The étale topos reconstructs varieties over sub-p-adic fields

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Abstract. Let K be a sub-p-adic field. We show that the functor sending a finite type K-scheme to its étale topos is fully faithful after localizing at the class of universal homeomorphisms. This generalizes a result of Voevodsky, who proved the analogous theorem for fields finitely generated over $\mathbb{Q}$. Our proof relies on Mochizuki's Hom-theorem in anabelian geometry, and a study of point-theoretic morphisms of fundamental groups of curves.

1. Introduction

In his 1983 letter to Faltings [Gro97], Grothendieck conjectured as one of his anabelian conjectures that, for fields K finitely generated over their prime field, *any* scheme X of finite type over K can be reconstructed, up to universal homeomorphism, from its étale topos $X_{\text{ét}}$. Voevodsky proved in [Voe90] that Grothendieck's conjecture is true when X is normal and K is finitely generated of characteristic zero, and Carlson–Haine–Wolf [CHW24], building on techniques of Voevodsky, proved Grothendieck's conjecture in characteristic zero, and also proved the conjecture in the case when K is a finitely generated field of positive transcendence degree and positive characteristic. In this paper, we show that Grothendieck's anabelian conjecture for étale topoi holds whenever K is a sub-p-adic field, i.e., K is a subfield of a finitely generated extension of $\mathbb{Q}_p$. This reproves Voevodsky's result.

We now state our results more precisely. Let $\mathbf{Sch}_K^{\text{ft}}$ be the category of schemes of finite type over K, and let $\mathbf{Sch}_K^{\text{ft}}[\text{UH}^{-1}]$ be the localization along the universal homeomorphims. Write $\mathbf{RTop}_K^{\text{pin}}$ for the category of topoi over the étale topos $\operatorname{Spec}(K)_{\text{ét}}$ and **pinned** geometric morphisms, i.e., morphisms such that the induced map $|\mathcal{X}| \to |\mathcal{Y}|$ on topological spaces takes closed points to closed points.

Theorem A (see Theorem 4.1)**.** *Let K be a sub-p-adic field. Then the functor*

$$(-)_{\text{ét}} \colon \mathbf{Sch}_K^{\text{ft}}[\text{UH}^{-1}] \longrightarrow \mathbf{RTop}_K^{\text{pin}}$$

sending a scheme $X \to \operatorname{Spec}(K)$ of finite type to its étale topos $X_{\text{ét}} \to \operatorname{Spec}(K)_{\text{ét}}$ is fully faithful.

For any two seminormal schemes X and Y of finite type over a field K of characteristic zero, any universal homeomorphism is in fact an isomorphism. If we denote by $\mathbf{Sch}_K^{\text{sn}}$ the category of seminormal schemes of finite type over K, the inclusion $\mathbf{Sch}_K^{\text{sn}} \to \mathbf{Sch}_K$ admits a right adjoint, which induces an equivalence $\mathbf{Sch}_K^{\text{ft}}[\text{UH}^{-1}] \to \mathbf{Sch}_K^{\text{sn}}$, see [CHW24, Theorem 1.13]. Thus, Theorem A can be translated as follows ($\operatorname{Hom}_K^{\text{pin}}$ denotes the Hom-groupoid in $\mathbf{RTop}_K^{\text{pin}}$, see § 2).

Theorem B. *Let K be a sub-p-adic field. Let X and Y be schemes of finite type over K and assume that X is seminormal. Then the natural map*

$$(-)_{\text{ét}} \colon \operatorname{Hom}_K(X, Y) \longrightarrow \operatorname{Hom}_K^{\text{pin}}(X_{\text{ét}}, Y_{\text{ét}})$$

is an equivalence of groupoids.

Following [CHW24], we know that $\operatorname{Hom}_K^{\text{pin}}(X_{\text{ét}}, Y_{\text{ét}})$ is equivalent to a set, and moreover, we reduce the proof of Theorem B to showing that the natural map

$$\operatorname{Hom}_K(X, \mathbb{P}^1_K \smallsetminus \{0,1,\infty\}) \longrightarrow \operatorname{Hom}_K^{\text{pin}}(X_{\text{ét}}, (\mathbb{P}^1_K \smallsetminus \{0,1,\infty\})_{\text{ét}})$$

Date: October 29, 2024.

is bijective whenever X is a smooth, connected, affine K-scheme. The main task is to show that a pinned morphism is either constant, or induces an open map on étale fundamental groups. In the latter case, we use Mochizuki's Hom-theorem to conclude [Moc99, Theorem A]. Generalizing an argument of Creutz and Voloch [CV22, Theorem 1.5], we establish the dichotomy by proving the following.

Proposition C (see Proposition 3.17)**.** *Let X and Y be geometrically connected hyperbolic curves over a Kummer-faithful field K. Suppose $f\colon \pi_1(X) \to \pi_1(Y)$ is a point-theoretic map over Gal_K. Then either f is open, or the image is contained in a single decomposition group.*

Acknowledgments. The first author thanks Eric Ahlqvist, Clark Barwick, Peter Haine, and Sebastian Wolf for helpful discussions. He especially wishes to thank Peter Haine and Sebastian Wolf, from whom he learned a lot on the topic of étale reconstruction, particularly through their joint work [CHW24]. Both of the authors of this paper gratefully acknowledge support from the Deutsche Forschungsgemeinschaft (DFG) through the Collaborative Research Centre TRR 326 "Geometry and Arithmetic of Uniformized Structures", project number 444845124.

2. Preliminaries on topoi

We recall some well-known results on étale topoi of schemes. Much of the material here is already recollected in [CHW24, Section 2], to which we refer the reader for more details.

2.1. **Topoi and pinned morphisms.** Let $\mathbf{RTop}$ denote the $(2,1)$-category of topoi with geometric 1-morphisms. We recall the relative notion of the $(2,1)$-category $\mathbf{RTop}_{\mathcal{S}}$ of topoi sliced over a particular topos $\mathcal{S}$:

(i) Objects of $\mathbf{RTop}_{\mathcal{S}}$ are geometric 1-morphisms of topoi $p_*\colon \mathcal{X} \to \mathcal{S}$.

(ii) Given $p_*\colon \mathcal{X} \to \mathcal{S}$ and $q_*\colon \mathcal{Y} \to \mathcal{S}$, the Hom-groupoid $\mathrm{Hom}_{\mathcal{S}}\big(p_*, q_*\big)$ has as object pairs (φ_*, α) where $\varphi_*\colon \mathcal{X} \to \mathcal{Y}$ is a geometric morphism and $\alpha\colon q_*\varphi_* \xrightarrow{\sim} p_*$ is a natural isomorphism. The morphisms from $(\varphi_*, \alpha)\colon \mathcal{X} \to \mathcal{Y}$ to $(\psi_*, \beta)\colon \mathcal{X} \to \mathcal{Y}$ are the natural isomorphisms $\vartheta\colon \varphi_* \xrightarrow{\sim} \psi_*$ such that

$$\begin{tikzcd} & q_*\psi_* \arrow[dr, "\beta"] & \\ q_*\varphi_* \arrow[ur, "q_*\vartheta"] \arrow[rr, "\alpha"'] & & p_* \end{tikzcd}$$

commutes.

