}"] & & \mathscr{S}_{\pi_1(\mathbb{P}^1_k - \{0,1,\infty\}/k)}
\end{tikzcd}$$

The map $\pi_1 : (X/k, \underline{y}) \mapsto \pi_1(U/k)$ with $U = X \setminus \{\underline{y}\}$ and equipped with the specified extra structure is surjective by definition and bijective if curves of type $(0,4)$ over k are anabelian in a weak sense.

The map Θ has an anabelian definition as follows. The quotient of $\pi_1(U/k)$ by the normal subgroup generated by the inertia subgroups above y_1 is geometric and canonically isomorphic to $\pi_1(\mathbb{P}^1_k - \{0,1,\infty\}/k)$. A cuspidal section associated to y_1 of $\pi_1(U/k)$ maps to a unique geometric section $\Theta(\pi_1(U/k))$ of $\pi_1(\mathbb{P}^1_k - \{0,1,\infty\}/k)$. The diagram commutes, because the curve $\left(\mathbb{P}^1_k, \{\lambda, 1, 0, \infty\}\right)$ maps to the section s_λ of $\pi_1(\mathbb{P}^1_k - \{0,1,\infty\}/k)$, where $\lambda = \mathrm{DV}(\lambda, 1; 0, \infty)$ equals the double ratio of the k-point $\left(\mathbb{P}^1_k, \{\lambda, 1, 0, \infty\}\right) \in \mathscr{M}_{0,4}(k)$.

Consider the map $\mathscr{M}_{0,5} \to \mathscr{M}_{0,4}$ that forgets the fifth point. We can identify $\mathscr{M}_{0,5}$ with the complement of the 4 marked points in the universal curve $\mathscr{C}_{0,4} \to \mathscr{M}_{0,4}$ of type $(0,4)$. We abreviate by $\overline{\pi}_{0,4}$ the fundamental group of the complement of the four marked points in a chosen geometric fibre of $\mathscr{M}_{0,5} \to \mathscr{M}_{0,4}$. There is an extension, see [8] Lemma 2.1,

$$1 \to \overline{\pi}_{0,4} \to \pi_1(\mathscr{M}_{0,5}) \to \pi_1(\mathscr{M}_{0,4}) \to 1. \tag{10.1}$$

We endow $\mathscr{C}_{0,4}$ with the regular log structure defined by the divisor given by the 4 universal sections. Then $\mathscr{M}_{0,5} = \mathscr{C}_{0,5}^{\mathrm{triv}}$ is the locus of triviality of the log structure and thus by purity $\pi_1(\mathscr{M}_{0,5}) = \pi_1^{\log}(\mathscr{C}_{0,4})$. The pullback along the section y_i defines a log structure M_i on $\mathscr{M}_{0,4}$ with constant rank 1. We obtain a diagram of (logarithmic) fundamental groups

$$\begin{tikzcd}
1 \arrow[r] & \hat{\mathbb{Z}}(1) \arrow[r] \arrow[d, "y_i"] & \pi_1^{\log}(\mathscr{M}_{0,4}, M_i) \arrow[r] \arrow[d, "\pi_1(y_i)"] & \pi_1(\mathscr{M}_{0,4}) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \overline{\pi}_{0,4} \arrow[r] & \pi_1(\mathscr{M}_{0,5}^{\mathrm{triv}}) \arrow[r] & \pi_1(\mathscr{M}_{0,4}) \arrow[r] & 1,
\end{tikzcd}$$

where y_i maps $\hat{\mathbb{Z}}(1)$ injectively onto an inertia group above y_i and the extension in the top row is the first Chern class extension of $y_i^*\left(\mathcal{O}(y_i)\right)$. The top row splits as $\mathrm{Pic}(\mathscr{M}_{0,4}) = 1$. Any splitting leads to a section of the bottom row which cyclotomically normalises the inertia at y_i and thus is a cuspidal section at y_i.

As a converse map to Θ we define for a section $s \in \mathscr{S}_{\pi_1(\mathscr{M}_{0,4}/k)}$ the extension $\prod_s$ as the pullback via s of the universal extension (10.1), which is an extension of Gal_k by $\overline{\pi}_{0,4}$ endowed with an anabelian weight filtration and 4 ordered cuspidal sections. It is not difficult to see that $s \mapsto \prod_s$ restricted to geometric sections and Θ are mutually inverse. It is tempting to believe that a more refined study of (10.1) may even give a proof of the section conjecture for $\mathscr{M}_{0,4}$.

