Étale contractible varieties in positive characteristic

ARMIN HOLSCHBACH, JOHANNES SCHMIDT, AND JAKOB STIX

Abstract — Unlike in characteristic 0, there are no non-trivial smooth varieties over an algebraically closed field $k$ of characteristic $p > 0$ that are contractible in the sense of étale homotopy theory.

INTRODUCTION

Homotopy theory is founded on the idea of contracting the interval, either as a space, or as an actual homotopy, i.e., a path in a space of maps. In algebraic geometry, the affine line $\mathbb{A}_k^1$ serves as an algebraic equivalent of the interval, at least in characteristic 0.

Matters differ in characteristic $p > 0$ where $\pi_1(\mathbb{A}_k^1)$ is an infinite group: a group $G$ occurs as a finite quotient of $\pi_1(\mathbb{A}_k^1)$ precisely if $G$ is a quasi-$p$ group due to Abhyankar’s conjecture for the affine line as proven by Raynaud. This raises the question whether there is an étale contractible variety in positive characteristic.

Theorem 1. Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $U/k$ be a smooth variety. Then $U$ is étale contractible, if and only if $U = \text{Spec}(k)$ is the point.

We recall the relevant terminology and results of étale homotopy from [AM69].

Definition 2. Let $k$ be an algebraically closed field.
(1) A variety $U$ over $k$ is (étale) contractible if the map $U_{\text{ét}} \to \text{Spec}(k)_{\text{ét}}$ is a weak equivalence of its étale homotopy type $U_{\text{ét}}$ with $\text{Spec}(k)_{\text{ét}}$.
(2) A variety $U$ over $k$ is (étale) $n$-connected for an $n \in \mathbb{N}$, if its étale homotopy groups $\pi_i^{\text{ét}}(U)$ vanish for all $i \leq n$.

The étale Hurewicz and Whitehead theorems, see [AM69, §4], lead to the following equivalent characterisations. The variety $U/k$ is contractible if and only if it is $n$-connected for all $n \in \mathbb{N}$. Moreover, for $n \geq 1$, a normal variety $U/k$ is $n$-connected if and only if
(i) $U$ is connected, and
(ii) $U$ is simply connected: $\pi_1^{\text{ét}}(U) = 1$, and
(iii) $U$ is cohomologically acyclic in degrees $\leq n$: for all $0 < i \leq n$ the groups $H_i^{\text{ét}}(U, A)$ vanish for all finite abelian groups $A$.

It turns out that our discussion in positive characteristic uses only $H^1$ and $H^2$, and moreover covers more than just smooth varieties. Here is the more precise result which proves Theorem 1 because smooth varieties are locally factorial and hence locally $\mathbb{Q}$-factorial everywhere.

Theorem 3. Let $k$ be an algebraically closed field of characteristic $p > 0$ and let $U/k$ be a 2-connected normal variety such that one of the following holds:
(a) $U$ is quasi-projective, or
(b) $U$ is locally $\mathbb{Q}$-factorial everywhere, or
(c) $\dim(U) \leq 2$.
Then $U$ has dimension 0.

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$^1$A finite group is a quasi-$p$ group if it is generated by its $p$-Sylow subgroups.
More precisely, we can prove the following Theorem 4 that has Theorem 3 with the assumptions (a) or (b) as a special case. Proposition 6 shows the existence of big Cartier divisors in this case. We recall big Cartier divisors on general varieties in Section §1.

**Theorem 4.** Let \( k \) be an algebraically closed field of characteristic \( p > 0 \) and let \( U/k \) be a connected normal variety with a big Cartier divisor and such that

(i) the group \( H^1_{\text{ét}}(U, \mathbb{F}_p) \) vanishes, and

(ii) there is a prime number \( \ell \neq p \) such that \( H^2_{\text{ét}}(U, \mu_\ell) = 0 \).

Then \( U \) has dimension 0.

The proof of Theorem 3 with the assumption (c) will be completed in Section 3 for normal surfaces, the case of normal curves being covered by assumption (a).