To a topos $\mathcal{X}$ one can functorially associate a topological space $|\mathcal{X}|$, by considering the locale of subobjects of the terminal object [CHW24, §2.4]. Given a map $\varphi\colon \mathcal{X} \to \mathcal{Y}$ of topoi, the natural map $|\varphi|\colon |\mathcal{X}| \to |\mathcal{Y}|$ is continuous. Call a geometric morphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$ **pinned** if the underlying map of topological spaces takes closed points to closed points. Given topoi $\mathcal{X}, \mathcal{Y} \in \mathbf{RTop}_{\mathcal{S}}$, write

$$\mathrm{Hom}_{\mathcal{S}}^{\mathrm{pin}}(\mathcal{X}, \mathcal{Y}) \subseteq \mathrm{Hom}_{\mathcal{S}}(\mathcal{X}, \mathcal{Y})$$

for the full subgroupoid spanned by the pinned geometric morphisms. Define $\mathbf{RTop}_{\mathcal{S}}^{\mathrm{pin}}$ as the $(2,1)$-category with the same objects as $\mathbf{RTop}_{\mathcal{S}}$, pinned geometric morphisms as 1-morphisms and 2-morphisms natural isomorphisms between pinned geometric morphisms.

For us, the topos $\mathcal{S}$ will always be the étale topos $\mathcal{S} = \mathrm{Spec}(K)_{\text{ét}}$ of a field K that we abbreviate by $K_{\text{ét}}$. We then use the shorthand

$$\mathbf{RTop}_K^{\mathrm{pin}} := \mathbf{RTop}_{\mathrm{Spec}(K)_{\text{ét}}}^{\mathrm{pin}} \qquad \text{and} \qquad \mathrm{Hom}_K^{\mathrm{pin}}(\mathcal{X}, \mathcal{Y}) := \mathrm{Hom}_{\mathrm{Spec}(K)_{\text{ét}}}^{\mathrm{pin}}(\mathcal{X}, \mathcal{Y}).$$

As is shown in [CHW24, Proposition 2.22], for X and Y locally of finite type over K, the groupoid $\mathrm{Hom}_K^{\mathrm{pin}}(X_{\text{ét}}, Y_{\text{ét}})$ is equivalent to a set. Thus, the $(2,1)$-category spanned by pinned geometric morphisms between étale topoi of schemes locally of finite type over K is equivalent to a 1-category.

2.2. **Base points of topoi.** Recall [Joh77, Section 8.4] that given a connected topos $\mathcal{X}$ and a point $p\colon \mathbf{Set} \to \mathcal{X}$, one has a Galois category $\mathcal{X}_{\text{fét}}$, the full subcategory of $\mathcal{X}$ consisting of objects of $\mathcal{X}$ which are locally constant constructible, and the fiber functor is given by $p^*\colon \mathcal{X}_{\text{fét}} \to \mathbf{Set}$. Further, a geometric morphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$ gives rise to a map $\varphi^*\colon (\mathcal{Y}, p^* \circ \varphi^*) \to (\mathcal{X}, p^*)$ of Galois categories. In the case that $X_{\text{ét}}$ and $Y_{\text{ét}}$ are étale topoi of connected schemes and $\varphi\colon X_{\text{ét}} \to Y_{\text{ét}}$ is a geometric morphism of étale topoi, and $\overline{x}$ is a basepoint for X, one gets, by considering automorphisms groups of fiber functors, a map

$$\varphi_*\colon \pi_1(X, \overline{x}) \to \pi_1(Y, \overline{x} \circ \varphi)$$

of fundamental groups.

Recollection 2.1. Suppose that X and Y are of finite type over a perfect field K and that $\varphi\colon X_{\text{ét}} \to Y_{\text{ét}}$ is a pinned morphism of étale topoi. Then for any algebraic extension L/K and any morphism $x\colon \operatorname{Spec} L \to X$ over K, the composite

$$\varphi \circ x_{\text{ét}}\colon L_{\text{ét}} \longrightarrow Y_{\text{ét}}$$

comes from a unique morphism of schemes $\operatorname{Spec} L \to Y$ over K [CHW24, Proposition 3.6]. This implies that we have, for every algebraic extension L/K, a map $\varphi(L)\colon X(L) \to Y(L)$. In the colimt over all L we obtain a unique map

$$\varphi(\overline{K})\colon X(\overline{K}) \longrightarrow Y(\overline{K})$$

such that for all points $a \in X(L) \subseteq X(\overline{K})$ we have $\varphi \circ a_{\text{ét}} = b_{\text{ét}}$ for $b = \varphi(L)(a)$.

Further, for any quasi-compact and separated étale open $j\colon U \to Y$, we know by [CHW24, Theorem A.8] that $\varphi^* U \in X_{\text{ét}}$ is represented by a quasi-compact and separated étale open $j'\colon \varphi^* U \to Y$. By taking slice topoi, we get a geometric morphism $(\varphi^* U)_{\text{ét}} \to U_{\text{ét}}$, the restriction of φ, which is again pinned. This follows from the fact that φ, $j_{\text{ét}}$ and $j'_{\text{ét}}$ are pinned, and the fact that, since $j : U \to Y$ is étale and quasicompact, it is quasi-finite.

3. Morphisms between étale topoi and fundamental groups

The purpose of this section is to show (Proposition 3.18) that given a non-constant map $\varphi\colon U_{\text{ét}} \to V_{\text{ét}}$ over $K_{\text{ét}}$ between the étale topoi of smooth geometrically connected curves U and V over a Kummer-faithful field K, see Definition 3.8, that the induced map of fundamental groups $\varphi_*\colon \pi_1(U, \bar{u}) \to \pi_1(V, \bar{v})$ is open, where $\bar{v} = \varphi(\bar{u})$.

Throughout §3, we fix a field K with a fixed algebraic closure $\overline{K}$, i.e. a base point for $\operatorname{Spec}(K)$. As soon as we have defined the notion of a Kummer-faithful field, the field K will be assumed to be Kummer-faithful. We will further assume for the main result of this section that K has characteristic 0.

