ALGEBRAIC AND GEOMETRIC COMBINATORICS

SUMMER TERM 2025

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Contents

↓ 1. Vorlesung, 22.4.2025 ↓

We write $\mathbb{N} = \mathbb{Z}_{\geq 0} = \{0, 1, \dots\}$, $\mathbb{P} = \mathbb{Z}_{>0} = \{1, 2, \dots\}$,
$[n] = \{1, 2, \dots, n\}$ and $[a, b] = \{a, a+1, \dots, b\}$.

1. A motivation

1.1. **Coloring graphs and partially ordered sets.** Let $G = (V, E)$ be an undirected graph. A **k-coloring** is a map $c : V \to [k]$ such that $c(u) \neq c(v)$ for all edges $uv \in E$. Define $\chi_G(k)$ as the number of k-colorings of G. The function $\chi_G : \mathbb{P} \to \mathbb{Z}_{\geq 0}$ was defined by Birkhoff in the hope of proving the 4-color theorem: If G is planar, then $\chi_G(4) > 0$. The reason why this might be a promising approach is that χ_G is a *nice* function.

Proposition 1.1 (Birkhoff)**.** *Let G be a graph with n nodes. Then there are $w_0, \dots, w_n \in \mathbb{Z}$ such that*

$$\chi_G(k) \;=\; w_n k^n + w_{n-1} k^{n-1} + \cdots + w_1 k + w_0$$

for all $k \in \mathbb{P}$.

We identify χ_G with this polynomial and write

$$\chi_G(t) \;=\; w_n t^n + w_{n-1} t^{n-1} + \cdots + w_1 t + w_0$$

where t is an indeterminate.

In order to prove this result, recall the *Principle of Inclusion-Exclusion.*

Theorem 1.2. *Let S be a finite set and $A_e \subseteq S$ subsets indexed by a finite set E. Then*

$$\Big| S \setminus \bigcup_{e \in E} A_e \Big| \;=\; |S| + \sum_{\varnothing \neq I \subseteq E} (-1)^{|I|} A_I \qquad \text{where } A_I := \bigcap_{e \in I} A_e \,.$$

Date: April 29, 2025.

This is easily proved by induction on $|E|$ but we will look into conceptual proofs later.

To apply this to our situation, fix $k \in \mathbb{P}$ and let $S = \{c : V \to [k]\}$. For an edge $e = uv \in E$ define $A_e := \{c \in S : c(u) = c(v)\}$. That is, A_e is the set of all labellings $c : V \to [k]$ that fail to be a coloring (at least) at the edge $e = uv$. Thus the k-colorings are precisely $S \setminus \bigcup_{e \in E} A_e$.

For a fixed $I \subseteq E$, consider the graph $G[I] := (V, I)$. By construction, A_I is the set of c such that $c(u) = c(v)$ whenever u, v are nodes in the same connected component of $G[I]$. Define $cc(I)$ to be the number of connected components of $G[I]$. Then $|A_I| = k^{cc(I)}$. Together this shows

$$\chi_G(k) = k^n + \sum_{\varnothing \neq I \subseteq E} (-1)^{|I|} k^{cc(I)} . \tag{1.1}$$

For the Petersen P_{10} graph this, for example, gives

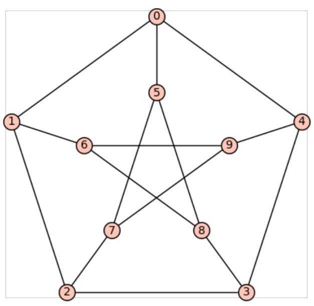
$$\chi_{P_{10}}(t) \;=\; t^{10} - 15t^9 + 105t^8 - 455t^7 + 1353t^6 - 2861t^5 + 4275t^4 - 4305t^3 + 2606t^2 - 704t$$

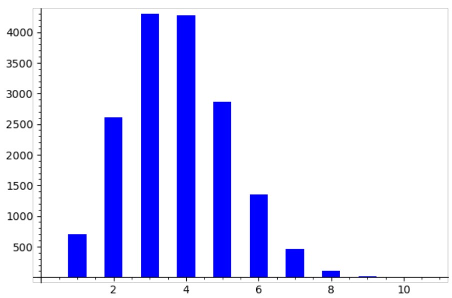
Things to observe:

- the coefficients are integers (clear);
- $\deg \chi_G(t) = |V|$, also clear, as $cc(I) < |V|$ for $I \neq \varnothing$;
- coefficient of $t^{|V|}$ is 1 (clear);
- there is no constant term (think for a second),
- the coefficients alternate in sign (not clear!).

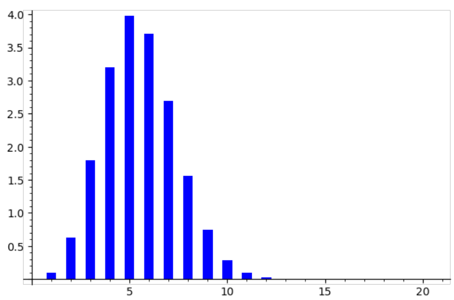
That's a fact that we will prove later. For now, let us write $\chi_G^+(t) = (-1)^{|V|}\chi_G(-t)$ for the polynomial all whose coefficients are positive. What do the coefficients count with respect to G? How are the individual coefficients related?

