GOETHE UNIVERSITÄT FRANKFURT AM MAIN

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Bewertungen

http://tinygu.de/VL-Bew19

2. Übungsblatt — Besprechung am 11. November 2019

Exercise 1. Show that $I := \mathbb{Z}\text{-span}\{a \vee b - a - b + a \wedge b : a, b \in L\} \subseteq \mathbb{Z}L$ is an ideal in the ring $(\mathbb{Z}L, +, \cdot)$, that is for all $f \in \mathbb{Z}L$, $g \in I$:

$$f \cdot g \in I\,.$$

Exercise 2. Let R be a ring with augmentation ε and for $f, g \in R$ define

$$f * g \;:=\; \varepsilon(g)f + \varepsilon(f)g - f \cdot g\,.$$

i) Show that $(R, +, *)$ is a ring, too.

ii) If $R = V(L)$ for some distributive lattice L, show that $a * b = a \vee b$.

Suppose that $\widehat{0}, \widehat{1} \in L$ are the minimal and maximal elements of L and define $\tau : V(L) \to V(L)$ by

$$\tau(f) \;:=\; \varepsilon(f)(\widehat{0} + \widehat{1}) - f \text{ for } f \in V(L).$$

iii) Show that τ is an **involution**, $\tau^2 = \mathrm{id}$, and thus a bijection. Furthermore show that $\tau(f \cdot g) = \tau(f) * \tau(g)$.

iv) For $a \in L$, we call an element $t \in L$ a **complement**[1], if $a \wedge t = \widehat{0}$ and $a \vee t = \widehat{1}$. Show that if a has a complement t, then $\tau(a) = t$, i.e. $\tau(a) - t \in I$.

Exercise 3. For a poset $(P, \preceq)$, an **ideal** is a set $I \subset P$ with $b \in I$ and $a \preceq b$ implies $a \in I$. The collection $\mathcal{J}(P) := \{I \subseteq P \text{ ideal}\}$ is a poset with respect to inclusion and with the usual set operations $\cup, \cap$ a distributive lattice. Show the converse: For every finite distributive lattice L, there is a poset P such that $L \cong \mathcal{J}(P)$. Use the theory we developed for the valuation ring $V(L)$.

[1] In general a complement does not need to be unique. However for distributive lattices it is, since $\iota : L \to V(L)$ is injective.