3.1. **Quasi-sections and point-theoretic maps.** Let X/K be a geometrically connected scheme of finite type and endow X with a geometric point $\bar{x}$ compatible with $\overline{K}$. For any finite extension L/K inside $\overline{K}$ the group $\mathrm{Gal}_L = \mathrm{Gal}(\overline{K}/L)$ is an open subgroup of $\mathrm{Gal}_K = \mathrm{Gal}(\overline{K}/K)$. We denote by

$$\mathscr{S}_{X/K}(L) := \{s\colon \mathrm{Gal}_L \to \pi_1(X, \bar{x}) \ ; \ \text{over } \mathrm{Gal}_K\}/_{\pi_1(X_{\bar{K}}, \bar{x})}$$

the set of $\pi_1(X_{\bar{K}}, \bar{x})$ conjugacy classes of L-**rational quasi-sections** (i.e., sections only defined on the open subgroup Gal_L). Functoriality of the étale fundamental group yields the non-abelian Kummer map

$$X(L) \longrightarrow \mathscr{S}_{X/K}(L), \qquad a \mapsto \big(\pi_1(a)\colon \mathrm{Gal}_L \to \pi_1(X, \bar{x})\big).$$

We may pass to the limit over all $L \subseteq \overline{K}$ that are finite over K and obtain

$$\kappa\colon X(\overline{K}) \longrightarrow \mathscr{S}_{X/K}(\overline{K}) := \operatorname*{colim}_L \mathscr{S}_{X/K}(L),$$

the non-abelian Kummer map from $\overline{K}$-rational points to the set of **quasi-sections**. The quasi-sections in the image of κ are called **geometric quasi-sections**.

Remark 3.1. (1) Any L-rational geometric quasi-section defines a $\pi_1(X_{\bar{K}}, \bar{x})$-conjugacy class of closed subgroups of $\pi_1(X, \bar{x})$ as the image of the section. The maximal subgroups among these images are nothing but the (conjugacy classses of) decomposition subgroups associated to closed points of X (or rather some choice of universal covering).

(2) Note that the sets of quasi-sections $\mathscr{S}_{X/K}(L)$ and $\mathscr{S}_{X/K}(\overline{K})$ do not depend on the choice of a base point of X up to canonical and compatible bijections.

Definition 3.2. Let X and Y be geometrically connected schemes of finite type over K. A homomorphism $\psi\colon \pi_1(X, \bar{x}) \to \pi_1(Y, \bar{y})$ over Gal_K is called **point-theoretic** if composition with ψ preserves geometric quasi-sections. In other words, for all finite L/K inside $\overline{K}$ and all points $a \in X(L)$ there is a finite extension L'/L inside $\overline{K}$ and a point $b \in Y(L')$ such that $\pi_1(b) = (\psi \circ \pi_1(a))|_{\mathrm{Gal}_{L'}}$.

Lemma 3.3. *Let $\varphi\colon X_{\text{ét}} \to Y_{\text{ét}}$ be a pinned morphism of topoi over $K_{\text{ét}}$ between two geometrically connected varieites over K. Then the induced map $\varphi_*\colon \pi_1(X, \bar{x}) \to \pi_1(Y, \bar{y})$ is point-theoretic.*

Proof. Indeed, by Recollection 2.1, for every finite extension L/K and every point $a \in X(L)$ the composite

$$\varphi \circ a_{\text{ét}}\colon (\operatorname{Spec} L)_{\text{ét}} \longrightarrow X_{\text{ét}} \longrightarrow Y_{\text{ét}}$$

is induced by a morphism of schemes $b\colon \operatorname{Spec} L \to Y$. Hence also

$$\varphi_* \circ \pi_1(a) = (\varphi \circ a_{\text{ét}})_*(b_{\text{ét}})_* = \pi_1(b). \qquad \square$$

3.2. **Galois sections of the generalized Jacobian.** In this subsection we ask the field K to be of characteristic zero. We extend a construction proposed in [Sti13, §13.5] only in the case of smooth projective curves.

Remark 3.4. (1) If X is a disjoint union $X = \amalg_i X_i$ of geometrically connected schemes of finite type X_i, then we can extend the definition of the sets of geometric quasi-sections as

$$\mathscr{S}_{X/K}(L) := \amalg_i \mathscr{S}_{X_i/K}(L)$$

and similarly with coefficients in $\overline{K}$.

(2) Let X and Y be geometrically connected schemes of finite type over K. The Künneth-formula yields an isomorphism

$$(\mathrm{pr}_{1\,*}, \mathrm{pr}_{2,*})\colon \pi_1(X \times_K Y, (\bar{x}, \bar{y})) \xrightarrow{\sim} \pi_1(X, \bar{x}) \times_{\mathrm{Gal}_K} \pi_1(Y, \bar{y}).$$

It follows that the projection maps induce canonical bijections

$$\mathscr{S}_{X \times_K Y/K}(L) \xrightarrow{\sim} \mathscr{S}_{X/K}(L) \times \mathscr{S}_{Y/K}(L),$$

and similarly with coefficients in $\overline{K}$.

Recollection 3.5. Let U be a smooth geometrically connected curve over K with smooth projective completion X/K. The generalized Picard scheme $\mathrm{Jac}_U^\bullet := \mathrm{Pic}_{X,D}$ parametrizes line bundles on X, together with a trivialization along $D = X \smallsetminus U$. The connected components of $\mathrm{Jac}_U^\bullet$ are the subschemes Jac_U^d of line bundles of degree $d \in \mathbb{Z}$. For $d = 0$ we recover the generalized Jacobian $\mathrm{Jac}_U = \mathrm{Jac}_U^0$ of U.

Tensor product of line bundles defines an abelian algebraic group structure on $\mathrm{Jac}_U^\bullet$. Therefore

$$\mathscr{S}_{\mathrm{Jac}_U^\bullet/K}(\overline{K})$$

is an abelian group. The degree map defines a short exact sequence

$$0 \to \mathscr{S}_{\mathrm{Jac}_U}(\overline{K}) \to \mathscr{S}_{\mathrm{Jac}_U^\bullet/K}(\overline{K}) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

For all integers d taking d-th tensor powers yields a map $\mathrm{Jac}_U^1 \to \mathrm{Jac}_U^d$ that induces an isomorphism

$$d_*\big(\pi_1(\mathrm{Jac}_U^1)\big) \xrightarrow{\sim} \pi_1(\mathrm{Jac}_U^d)$$

over Gal_K, where $d_*\big(\pi_1(\mathrm{Jac}^1_U)\big)$ is the pushout of the extension

$$1 \to \pi_1((\mathrm{Jac}^1_U)_{\bar{K}}) \to \pi_1(\mathrm{Jac}^1_U) \to \mathrm{Gal}_K \to 1$$

along the multiplication by d map on the abelian group $\pi_1((\mathrm{Jac}^1_U)_{\bar{K}})$. Note that for $d = 0$, the resulting base point in Jac^0_U is 0, and the extension of fundamental groups is canonically split.