In the proof of Theorem 4 one would like to work with a completion \( U \subseteq X \) and the geometry of line bundles on \( U \) versus \( X \). For that strategy to work, we need a completion that is locally factorial along \( Y = X \setminus U \). Since in characteristic \( p > 0 \) resolution of singularities is presently absent in dimension \( \geq 4 \) we resort to desingularisation by alterations due to de Jong. Unfortunately, the alteration typically destroys the étale contractibility assumption. The strategy consists in first deducing more coherent properties from étale 2-connectedness that can be transferred to the alteration.

The key difference with characteristic 0 comes from Artin–Schreier theory relating \( H^1_{\text{ét}}(U, \mathbb{F}_p) \) to global rational functions.

**Remark 5.** Some further comments on the situation in characteristic 0 in contrast to Theorem 1:

1. In characteristic 0 smoothness seems to be a crucial property to get an interesting classification problem. Indeed, let \( k = \mathbb{C} \), then the affine cone \( U \) over any projective variety has a \( \mathbb{C}^* \)-action with the cone point \( 0 \in U \) as its only attracting fixed point. It follows that \( U(\mathbb{C}) \) is homotopy equivalent to 0, and there are far too many (singular) contractible varieties for a manageable classification.

2. There are contractible complex smooth surfaces other than \( \mathbb{A}^2_{\mathbb{C}} \), the first such example is due to Ramanujam [Ra71, §3], see also tom Dieck and Petrie [tDP90] for explicit equations. All of them are affine and have rational smooth projective completions.

3. Smooth varieties \( U/\mathbb{C} \) different from affine space \( \mathbb{A}^n_{\mathbb{C}} \) but with \( U(\mathbb{C}) \) diffeomorphic to \( \mathbb{C}^n \) are known as exotic algebraic structures on \( \mathbb{C}^n \). These varieties are contractible and we recommend the Bourbaki talk on \( \mathbb{A}^n \) by Kraft [Kr94], or the survey by Zaidenberg [Z99]. A remarkable non-affine (but quasi-affine) example \( U \) was obtained by Winkelmann [Wi90] as a quotient \( U = \mathbb{A}^5/\Gamma \) and more concretely as the complement in a smooth projective quadratic hypersurface in \( \mathbb{P}^5_{\mathbb{C}} \) of the union of a hyperplane and a smooth surface.

**Notation.** We keep the following notation throughout the note: \( k \) will be an algebraically closed field. By definition, a variety over \( k \) is a separated scheme of finite type over \( k \). We will denote the étale fundamental group by \( \pi_1 \) and its maximal abelian quotient by \( \pi_1^{\text{ab}} \). The sheaf \( \mu_\ell \) for \( \ell \) different from the characteristic denotes the (locally) constant sheaf of \( \ell \)-th roots of unity.

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### 1. Big divisors on varieties

Recall that for a Cartier divisor \( D \) on a normal variety \( U/k \), its **Iitaka dimension** is

\[
\kappa(D) = \begin{cases} 
\max_{m \in \mathbb{N}(D)} \dim(\wp_{mD}(U)) & \text{if } \mathbb{N}(D) = \{ m \in \mathbb{N} \mid mD \neq \emptyset \} \neq \emptyset, \\
-\infty & \text{otherwise}.
\end{cases}
\]

Here \( \varphi_{|mD|} \) denotes the rational map associated to the linear system \(|mD|\), and \( \varphi_{|mD|}(U) \) denotes the closure of its image. A Cartier divisor is **big** if

\[
\kappa(D) = \dim(U),
\]

i.e., if \( \varphi_{|mD|} \) is generically finite for \( m \gg 0 \). The pullback of a big divisor under a generically finite morphism is obviously big itself.

**Proposition 6.** Let \( k \) be an algebraically closed field and let \( U/k \) be a variety such that one of the following holds:

(a) \( U \) is quasi-projective.

(b) \( U \) is normal and locally \( \mathbb{Q} \)-factorial everywhere.

Then \( U \) has a big Cartier divisor.

