GOETHE UNIVERSITÄT FRANKFURT AM MAIN

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Bewertungen

http://tinygu.de/VL-Bew19

5. Übungsblatt — Besprechung am 20. Januar 2020

Exercise 1. Let S, T be d-simplices such that $S \cup T \in \mathcal{T}^d$, the set of all simplices in $\mathbb{R}^d$. Show that S, T and $S \cap T$ are the parts of an elementary move of $S \cup T$.

Exercise 2. Let $Q \subset \mathbb{R}^d$ be a polyhedron and $p \in \mathbb{R}^d$. Show the following properties of the tangent cone:

(a) $T_pQ = \varnothing \iff p \notin Q$.

(b) Let $F \subseteq Q$ be a face. Then T_pF is a face of T_pQ.

(c) Let Q, Q' be two polyhedra. Then $T_p(Q \cap Q') = (T_pQ) \cap (T_pQ')$.

Exercise 3. i) For $c \in \mathbb{R}^d$ and a polyhedron $Q \subseteq \mathbb{R}^d$, let $Q^c := \{x \in Q : c^t x \geq c^t y \text{ for all } y \in Q\}$ be the (possibly empty) face of Q which maximizes c. Show, that the map $F_c : \mathcal{Q}^d \to \mathrm{Ch}(\mathcal{Q}^d)$ given by $Q \mapsto [Q^c]$ is a valuation.

ii) Let $P, Q \subset \mathbb{R}^2$ be polytopes of dimension ≤ 2. We write $P \sim Q$ if there are polytopes $R_i \in \mathcal{P}^2$ and $a_i \in \mathbb{Z}$ such that

$$[P] \;=\; \sum_{i=1}^{m} a_i[R_i] \quad \text{and} \quad [Q] \;=\; \sum_{i=1}^{m} a_i[t_i + R_i]$$

for some $t_i \in \mathbb{R}^2$. Show that $P \sim Q$ if and only if $P = Q + t$.

Exercise 4. A **lattice polytope** in the plane is a polytope $P \subset \mathbb{R}^2$ of the form $P = \mathrm{conv}(V)$ for $V \subset \mathbb{Z}^2$. A lattice simplex P of dimension ≤ 2 is unimodular if its vertices are the only lattice points in P.

i) Show that for any unimodular lattice simplex $S \subseteq \mathbb{Z}^2$ of dimension $d \leq 2$ one has for all $t \in \mathbb{Z}_{\geq 0}$

$$|tS \cap \mathbb{Z}^2| \;=\; \binom{t+1}{d}.$$

ii) Show that any lattice polygon P has a triangulation into unimodular simplices (that is, points, segments, and triangles).

iii) Conclude that $E_P(t) := |t \cdot P \cap \mathbb{Z}^2|$ agrees with a polynomial in t of degree ≤ 2. It is called the **Ehrhart polynomial**.

iv) Show Pick's Theorem: For any lattice polygon with b lattice points on the boundary and i lattice points in the interior one has

$$\mathrm{vol}(P) \;=\; i + \frac{b}{2} - 1\,.$$

First, show that a two dimensional unimodular simplex has volume $\frac{1}{2}$.

v) A **unimodular transformation** has the form $f(x) = Ax + t$ where $t \in \mathbb{Z}^2$ and $A \in \mathbb{Z}^{2\times 2}$ such that $\det(A) = \pm 1$. Call two lattice polygons P, P' **scissor congruent**, if they can be partitioned into relatively open lattice simplices $S_1, \ldots, S_m$ and $S'_1, \ldots, S'_m$ such that $S_i = f_i(S'_i)$ for some unimodular transformation f_i. Show that this is precisely the case if P and P' have the same Ehrhart polynomial.