Recollection 3.6. The generalized Abel-Jacobi map,

$$j_U \colon U \longrightarrow \mathrm{Jac}^1_U, \qquad P \mapsto \mathcal{O}_X(P),$$

is the universal map into a torsor under a semi-abelian variety, the semi-abelian Albanese morphism. The non-abelian Kummer map is compatible with the Abel-Jacobi map. It moreover factors over the group of 0-cycles on $U_{\bar{K}}$ as follows:

$$\begin{tikzcd} U(\overline{K}) \arrow[r] \arrow[d, "\kappa"] & Z_0(U_{\bar{K}}) \arrow[r, "j_U"] & \mathrm{Jac}^\bullet_U(\overline{K}) \arrow[d, "\kappa"] \\ \mathscr{S}_{U/K}(\overline{K}) \arrow[rr, "j_{U,*}"] & & \mathscr{S}_{\mathrm{Jac}^\bullet_U/K}(\overline{K}) \end{tikzcd}$$

From [KL81, Section II] we deduce that $\pi_1(j_U)$ induces an isomorphism

$$\pi_1^{(\mathrm{ab})}(U, \bar{u}) \to \pi_1(\mathrm{Jac}^1_U, \bar{u}_1),$$

where $\pi_1^{(\mathrm{ab})}(U, \bar{u})$ denotes the geometric abelianization of $\pi_1(U, \bar{u})$, and $\bar{u}_1 = j_U(\bar{u})$. It follows from the description of $\pi_1(\mathrm{Jac}^d_U)$ given above, that for all integers d the extension $\pi_1(\mathrm{Jac}^d_U)$ can be reconstructed from $\pi_1(U, \bar{u})$. This proves the following lemma.

Lemma 3.7. *Let U, V be smooth, geometrically connected curves over K, and let*

$$\psi \colon \pi_1(U, \bar{u}) \to \pi_1(V, \bar{v})$$

be a homomorphism over Gal_K*. Then there are unique group homomorphisms*

$$\psi^{\mathrm{ab},d} \colon \pi_1(\mathrm{Jac}^d_U, \bar{u}_d) \to \pi_1(\mathrm{Jac}^d_V, \bar{v}_d)$$

over Gal_K*, where $\bar{u}_d = d(j_U(\bar{u}))$ and similarly $\bar{v}_d$ such that*

$$\psi^{\mathrm{ab},1} \circ \pi_1(j_U) = \pi_1(j_V) \circ \psi$$

and the $\psi^{\mathrm{ab},d}$'s are compatible with the maps induced by multiplication on $\mathrm{Jac}^\bullet_U$ *and* $\mathrm{Jac}^\bullet_V$*.*

In particular, we obtain a commutative diagram

$$\begin{tikzcd} \mathscr{S}_{U/K}(\overline{K}) \arrow[r, "\psi"] \arrow[d, "j_{U,*}"] & \mathscr{S}_{V/K}(\overline{K}) \arrow[d, "j_{V,*}"] \\ \mathscr{S}_{\mathrm{Jac}^\bullet_U/K}(\overline{K}) \arrow[r, "\psi^{\mathrm{ab},\bullet}"] & \mathscr{S}_{\mathrm{Jac}^\bullet_V/K}(\overline{K}) \end{tikzcd}$$

in which the bottom map is a group homomorphism.

3.3. **Sections over Kummer-faithful fields.** We recall the definition [Moc15, Definition 1.5] for characteristic 0, see [Hos17, Definition 1.2] for a version over perfect fields.

Definition 3.8. A **Kummer-faithful** field in characteristic 0 is a field K of characteristic 0 such that for all semi-abelian varieties B/K the following equivalent conditions hold.

(a) The Kummer map $B(K) \to \mathrm{H}^1(K, \pi_1^{\mathrm{ab}}(B_{\bar{K}}))$ is injective.

(b) The intersection $\bigcap_{n\geq 1} nB'(K) = 0$ is trivial, i.e. $B(K)$ does not contain arbitrarily divisible elements.

The equivalence is a consequence of the exact sequence of Kummer theory.

Remark 3.9. (1) Note that the Kummer map of Kummer theory applied to B/K agrees (by comparing with the 0-section) with the non-abelian Kummer map of the section conjecture

$$B(K) \to \mathscr{S}_{B/K}(K) = \mathrm{H}^1(K, \pi_1(B_{\bar{K}})) = \mathrm{H}^1(K, \pi_1^{\mathrm{ab}}(B_{\bar{K}})),$$

see for example [Sti13, Corollary 71].

(2) Sub-p-adic fields are Kummer-faithful, but not conversely, see for example [Moc15, Remark 1.5.4], [Hos17, Remark 1.2.3] and [Oht22; Oht23].

(3) Weil restriction shows that finite extensions of Kummer-faithful fields are Kummer-faithful.

From now on K is assumed to be a Kummer-faithful field.

Lemma 3.10. *Let X/K be a geometrically connected scheme of finite type that admits an injective map $X \to W$ into a torsor W/K under a semiabelian variety B/K. Then for any finite extension L/K (inside $\overline{K}$) the non-abelian Kummer map*

$$\kappa\colon X(L) \longrightarrow \mathscr{S}_{X/K}(L)$$

is injective and induces a bijection of $X(L)$ with the set of geometric L-rational quasi-sections.

Proof. This follows at once from the naturality of the non-abelian Kummer map, and the semi-abelian analogue of [Sti13, Proposition 73]. □

The proof of the following is immediate in view of Lemma 3.10.

Corollary 3.11. *Let X and Y be a geometrically connected schemes of finite type over K such that Y admits an injective map $Y \to W$ into a torsor W/K under a semiabelian variety B/K. Then for any point-theoretic homomorphism $\psi\colon \pi_1(X, \bar{x}) \to \pi_1(Y, \bar{y})$ over Gal_K there is a unique map of sets*

$$\psi(\overline{K})\colon X(\overline{K}) \longrightarrow Y(\overline{K}),$$

which is the colimit of the maps $\psi(L)\colon X(L) \longrightarrow Y(L)$ defined for all $a \in X(L)$ by the equality $\pi_1(b) = \psi \circ \pi_1(a)$ for $b = \psi(L)(a)$, i.e., $\psi(\overline{K})$ agrees with $\psi \circ -$ on geometric quasi-sections.

Proposition 3.12. *Let X and Y be geometrically connected schemes of finite type over K such that Y admits an injective map $Y \to W$ into a torsor W/K under a semiabelian variety B/K. Let $\varphi\colon X_{et} \to Y_{et}$ be a pinned morphism of topoi over K. Then the two maps*

$$\varphi(\overline{K}), \varphi_*(\overline{K})\colon X(\overline{K}) \longrightarrow Y(\overline{K})$$

agree, where $\varphi_*\colon \pi_1(X, \bar{x}) \to \pi_1(Y, \bar{y})$ is the map induced by φ, and $\bar{y} = \bar{x} \circ \varphi$.

Proof. By Lemma 3.3 the homomorphism φ_* is point-theoretic, so that $\varphi_*(\overline{K})$ is well defined by Corollary 3.11. The proof of Lemma 3.3 actually shows directly the apparently stronger statement of the claim in view of the functoriality of the étale fundamental group. □

3.4. **Point-theoretic maps and the Abel-Jacobi map.** We remind the reader that we assume that K is a Kummer-faithful field of characteristic 0.