10.2. **Proper units.** In order to discuss the case of higher genus, in particular elliptic curves, we need to address again Proposition 48. A cuspidal ratio depends only on the restriction of the unit to its divisor in

$$\mathrm{Div}^0_Y(X) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} = \mathrm{Hom}_k\left(W_{-2}(\pi_1^{\mathrm{ab}}(\overline{U})), \hat{\mathbb{Z}}(1)\right)$$

up to a cuspidal ratio that comes from a unit in the image of

$$\left(T\,\mathrm{Pic}^0_X\right)(k) \subset \hat{\mathcal{O}}^*\left(\pi_1(U/k)\right)/\widehat{k^*},$$

which we choose to call **proper units** and which are constructed as follows. To an $L \in \left(T\,\mathrm{Pic}^0_X\right)(k)$ belongs by the Weil-pairing a Galois invariant map $\pi_1^{\mathrm{ab}}(\overline{X}) \to \pi_1(\overline{\mathrm{Alb}}_X) \to \hat{\mathbb{Z}}(1)$, which defines a map of semidirect products $\pi_1(\mathrm{Alb}_X/k) \to \pi_1(\mathbb{G}_m/k)$ and thus the proper unit

$$f_L : \pi_1(U/k) \to \pi_1(X/k) \to \pi_1(\mathrm{Alb}_X/k) \to \pi_1(\mathbb{G}_m/k)$$

which has trivial divisor. Using Kummer theory for Alb_X we get a pairing

$$\left(T\,\mathrm{Pic}^0_X\right)(k) \times \widehat{\mathrm{Alb}_X(k)} \to \widehat{k^*}$$

that maps $(L, y'-y)$ to the cuspidal ratio $\lambda_{f_L;y',y}$. This immediately yields the following lemma.

Lemma 65. *The cuspidal ratio of a proper unit with respect to cusps that differ by n-torsion in Alb_X takes values in n-torsion of $\widehat{k^*}$.* □

Lemma 66. *The n-torsion of $\widehat{k^*}$ is generated by the image of $\mu_n(k)$.*

Proof: Let $a \in \widehat{k^*}$ with representatives $a_r \bmod (k^*)^r$ be n-torsion, i.e., for $r \in \mathbb{N}$ we have $\alpha_r \in k^*$ such that $a_{r+n}^n = \alpha_r^{r+n}$. For some $\zeta \in \mu_n(k)$ depending on r we thus have $a_{r+n}\zeta = \alpha_r^r$. As there are only finitely many n^{th} roots of unity, one ζ will do it for all r. Then $a_r\zeta \equiv a_{r+n}\zeta \equiv 1$ modulo $(k^*)^r$ shows that $a = \zeta^{-1}$. □

10.3. **The j-invariant.** We are going to describe the j-invariant[5]. of an elliptic cuvre E with origin $e \in E$ in terms of $\pi_1(E - \{e\}/k)$ under mild assumptions on k.

Theorem 67. *Let ℓ be a prime number and let k be a field such that the ℓ-adic cyclotomic character is non-Tate. Let E/k be an elliptic curve with origin $e \in E(k)$, and let k'/k be the minimal finite extension such that the 2-torsion*

$$E[2] = \{P_0, P_1, P_\infty, e\}$$

of E is k'-rational. We further assume that one of the following holds:

(i) *$k'^* \hookrightarrow \widehat{k'^*}$ is injective, or*

(ii) *k/k_0 is a function field and E is not a twist of an elliptic curve defined over k_0.*

Then the j-invariant of E is encoded group theoretically in the fundamental group extension $\pi_1(E - \{e\}/k)$ as

$$j(\pi_1(E - \{e\}/k)) = j(\lambda) = 1728 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1-\lambda)^2} \in k$$

where $\lambda \in k'$ is the unique element such that, for units

$$x_i : \pi_1(E \otimes k' - E[2]/k') \to \pi_1(\mathbb{G}_m/k')$$

with divisor $2P_i - 2e$ and $i \in \{0, 1, \infty\}$, the equations

$$\lambda_{x_0;P_\infty,P_1} = \pm\lambda$$

$$\lambda_{x_1;P_\infty,P_0} = \pm(1-\lambda)$$

describe these cuspidal ratios in $\widehat{k'^*}$ up to sign.

Remark 68. The notion of j being *encoded group theoretically* means that all the data required to describe j as an element of k can be extracted from $\pi_1(E - \{e\}/k)$ as an extension of Gal_k in group theoretical terms, i.e., forgetting that we have the underlying punctured elliptic curve.