**Proof.** Let us first assume (a). Since any ample divisor is big, the conclusion holds.

If (b) holds, then we first choose a dense affine open \( V \subseteq U \) and an effective big Cartier divisor \( D \) on \( V \) by the above, since \( V \) is quasi-projective. Let \( B = U \setminus V \) be the boundary, in fact a Weil-divisor since \( V \) is affine, and let \( D' \) be the Zariski closure of \( D \) as a Weil divisor on \( U \). By assumption (b) there is an \( m \geq 1 \) such that \( mD' \) and \( mB \) are both effective Cartier divisors and there are sections \( s_0, \ldots, s_d \in H^0(V, mD) \) such that the induced map \( V \to \mathbb{P}^d_k \) is generically finite. For \( r \gg 0 \) the sections \( s_i \) extend to sections of

\[
H^0(U, mD + mrB)
\]

so that \( mD + mrB \) is the desired big Cartier divisor on \( U \).

\[ \square \]

2. **Geometry of varieties with vanishing \( H^1 \) and \( H^2 \)**

2.1. **Line bundles.** Let \( U \) be a variety over \( k \) with \( H^2_{\text{ét}}(U, \mu_\ell) = 0 \) for some prime number \( \ell \) different from the characteristic of \( k \). The Kummer sequence \( 0 \to \mu_\ell \to \mathbb{G}_m \to \mathbb{G}_m \to 0 \) on \( U \) yields in étale cohomology the exact sequence

\[
\text{Pic}(U) \xrightarrow{\ell} \text{Pic}(U) \to H^2_{\text{ét}}(U, \mu_\ell) = 0.
\]

So \( \text{Pic}(U) \) is an \( \ell \)-divisible abelian group.

2.2. **Regular functions.** The following argument crucially depends on \( k \) being a field of positive characteristic.

**Proposition 7.** Let \( k \) be of characteristic \( p > 0 \) and let \( U/k \) be a connected variety such that \( \pi_1^{\text{ab}}(U) \otimes \mathbb{F}_p \) is finite. Then there is no non-constant map \( f : U \to \mathbb{A}^1_k \). In other words

\[
H^0(U, \mathcal{O}_U) = k.
\]

**Proof.** We argue by contradiction and assume that there is a dominant map \( f : U \to \mathbb{A}^1_k \). Then the induced map

\[
f_* : \pi_1^{\text{ab}}(U) \otimes \mathbb{F}_p \to \pi_1^{\text{ab}}(\mathbb{A}^1_k) \otimes \mathbb{F}_p
\]

has image of finite index in the infinite group \( \pi_1^{\text{ab}}(\mathbb{A}^1_k) \otimes \mathbb{F}_p \), a contradiction. \[ \square \]

By the tautological duality \( H^1_{\text{ét}}(U, \mathbb{F}_p) = \text{Hom}(\pi_1^{\text{ab}}(U), \mathbb{F}_p) \) the vanishing of \( H^1_{\text{ét}}(U, \mathbb{F}_p) \) implies the assumption of Proposition 7.
2.3. Using alterations. Clearly, the content of Sections §2.1 and §2.2 reduce the proof of Theorem 4 to the following proposition.

Proposition 8. Let $k$ be an algebraically closed field and let $U/k$ be a connected normal variety with a big Cartier divisor and such that

(i) $H^0(U, \mathcal{O}_U) = k$, and
(ii) there is a prime number $\ell$ such that $\text{Pic}(U)$ is $\ell$-divisible.

Then $U$ has dimension 0.

Proof. By [dJ96, Theorem 7.3], there exists an alteration, i.e., a generically finite projective map $h : \tilde{U} \to U$ such that $\tilde{U}$ can be embedded into a connected smooth projective variety $\tilde{X}$.

Step 1: Since $U$ is normal, the maximal open $V \subset U$ such that the restriction $h|_V : \tilde{V} = \pi^{-1}(V) \to V$ is a finite map has boundary $U \setminus V$ of codimension at least 2.

The $k$-algebra $H^0(\tilde{V}, \mathcal{O}_{\tilde{V}})$ is an integral domain inside the function field of $\tilde{V}$. The minimal polynomial for a section $s \in H^0(\tilde{V}, \mathcal{O}_{\tilde{V}})$ with respect to the function field of $V$ has coefficients that are regular functions on $V$ by normality and uniqueness of the minimal polynomial. Hence these coefficients are elements of $H^0(V, \mathcal{O}_V) = H^0(U, \mathcal{O}_U) = k$, and so $H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}) = k$.