Lemma 3.13. *Let U be a geometrically connected smooth curve over K. Then the non-abelian Kummer map*

$$\kappa\colon \mathrm{Jac}_U^{\bullet}(\overline{K}) \longrightarrow \mathscr{S}_{\mathrm{Jac}_U^{\bullet}/K}(\overline{K})$$

is injective.

Proof. This follows from the definition of Kummer-faithful in combination with Remark 3.9 since the generalized Jacobian Jac_U^0 is a semi-abelian variety over K. □

Proposition 3.14. *Let U and V be smooth, geometrically connected curves over K, and suppose that $\pi_1(V_{\overline{K}}, \bar{v})$ is non-trivial. Given a point-theoretic morphism*

$$\psi \colon \pi_1(U, \bar{u}) \to \pi_1(V, \bar{v})$$

over Gal_K, then there is a unique group homomorphism

$$\psi^{\mathrm{ab}}(\overline{K}) \colon \operatorname{Jac}^{\bullet}_U(\overline{K}) \to \operatorname{Jac}^{\bullet}_V(\overline{K})$$

which makes the diagram

$$\begin{tikzcd}
U(\overline{K}) \arrow[r, "\psi(\overline{K})"] \arrow[d, "j_U"] & V(\overline{K}) \arrow[d, "j_V"] \\
\operatorname{Jac}^{\bullet}_U(\overline{K}) \arrow[r, "\psi^{\mathrm{ab}}(\overline{K})"] & \operatorname{Jac}^{\bullet}_V(\overline{K})
\end{tikzcd}$$

commutative.

Proof. From Recollection 3.6 we know that the top and bottom parts commute in the diagram

$$\begin{tikzcd}
U(\overline{K}) \arrow[rr] \arrow[dr, "\kappa"] \arrow[dd, "\psi(\overline{K})"] & & Z_0(U_{\bar{K}}) \arrow[r, two heads, "j_U"] \arrow[dd, "\mathbb{Z}[\psi(\overline{K})]"] & \operatorname{Jac}^{\bullet}_U(\overline{K}) \arrow[dr, "\kappa"] \arrow[dd, dotted, "\psi^{\mathrm{ab}}(\overline{K})"] & \\
& \mathscr{S}_{U/K}(\overline{K}) \arrow[rrr, "j_{U,*}"] \arrow[dd, "\psi"] & & & \mathscr{S}_{\operatorname{Jac}^{\bullet}_U/K}(\overline{K}) \arrow[dd, "\psi^{\mathrm{ab},\bullet}"] \\
V(\overline{K}) \arrow[rr] \arrow[dr, "\kappa"] & & Z_0(V_{\bar{K}}) \arrow[r, "j_V"] & \operatorname{Jac}^{\bullet}_V(\overline{K}) \arrow[dr, hook, "\kappa"] & \\
& \mathscr{S}_{V/K}(\overline{K}) \arrow[rrr, "j_{V,*}"] & & & \mathscr{S}_{\operatorname{Jac}^{\bullet}_V/K}(\overline{K}) .
\end{tikzcd}$$

As $\pi_1(V_{\overline{K}})$ is non-trivial by assumption, the map $j_V \colon V \to \operatorname{Jac}^1_V$ is injective. Therefore, since ψ is point-theoretic and K is Kummer-faithful, Corollary 3.11 provides the map $\psi(\overline{K})$ such that the left face commutes.

The map $\psi^{\mathrm{ab},\bullet}$ was constructed in Lemma 3.7 such that the front face commutes.

The map $\mathbb{Z}[\psi(\overline{K})]$ is the unique group homomorphism extending the map $\psi(\overline{K})$. By [Ser88, Chapter V, Proposition 3], the natural map $j_U \colon Z_0(U_{\overline{K}}) \to \operatorname{Jac}^{\bullet}_U(\overline{K})$ is surjective. Furthermore, the bottom maps κ on the right hand side is injective by Lemma 3.13. Thus a simple diagram chase, using that $Z_0(U_{\bar{K}})$ is the free abelian group on $U(\overline{K})$, shows the existence and uniqueness of the claimed homomorphism $\psi^{\mathrm{ab}}(\overline{K})$. $\square$

We will now essentially follow the argument of Creutz and Voloch [CV22, Theorem 1.5] to show that, any point-theoretic map $\psi \colon \pi_1(U, \bar{u}) \to \pi_1(V, \bar{v})$ of hyperbolic curves is either open, or the image of ψ is contained in a single decomposition group.

For a geometrically connected curve U over K with smooth projective compactification X, let g_U be the (geometric) genus of X, and let $n_U = \deg(X_{\overline{K}} \smallsetminus U_{\overline{K}})$ the degree of the boundary divisor. Finally, set $\varepsilon_U = 1$ if U is affine and 0 otherwise. The ℓ-adic Euler characteristic of $U_{\bar{K}}$ is

$$\chi_U \coloneqq \chi(U_{\bar{K}}, \mathbb{Q}_\ell) = 2 - 2g_U - n_U.$$

By [Ser88, Chapter V, Theorem 1], the dimension of Jac^0_U is

$$r_U \coloneqq \dim \operatorname{Jac}^0_U = g_U + n_U - \varepsilon_U = 1 - \varepsilon_U + \frac{1}{2}(n_U - \chi_U). \tag{3.15}$$

Proposition 3.16. *Let U and V be geometrically connected smooth curves over a Kummer-faithful field K of characteristic zero, and assume that $\pi_1(V_{\overline{K}}, \bar{v})$ is non-trivial. Let*

$$\psi \colon \pi_1(U, \bar{u}) \to \pi_1(V, \bar{v})$$

be a point-theoretic morphism over Gal_K.

Then, if $\chi_V \leq -4g_U - 4n_U$*, the image of* $\psi(\overline{K})\colon U(\overline{K}) \to V(\overline{K})$ *is finite.*

Proof. We use (3.15) to translate the numerical assumption into

$$2r_U - r_V = 2(g_U + n_U - \varepsilon_U) + \frac{1}{2}(\chi_V - n_V) + \varepsilon_V - 1 \leq -2\varepsilon_U - \frac{1}{2}n_V + \varepsilon_V - 1 < 0,$$

so that the estimate $2r_U < r_V$ holds.

We may replace the base points by $\bar{u} \in U(\overline{K})$ and $\bar{v} = \psi(\overline{K})(\bar{u}) \in V(\overline{K})$. Moreover, we consider the Abel-Jacobi maps with respect to these base points

$$j_{\bar{u}} : U_{\bar{K}} \to \mathrm{Jac}^0_{U,\bar{K}}, \quad z \mapsto j_U(z) - \bar{u} \quad \text{and} \quad j_{\bar{v}} : V_{\bar{K}} \to \mathrm{Jac}^0_{V,\bar{K}}, \quad z \mapsto j_V(z) - \bar{v}.$$

Then the diagram of Proposition 3.14 gives rise to a commutative diagram

$$\begin{array}{ccc} U(\overline{K}) & \xrightarrow{\psi(\overline{K})} & V(\overline{K}) \\ \downarrow{\scriptstyle j_x} & & \downarrow{\scriptstyle j_y} \\ \mathrm{Jac}^0_U(\overline{K}) & \xrightarrow{\psi^{\mathrm{ab},0}} & \mathrm{Jac}^0_V(\overline{K}) \end{array}$$

where the map $\psi^{\mathrm{ab},0}$ is a group homomorphism.