Proof: By Corollary 32 the assumptions on k suffice to characterise cuspidal sections group theoretically, even after finite scalar extension, and thus also characterise the Gal_k-set of cusps.

We consider $\pi_1(E - \{e\}/k)$ as equipped with a choice of cuspidal section s_e based at e. The finite étale cover

$$E - E[2] \ \to \ E - \{e\}$$

which is given by multiplication by 2 is characterized as the unique $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-cover which is unramified over E together with a choice of a lift of s_e. The set of cusps of $E - E[2]$ as a Gal_k-set thus has a distinguished element, again denoted e and three other elements P_0, P_1 and P_∞. Let k'/k be the smallest Galois extension such that all cusps of $E - E[2]$ are rational over k', i.e., such that the Gal_k-set of cusps of $\pi_1(E - E[2]/k)$ has trivial $\mathrm{Gal}_{k'}$-action.

Fixing the points of order 2 imposes on $E \otimes k'$ the structure of a twist of the Legendre form as

$$E \otimes k' = \{cY^2 = X(X-1)(X-\lambda)\}$$

such that $P_0 = (0,0), P_1 = (1,0)$ and $P_\infty = (\lambda, 0)$ for some $\lambda \in k'^* - \{1\}$ and e is the point at infinity. Let x_i be a unit on $\pi_1(E \otimes k' - E[2]/k')$ with divisor $2P_i - 2e$. The functions X, $1 - X$, and $\lambda - X$ are all units on $E \otimes k' - E[2]$, and we have $x_0 = \pi_1(X)$, $x_1 = \pi_1(1-X)$ and $x_\infty = \pi_1(\lambda - X)$ up to a multiplicative constant and a proper unit.

Since the unit x_i is only well defined up to a proper unit and a multiplicative constant, and since we evaluate with respect to cusps that differ by 2-torsion, by Lemma 65 and Lemma 66 the cuspidal ratio

$$\lambda_{x_0;P_\infty,P_1} = x_0(P_\infty)/x_0(P_1)$$

[5]The author acknowledges a discussion with Hiroaki Nakamura held in 2005 where the question how to recover the j-invariant was addressed while the author enjoyed the hospitality of the University of Okayama.

considered as an element in $\widehat{k'^*}/\pm 1$ is well defined independently of the choice of x_0. With the λ from the Legendre form we find

$$\lambda = \lambda_{x_0;P_\infty,P_1} \in \widehat{k'^*}/\pm 1$$

so that in particular the cuspidal ratio takes values in

$$k'^*/\pm k'^*_{\mathrm{div}} \subseteq \widehat{k'^*}/\pm 1.$$

Similarly, the cuspidal ratio of x_1 with respect to P_∞ and P_0 gives

$$1-\lambda = \lambda_{x_1;P_\infty,P_0} \in k'^*/\pm k'^*_{\mathrm{div}}.$$

It remains to show uniqueness of λ. Arguing by contradiction, let us assume that $\mu \in k'^* - \{1\}$ with $\mu \neq \lambda$ also gives rise to these values of cuspidal ratios in $k'^*/\pm k'^*_{\mathrm{div}}$. Then there are elements $\zeta, \xi \in \pm k'^*_{\mathrm{div}}$ such that

$$\lambda = \zeta\mu \quad \text{ and } \quad 1-\lambda = \xi(1-\mu).$$

Therefore $\zeta \neq \xi$ and

$$\lambda = \frac{\zeta - \zeta\xi}{\zeta - \xi} = \mathrm{DV}(\zeta^{-1}, \infty; 1, \xi^{-1}) \qquad \mu = \frac{1-\xi}{\zeta-\xi} = \mathrm{DV}(\xi, \infty; 1, \zeta)$$

are double ratios of elements from $\pm k'^*_{\mathrm{div}} \cup \{\infty\}$. Under assumption (i) we have $k'_{\mathrm{div}} = 1$ and thus $\{\zeta, \xi\} = \{1, -1\}$ showing that $\mu = \lambda$. Under assumption (ii) the set $\pm k'^*_{\mathrm{div}}$ is algebraic over k_0, hence λ is algebraic over k_0, therefore $j(E) \in k_0$ and E/k is a twist of an elliptic curve defined over k_0, contradicting assumption (ii). □

Remark 69. A similar construction gives the moduli parameters for a hyperelliptic curve X over k from the fundamental group extension $\pi_1(U/k)$ of the complement U in X of the branch locus of the hyperelliptic double cover $X \to \mathbb{P}^1_k$. In fact, the moduli parameters in question are the double ratios of 4-tuples of branch points.

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