Step 2: The Picard scheme of $\tilde{X}$ exists by [Kl05, Theorem 4.18.2] and satisfies the theorem of the base by [Kl71, Theorem 5.1], see also [Kl05, Theorem 6.16 and Remark 6.19], namely the Néron–Severi group

$$\text{NS}(\tilde{X}) = \text{Pic}(\tilde{X})/\text{Pic}^0(\tilde{X})$$

is a finitely generated abelian group. Since the restriction map $\text{Pic}(\tilde{X}) \to \text{Pic}(\tilde{U})$ is surjective the induced composite map

$$h^* : \text{Pic}(U) \to \text{coker}(\text{Pic}^0(\tilde{X}) \to \text{Pic}(\tilde{U}))$$

maps an $\ell$-divisible group to a finitely generated abelian group, hence has finite image of order prime to $\ell$.

Step 3: Let $D$ be a big Cartier divisor on $U$. Since $h : \tilde{U} \to U$ is generically finite, also the divisor $h^*D$ is a big Cartier divisor on $\tilde{U}$. Moreover, as in the proof of Proposition 6, there is a big divisor $\tilde{D}$ on $\tilde{X}$ that restricts to $h^*D$ on $\tilde{U}$. Upon replacing $D$ and $\tilde{D}$ by a positive multiple we may assume by the finiteness of the image of the map (2.1) that $\tilde{D}$ is algebraically and thus numerically equivalent to a divisor $B$ on $X$ that is supported in $X \setminus \tilde{U}$.

Since bigness on projective varieties only depends on the numerical equivalence class, see [Laz04, Corollary 2.2.8], the divisor $B$ is also big. But by restriction to $\tilde{V}$

$$\bigcup_{n \geq 0} H^0(\tilde{X}, \mathcal{O}_X(nB)) \subseteq H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}) = k$$

by Step 1 above, and we conclude that \(\dim(U) = \dim(\tilde{X}) = \kappa(B) \leq 0\). □

2.4. Complementing examples. We present a few examples that illustrate the assumptions in Theorem 4 or Proposition 8 or properties of the variety $U$ in question that can easily be derived from the cohomology vanishing assumption.
No non-constant functions and no proper curves. If we assume that $U$ is normal and quasi-projective and the assumptions (i) and (ii) of Proposition 8 hold, then $U$ cannot contain a curve $C \rightarrow U$ that is proper over $k$ (by arguments that are more elementary than the proof of Proposition 8). Indeed, let $L$ be an ample line bundle on $U$, so that $\deg L|_C > 0$. On the other hand, we have $\deg L|_C = 0$, because $L \in \text{Pic}(U)$ is $\ell$-divisible and zero is the only $\ell$-divisible value in $\mathbb{Z}$, contradiction.

It is therefore tempting to raise the following question: is necessarily $U = \text{Spec}(k)$ for a connected normal variety $U/k$ over an algebraically closed field $k$ such that

(i) $H^0(U, \mathcal{O}_U) = k$, and

(ii) there is no non-constant map $C \rightarrow U$ from a proper smooth curve $C/k$?

The answer is no as the following example based on work of Totaro shows.

Example 9. Let $k$ be the algebraic closure of $\mathbb{F}_p$ and consider a smooth curve $D \rightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k$ of bi-degree $(2,3)$. Let $Y$ be the strict transform of $D$ in the blow-up $\sigma : X \rightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k$ of 12 points on $D$. Let $U$ be the complement $X \setminus Y$. For a choice of parameters (the curve $D$ and the 12 points) in a non-empty Zariski open of the space of all parameters, Totaro shows in [To09] that

$$H^0(U, \mathcal{O}_U) = \bigcup_{n \geq 0} H^0(X, \mathcal{O}_X(nY)) = k$$

contains only the constants, and $Y$ is nef. In fact, $Y$ meets all curves except those $C \rightarrow X$ such that $\sigma(C)$ meets $D$ only in a subset of the 12 points and only at most transversally. Let $\sigma(C)$ be such a curve of bi-degree $(a,b)$ moving in a family of dimension $(a+1)(b+1) - 1$. Then $2b + 3a = (\sigma(C) \cdot Y) \leq 12$ and for generic parameters the subspace with the described intersection locus has dimension

$$d = (a+1)(b+1) - 1 - (2b + 3a) = (a-1)(b-2) - 2.$$

In the range $0 < 2b + 3a \leq 12$ and $a, b \geq 0$ we have $d < 0$ and so there are simply no such curves $C$ for generic parameter values.