We further know by [Ser88, Chapter V, Theorem 1], that the natural map from the r_U-fold product $U_{\bar{K}}^{r_U} \to \mathrm{Jac}^0_{U,\bar{K}}$ is dominant. Since in a connected algebraic group any dense open subset generates the whole group by a sum of two elements, this implies that any element of $\mathrm{Jac}^0_U(\overline{K})$ is a sum of at most $2r_U$ many elements in the image of $j_{\bar{u}}$. It follows that the image of $\psi^{\mathrm{ab},0}$ is contained in the image of the natural map from the $2r_U$-fold product $V_{\bar{K}}^{2r_U} \to \mathrm{Jac}^0_{V,\bar{K}}$. By assumption and the calculation above, this map is not dominant. Thus the Zariski closure B of $\psi^{\mathrm{ab},0}(\mathrm{Jac}^0_U(\overline{K}))$ in $\mathrm{Jac}^0_{V,\bar{K}}$ is a proper subgroup (note that $\psi^{\mathrm{ab},0}$ is a group homomorphism and thus this Zariski closure is a group).

Now the image of $\psi(\overline{K})\colon U(\overline{K}) \to V(\overline{K})$ is contained in $j_{\bar{v}}^{-1}(B)$. As $j_{\bar{v}}(V)$ generates $\mathrm{Jac}^0_{V,\bar{K}}$ and B is a proper subgroup, the intersection $B \cap j_{\bar{v}}(V)$ must be finite, and so $\psi(\overline{K})$ has finite image. □

Proposition 3.17. *Let* U *and* V *be geometrically connected smooth curves over a Kummer-faithful field of characteristic zero, and assume that* $\pi_1(V_{\overline{K}}, \bar{v})$ *is non-trivial. Let*

$$\psi\colon \pi_1(U, \bar{u}) \to \pi_1(V, \bar{v})$$

be a point-theoretic morphism over Gal_K.

Then either ψ *is open, or the image of* ψ *is contained in a single decomposition group and a fortiori* $\psi(\overline{K})$ *is constant.*

Proof. Let us suppose that ψ is not open: then

$$\psi(\pi_1(U, \bar{u})) = \bigcap_{i \in I} \pi_1(V_i, \bar{v}_i) \subseteq \pi_1(V, \bar{v})$$

for pointed finite étale coverings $V_i \to V$ with V_i geometrically connected the degree of $V_i \to V$ becomes arbitrarily large. Then, by the Riemann–Hurwitz formula, the Euler characteristic

$$\chi_{V_i} = \deg(V_i \to V) \cdot \chi_V$$

of the V_i becomes an arbitrarily "large" negative number. The induced map $\psi_i\colon \pi_1(U, \bar{u}) \to \pi_1(V_i, \bar{v}_i)$ is still point-theoretic. Hence, by Proposition 3.16 the image of $\psi_i(\overline{K})$ is finite, and thus also the image of $\psi(\overline{K})$ is finite.

By choosing $\bar{u}$ and $\bar{v}$ as in the proof of Proposition 3.16 we can view $V(\overline{K})$ as embedded in $\mathrm{Jac}^0_V(\overline{K})$. From what we just showed, the subgroup generated by the image of $\psi(\overline{K})$

$$\langle \mathrm{im}\left(\psi(\overline{K})\colon U(\overline{K}) \to V(\overline{K})\right)\rangle \subset \mathrm{Jac}^0_V(\overline{K})$$

is finitely generated. However, it is also a quotient of the divisible group $\mathrm{Jac}^0_U(\overline{K})$. Thus, it must be trivial, which in turn implies that the image of $\psi(\overline{K})$ is just one point, namely $\bar{v}$.

For any open subgroup $H \subset \pi_1(V, \bar{v})$ the same conclusion holds for the induced map $f^{-1}(H) \to H$ interpreted as the point-theoretic map between fundamental groups of respective finite étale covers. This implies that any decomposition group $D_{\tilde{x}} \subset \pi_1(U, \bar{u})$ has image contained in the same, fixed decomposition group $D_{\tilde{v}}$ of $\pi_1(V, \bar{v})$.

Let $\Delta \subset \pi_1(U, \bar{u})$ be the (closure) of the subgroup generated by all decomposition groups of $\pi_1(U, \bar{u})$. We see that $\psi(\Delta)$ is open in $D_{\tilde{v}}$. Furthermore, for any $\gamma \in \pi_1(U, \bar{u})$, we have

$$\psi(\gamma)\psi(\Delta)\psi(\gamma)^{-1} = \psi(\Delta).$$

This implies that $\psi(\gamma)$ lies in the normalizer $N_{\pi_1(U,\bar{u})}(\psi(\Delta))$ of the image $\psi(\Delta)$. But, for any open subgroup $D \subset D_{\tilde{v}}$, it is known[1] that

$$N_{\pi_1(U,\bar{u})}(D) \subset D_{\tilde{v}},$$

see [Moc05, Theorem 1.3]. Thus, $\psi(\pi_1(U, \bar{u}))$ is contained in $D_{\tilde{v}}$, and clearly $D_{\tilde{v}}$ is not open. $\square$

Proposition 3.18. *Let U and V be geometrically connected smooth curves over a Kummer-faithful field. Suppose $\varphi\colon U_{et} \to V_{et}$ is a pinned morphism of sites over $K_{\text{ét}}$ such that the map*

$$\varphi(\overline{K})\colon U(\overline{K}) \to V(\overline{K})$$

is non-constant. Then the induced map

$$\varphi_*\colon \pi_1(U, \bar{u}) \to \pi_1(V, \bar{v})$$

is open.

Proof. Lemma 3.3 shows that $\varphi_*\colon \pi_1(U, \bar{u}) \to \pi_1(V, \bar{v})$ is point-theoretic. We first assume that V is hyperbolic. Then $j_V : V \to \mathrm{Jac}^1_V$ is an embedding into a torsor under the semi-abelian variety Jac_V, so that Proposition 3.12 shows that φ and φ_* induce the same map on $\overline{K}$-points. Therefore $\varphi_*(\overline{K})$ is also non-constant. Hence Proposition 3.17 applies and concludes the proof.