Nevertheless, this $U$ is smooth and thus fails to be étale contractible by Theorem 1.

Absence of big divisors. The condition in Theorem 4 that $U$ contains a big divisor cannot be omitted.

We first recall two facts about complete toric varieties that are standard analytically over $\mathbb{C}$ and which have étale counterparts for toric varieties over arbitrary algebraically closed base fields, in particular of characteristic $p > 0$.

Lemma 10. Let $k$ be an algebraically closed field. Any complete toric variety $X/k$ is étale simply connected: $\pi_1(X) = 1$.

Proof. By toric resolution, see [CLS11, §11.1], there is a resolution of singularities $\tilde{X} \rightarrow X$ with a smooth projective toric variety $\tilde{X}$. Since $\tilde{X}$ is rational, birational invariance of the étale fundamental group shows $\pi_1(\tilde{X}) = \pi_1(\mathbb{P}^n_k) = 1$, and the surjection $\pi_1(\tilde{X}) \rightarrow \pi_1(X)$ shows that $X$ is étale 1-connected. \hfill $\Box$

Lemma 11. Let $k$ be an algebraically closed field of characteristic $p$, and let $X/k$ be a complete toric variety. Then for all $\ell \neq p$ we have

$$H^2_{\text{ét}}(X, \mathbb{Z}_\ell(1)) \simeq \text{Pic}(X) \otimes \mathbb{Z}_\ell.$$

Proof. In the context of toric varieties over $\mathbb{C}$ and with respect to singular cohomology this is [CLS11, Theorem 12.3.3]. The $\ell$-adic case for toric varieties over an algebraically closed field $k$ of characteristic $\neq \ell$ follows with a parallel proof. \hfill $\Box$

The examples showing that a big divisor is essential in Theorem 4 come from toric geometry.
Example 12. Let $U = X$ be a complete normal non-projective toric variety $X$ of dimension 3 with trivial Picard group. Such toric varieties have been constructed in [Eik92, Example 3.5], or [Ful93, pp. 25–26, 65]. These sources construct $X$ over $\mathbb{C}$ but the constructions work mutatis mutandis over any algebraically closed base field $k$. Then

(i) $H^2_{\acute{e}t}(X, \mathbb{F}_p) = 0$ by Lemma 10, and

(ii) $H^2_{\acute{e}t}(X, \mathbb{Z}_\ell(1)) = 0$ for all $\ell \neq p$ by Lemma 11, and since there is non-trivial torsion in $\ell$-adic cohomology only for finitely many primes [Ga83], we conclude that $H^2(X, \mu_\ell) = 0$ for almost all $\ell \neq p$.

Therefore the assumptions of Theorem 4 hold with the exception of the presence of a big Cartier divisor. Nevertheless, these toric varieties are not étale contractible since $\text{Pic}(X)$ is locally constant and commutes with arbitrary base change. Since its fibre in $\mathbb{P}^1_k$ with trivial Picard group. Such toric varieties have been constructed in [Eik92, Example 3.5],

(i) $H^{0}(X, \mathcal{O}_X) = k$, and

(ii) $\text{Pic}(X)$ is $\ell$-divisible for some prime number $\ell \neq p$,

since these were deduced from the étale cohomological vanishing without the help of a big Cartier divisor. Yet again, these examples are showing that one cannot conclude $X = \text{Spec}(k)$ in Proposition 8 without a big Cartier divisor.

3. Normal surfaces

In this section we give a proof of Theorem 3 in the case of assumption (c) for surfaces. Not every normal surface admits a big Cartier divisor, so something needs to be done. Examples of proper normal surfaces with trivial Picard group, in particular without big divisors, can be found in [Na58] and [Sch99]. Nevertheless, a specialisation argument allows us to conclude the existence of an ample Cartier divisor on a hypothetical normal 2-contractible surface in general.

Theorem 13. There is no normal connected surface $U/k$ over an algebraically closed field $k$ of characteristic $p > 0$ such that

(i) $H^1_{\acute{e}t}(U, \mathbb{F}_p) = 0$, and

(ii) $H^2_{\acute{e}t}(U, \mu_\ell) = 0$ for some prime number $\ell \neq p$.

Proof. By Nagata’s embedding theorem and resolution of singularities for surfaces, the variety $U$ is a dense open in a normal proper surface $X/k$ with boundary $Y = X \setminus U$ being a normal crossing divisor. In particular, the surface $X$ is smooth in a neighbourhood of $Y$. By the usual limit arguments, we may choose an integral scheme $S$ of finite type over $\mathbb{F}_p$ together with a proper flat $f : \mathcal{X} \to S$, a relative Cartier divisor $\mathcal{Y}$ in $\mathcal{X}/S$ with normal crossing relative to $S$ and complement $\mathcal{U} = \mathcal{X} \setminus \mathcal{Y}$ such that all fibres are normal proper surfaces and such that there is a point $\eta : \text{Spec}(k) \to S$ over the generic point of $S$ such that the fibre over $\eta$ agrees with the original $\mathcal{X}_\eta = X$ together with $\mathcal{U}_\eta = U$ and $\mathcal{Y}_\eta = Y$. We may further assume that the set of irreducible components of the fibres of $\mathcal{Y}$ forms a constant system, and each component of $\mathcal{Y}$ is a Cartier divisor.

It follows from [SGA4]_2, Finitude, Theorem 1.9] that we may further assume that $R^2 f|_{\mathcal{Y}}_* \mu_\ell$ is locally constant and commutes with arbitrary base change. Since its fibre in $\eta$

$$ (R^2 f|_{\mathcal{Y}}_* \mu_\ell)_\eta = H^2_{\acute{e}t}(U, \mu_\ell) = 0 $$

vanishes we conclude that for all geometric points $\bar{s} \in S$ we have $H^2_{\acute{e}t}(\mathcal{Y}_\bar{s}, \mu_\ell) = 0$, where $\mathcal{Y}_\bar{s}$ is the fibre of $\mathcal{Y} \to S$ in $\bar{s}$. As in the proof of Theorem 4 this implies that for every Cartier divisor $D$ on $\mathcal{X}_\bar{s}$ there is an $m \geq 1$ and a Cartier divisor $E$ on $\mathcal{X}_\bar{s}$ supported in $\mathcal{Y}_\bar{s}$ such that

$$ mD \equiv E $$

are numerically equivalent.
We apply this insight to geometric fibres $\mathcal{Z}_t$ above closed points $t \in S$. Since by a theorem of Artin [Ar62, Corollary 2.11], all proper normal surfaces over the algebraic closure of a finite field are projective, we conclude that there is a very ample Cartier divisor $H_t$ on $\mathcal{Z}_t$ with support contained in $\mathcal{Y}_t$.

Let $\mathcal{H} \hookrightarrow \mathcal{X}$ be the relative Cartier divisor with support in $\mathcal{Y}$ that specialises to $H_t$. By [EGA3, Theorem 4.7.1], the divisor $\mathcal{H}$ is ample relative to $S$ in an open neighbourhood of $t \in S$. Consequently, the normal proper surface $X$ is projective, and in particular $U$ admits a big divisor. The proof now follows from Theorem 4.

Remark 14. It follows from the proof of Theorem 13 that any proper non-projective normal surface $X$ with trivial Picard group, in particular the examples of [Na58] and [Sch99], must have $H^2(X, \mu_\ell) \neq 0$ and a fortiori must contain non-trivial $\ell$-torsion classes in the cohomological Brauer group $Br(X)$ for all $\ell$ different from the characteristic. The existence of nontrivial torsion classes in $Br(X)$ under the above assumptions was proven by different methods in [Sch01, proof of Theorem 4.1 on page 453].

References


[SGA4\frac{1}{2}] Deligne, P., Cohomologie étale (SGA4\frac{1}{2}), Lecture Notes in Mathematics 569, Springer, 1977.