In the general case, we may reduce to V being hyperbolic as follows. For a sufficiently small Zariski open $j'\colon V' \to V$ we consider $j\colon U' = \varphi^{-1}(V') \to U$, which is also Zariski open. The pinned map φ of topoi induces a pinned map $\varphi'\colon U'_{\text{ét}} \to V'_{\text{ét}}$ of topoi over $K_{\text{ét}}$. There is a commutative diagram

$$\begin{tikzcd}
\pi_1(U', \bar{u}') \arrow[r, "\varphi'_*"] \arrow[d, "j'_*"] & \pi_1(V', \bar{v}') \arrow[d, "j_*"] \\
\pi_1(U, \bar{u}) \arrow[r, "\varphi_*"] & \pi_1(V, \bar{v})
\end{tikzcd}$$

with surjective vertical maps. The upper horizontal map is open by the hyperbolic case, and so is the lower map. $\square$

4. Étale reconstruction for sub-p-adic fields

The category $\mathbf{Sch}^{\mathrm{ft}}_K$ of schemes of finite type over a field K localized along the class of universal homeomorphisms is denoted by

$$\mathbf{Sch}^{\mathrm{ft}}_K[\mathrm{UH}^{-1}].$$

We now prove Theorem A, the main result, stated again for the convenience of the reader.

Theorem 4.1. *Let K be a sub-p-adic field. Then the functor*

$$(-)_{\text{ét}}\colon \mathbf{Sch}^{\mathrm{ft}}_K[\mathrm{UH}^{-1}] \longrightarrow \mathbf{RTop}^{\mathrm{pin}}_K$$

sending an X/K of finite type to its étale topos $X_{\text{ét}} \to K_{\text{ét}}$ is fully faithful.

[1] Mochizuki proves the required claim for local fields, but the same proof goes through for Kummer-faithful fields.

Recall that the inclusion of the category of seminormal schemes of finite type over K into $\mathbf{Sch}_K^{\mathrm{ft}}$ has a right-adjoint, the semi-nomalization. This functor seminormalization realizes the localization functor

$$\mathbf{Sch}_K^{\mathrm{ft}} \longrightarrow \mathbf{Sch}_K^{\mathrm{ft}}[\mathrm{UH}^{-1}]$$

since K has characteristic 0, see [CHW24, Corollary 1.15.1]. Semi-normalization agrees with absolute weak normalization in characteristic 0, therefore [CHW24, Theorem 4.18.1/5] reduces the proof of Theorem 4.1 to the following statement (we omit the assumption of a K-rational point, because we do not need it).

Proposition 4.2. *Let K be a sub-p-adic field, and let X be a smooth connected scheme of finite type over K. Then for any pinned morphism $\varphi\colon X_{\text{ét}} \to \mathbb{A}^1_{K,\text{ét}}$ over $K_{\text{ét}}$ there is a map $f : X \to \mathbb{A}^1_K$ over K such that*

$$f(\overline{K})\colon X(\overline{K}) \to \mathbb{A}^1_K(\overline{K}) = \overline{K}$$

agrees with the map $\varphi(\overline{K})$.

We now replace $\mathbb{A}^1_K$ by the hyperbolic curve $\mathbb{P}^1_K \smallsetminus \{0, 1, \infty\}$ in Proposition 4.2.

Lemma 4.3. *Let K be a sub-p-adic field. Theorem 4.1 follows from Proposition 4.5 below.*

Proof. Suppose given a pinned morphism $\varphi\colon X_{\text{ét}} \to \mathbb{A}^1_{K,\text{ét}}$. We cover $\mathbb{A}^1_K$ by two Zariski opens $U_1 = \mathbb{A}^1_K \smallsetminus \{0, 1\}$ and $U_2 = \mathbb{A}^1_K \smallsetminus \{2, 3\}$, both of which are isomorphic to $\mathbb{P}^1_K \smallsetminus \{0, 1, \infty\}$. The preimages $X_i = \varphi^*(U_i)$ form a Zariski open cover $X = X_1 \cup X_2$ of X and φ restricts to pinned morphisms $\varphi_i : X_{i,\text{ét}} \to U_{i,\text{ét}}$. By assumption, Proposition 4.5 yields morphisms $f_i : X_i \to U_i$ over K that agree with $\varphi_i(\overline{K})$ on $\overline{K}$-points. It follows that f_1 and f_2 agree on $X_1 \cap X_2$ and thus glue to a map $f : X \to \mathbb{A}^1_K$ with $f(\overline{K}) = \varphi(\overline{K})$. This verifies the criterion for Theorem 4.1 provided by Proposition 4.2. □

Lemma 4.4. *Let X be a smooth geometrically connected scheme of finite type over the field K. On $X(\overline{K})$ we consider the equivalence relation generated by pairs of points being equivalent if there is a geometrically connected smooth curve C over K and a map $g : C \to X$ over K with $x, y \in g(C(\overline{K}))$. Then any two points are equivalent.*

Proof. In order to connect $x, y \in X(\overline{K})$ we consider their Galois orbits, which is $Z(\overline{K})$ for a 0-cycle $Z \subsetneq X$. By [CP16, Corollary 1.9] there is a geometrically irreducible curve $C' \subseteq X$ that passes through Z. Let $C \to C$ be the normalization. Then C is smooth and geometrically connected over K, and x and y are in $g(C(\overline{K}))$ for $g : C \to C' \to X$. □

Proposition 4.5. *Let K be a sub-p-adic field, and let X be a smooth connected scheme of finite type over K. Then for any pinned morphism $\varphi\colon X_{\text{ét}} \to (\mathbb{P}^1_K \smallsetminus \{0, 1, \infty\})_{\text{ét}}$ over $K_{\text{ét}}$ there is a map $f : X \to \mathbb{P}^1_K \smallsetminus \{0, 1, \infty\}$ over K such that*

$$f(\overline{K})\colon X(\overline{K}) \to \mathbb{P}^1_K \smallsetminus \{0, 1, \infty\}(\overline{K})$$

agrees with the map $\varphi(\overline{K})$.

Proof. By Lemma 3.3 the homomorphism

$$\varphi_* : \pi_1(X, \bar{x}) \to \pi_1(\mathbb{P}^1_K \smallsetminus \{0, 1, \infty\}, \overrightarrow{01})$$

is point-theoretic, so that $\varphi_*(\overline{K})$ is well defined by Corollary 3.11 and agrees with $\varphi(\overline{K})$ by Proposition 3.12. Therefore we will search for a map f with $f(\overline{K}) = \varphi_*(\overline{K})$.

If φ_* is open, then the celebrated theorem of Mochizuki [Moc99, Theorem A] shows that there is a map $f : X \to \mathbb{P}^1_K \smallsetminus \{0, 1, \infty\}$ over K such that φ_* equals $\pi_1(f)$ up to conjugation by an element of the geometric fundamental group of $\mathbb{P}^1_K \smallsetminus \{0, 1, \infty\}$. In particular, then $f(\overline{K})$ equals $\varphi_*(\overline{K})$, and the proof is complete in this case.

We may now assume that φ_* is not open, and we are going to prove that $\varphi(\overline{K})$ is constant. We first show that $\varphi(\overline{K})$ is constant on the image of a map $g\colon C \to X$ from a geometrically connected smooth curve C over K. The map $\varphi \circ g_{\text{ét}}\colon C_{\text{ét}} \to (\mathbb{P}^1_K \smallsetminus \{0,1,\infty\})_{\text{ét}}$ is pinned as a composition of pinned maps. The induced map

$$(\varphi \circ g_{\text{ét}})_* = \varphi_* \circ \pi_1(g)\colon \pi_1(C, \bar{c}) \to \pi_1(\mathbb{P}^1_K \smallsetminus \{0,1,\infty\}, \overrightarrow{01})$$

is not open because its image is contained in the image of φ_*. Hence, by Proposition 3.18 applied to $\varphi \circ g_{\text{ét}}$, the map

$$\varphi(\overline{K}) \circ g_{\text{ét}}(\overline{K}) = (\varphi \circ g_{\text{ét}})(\overline{K})\colon C(\overline{K}) \to \mathbb{P}^1_K \smallsetminus \{0,1,\infty\}(\overline{K})$$

is constant. If X is geometrically connected, then Lemma 4.4 shows that $\varphi(\overline{K})$ is constant. If X is not geometrically connected, then no such map $g\colon C \to X$ exists.

Assume first that X is geometrically connected. By what we just proved then $\varphi(\overline{K})$ is constant, lets say with image $a \in \big(\mathbb{P}^1_K \smallsetminus \{0,1,\infty\}\big)(\overline{K})$. Pick any of the maps $g\colon C \to X$ used in the previous paragraph. Then $\varphi_* \circ \pi_1(g)$ factors over one of the decompositon groups D_a in the conjugacy class of decomposition subgroups associated to the point a. As C is geometrically connected over K the composition

$$\pi_1(C, \bar{c}) \xrightarrow{\varphi_* \circ \pi_1(g)} D_a \subset \pi_1(\mathbb{P}^1_K \smallsetminus \{0,1,\infty\}, \overrightarrow{01}) \xrightarrow{\text{pr}_*} \text{Gal}_K$$

is surjective. This implies that $a \in X(K)$ is a K-rational point. Hence there is a map

$$f\colon X \longrightarrow \text{Spec}(K) = \text{Spec}(k(a)) \longrightarrow \mathbb{P}^1_K \smallsetminus \{0,1,\infty\},$$

which agrees with $\varphi(\overline{K})$ on $\overline{K}$-points. Thus, we have proven our statement when X is geometrically connected.

On the other hand, if X is not geometrically connected, then by a base change to some finite Galois extension L/K contained in $\overline{K}$, we can assume that all connected components of the base change X_L have an L-rational point and therefore are geometrically connected over L. Hence, by performing the above argument to each connected component, we can construct a map

$$f_L\colon X_L \longrightarrow \mathbb{P}^1_L \smallsetminus \{0,1,\infty\}$$

such that $\varphi_L(\bar{K}) = f_L(\bar{K})$, where φ_L is the base change $(X_L)_{\text{ét}} \to (\mathbb{P}^1_L \smallsetminus \{0,1,\infty\})_{\text{ét}}$ of φ. Now φ_L is invariant under $\text{Gal}(L/K)$, so also $f_L(\overline{K})$ is Galois invariant and thus, moreover, the map f_L is Galois invariant. By Galois descent, the map f_L descends to a map $f\colon X \to \mathbb{P}^1_K \smallsetminus \{0,1,\infty\}$ such that $\varphi(\bar{K}) = f(\bar{K})$.

This also completes the proof of Theorem 4.1 and thus the main result of the paper. □

References

[CHW24] Magnus Carlson, Peter J. Haine, and Sebastian Wolf. *Reconstruction of schemes from their étale topoi*. 2024. arXiv: `2407.19920 [math.AG]`.

[CP16] François Charles and Bjorn Poonen. "Bertini irreducibility theorems over finite fields". In: *J. Am. Math. Soc.* 29.1 (2016), pp. 81–94. DOI: `10.1090/S0894-0347-2014-00820-1`.

[CV22] Brendan Creutz and José Felipe Voloch. "The Brauer-Manin obstruction for constant curves over global function fields". In: *Ann. Inst. Fourier* 72.1 (2022), pp. 43–58. DOI: `10.5802/aif.3473`.

[Gro97] Alexander Grothendieck. "Brief an G. Faltings". In: *Geometric Galois actions, 1*. Vol. 242. London Math. Soc. Lecture Note Ser. With an English translation on pp. 285–293. Cambridge: Cambridge Univ. Press, 1997, pp. 49–58.

[Hos17] Yuichiro Hoshi. "On the Grothendieck conjecture for affine hyperbolic curves over Kummer-faithful fields". In: *Kyushu J. Math.* 71.1 (2017), pp. 1–29. DOI: `10.2206/kyushujm.71.1`.

[Joh77] P. T. Johnstone. *Topos theory.* Vol. 10. Lond. Math. Soc. Monogr. Academic Press, London, 1977.

[KL81] Nicholas M. Katz and Serge Lang. "Finiteness theorems in geometric classfield theory. (With an appendix by Kenneth A. Ribet)". In: *Enseign. Math. (2)* 27 (1981), pp. 285–314, 315–319.

[Moc99] Shinichi Mochizuki. "The local pro-p anabelian geometry of curves". In: *Invent. Math.* 138.2 (1999), pp. 319–423. DOI: 10.1007/s002220050381.

[Moc05] Shinichi Mochizuki. "Galois sections in absolute anabelian geometry". In: *Nagoya Math. J.* 179 (2005), pp. 17–45. DOI: 10.1017/S0027763000025599.

[Moc15] Shinichi Mochizuki. "Topics in absolute anabelian geometry III: global reconstruction algorithms". In: *J. Math. Sci. Univ. Tokyo* 22.4 (2015), pp. 939–1156.

[Oht22] Sachiko Ohtani. "Kummer-faithful fields which are not sub-p-adic". In: *Res. Number Theory* 8.1 (2022). Id/No 15, p. 7. DOI: 10.1007/s40993-022-00311-2.

[Oht23] Sachiko Ohtani. "Corrigendum to: Kummer-faithful fields which are not sub-p-adic". In: *Res. Number Theory* 9.2 (2023), Paper No. 36, 7. DOI: 10.1007/s40993-023-00440-2.

[Ser88] Jean-Pierre Serre. *Algebraic groups and class fields.* Vol. 117. Graduate Texts in Mathematics. Translated from the French. Springer-Verlag, New York, 1988, pp. x+207. DOI: 10.1007/978-1-4612-1035-1.

[Sti13] Jakob Stix. *Rational points and arithmetic of fundamental groups. Evidence for the section conjecture.* Vol. 2054. Lect. Notes Math. Berlin: Springer, 2013. DOI: 10.1007/978-3-642-30674-7.

[Voe90] V. A. Voevodskiĭ. "Étale topologies of schemes over fields of finite type over $\mathbf{Q}$". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 54.6 (1990), pp. 1155–1167. DOI: 10.1070/IM1991v037n03ABEH002156.

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