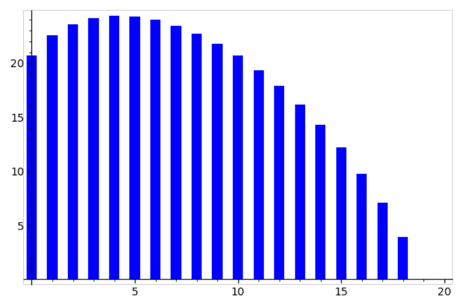
If we plot the coefficients of $\chi^+_{P_{10}}$, we get





This is not a coincidence. For a random graph with 20 nodes, the picture looks like this. Even stronger, if we plot the *logarithms* of the coefficients, we see





A sequence $a = (a_0, a_1, \dots, a_m)$ of positive integers is **unimodal** if there is an i such that $a_0 \le a_1 \le \cdots \le a_i \ge a_{i+1} \ge \cdots \ge a_m$. The sequence is a is **log-concave** if $a_i^2 \ge a_{i-1}a_{i+1}$ for all $0 < i < m$.

Exercise 1.1. If a sequence a of positive integers is log-concave, then a is unimodal.

What we are seeing in the plots is that the sequence of absolute values of coefficients is log-concave and hence unimodal. In particular, the individual entries are not independent of each other!

Example 1.1 (Complete graphs)**.** Let K_n be the complete graph on nodes $[n]$. Coloring the nodes one at a time, we see that the chromatic polynomial is simply $\chi_{K_n}(t) = t(t-1)(t-2)\cdots(t-n+1)$ and

$\chi^+_{K_n}(t) = t(t+1)\cdots(t+n-1)$. Recall that every permutation can be uniquely decomposed into cycles. For example the permutation $(\tau(1), \tau(2), \tau(3)) = (3, 2, 1)$ has cycles $(1, 3)$ and (2). The **(unsigned) Stirling number of the first kind** $\bar{s}_{n,k}$ counts the number of permutations with exactly k cycles. For $0 < k \leq n$, they satisfy the recurrence relation

$$\bar{s}_{n,k} \;=\; \bar{s}_{n-1,k-1} + (n-1)\bar{s}_{n-1,k}\,,$$

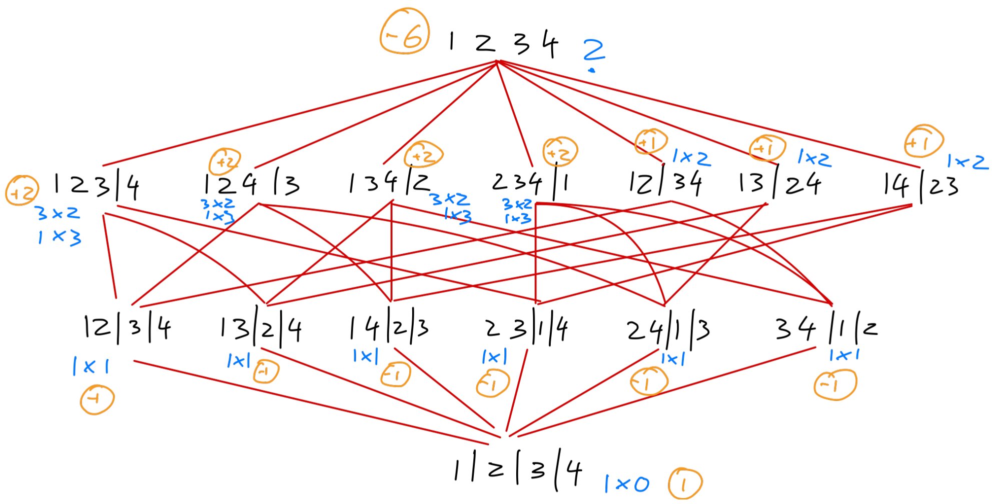
and together with $\bar{s}_{n,1} = (n-1)!$ and $\bar{s}_{n,0} = 0$, ones verifies that

$$\chi^+_{K_n}(t) \;=\; \bar{s}_{n,n}t^n + \cdots + \bar{s}_{n,2}t^2 + \bar{s}_{n,1}t\,. \qquad \diamond$$

Our derivation of $\chi_G(t)$ had $2^{|E|}$ terms to produce a polynomial with exactly $|V|$ terms. So, there *has* to be a better way to compute $\chi_G(t)$. A more insightful way is to observe that every $I \subseteq E$ defines an equivalence relation on V: call nodes $u, v \in V$ equivalent if they are in the same connected component of $G[I]$. This gives a decomposition $V = V_1 \uplus V_2 \uplus \cdots \uplus V_k$ into equivalence classes. We call $P(I) = \{V_1, \ldots, V_k\}$ an **unordered partition** of V and we write $\mathrm{Par}(G) := \{P(I) : I \subseteq E\}$. For $I = \varnothing$, we obtain $P(\varnothing) = \{\{v\} : v \in V\}$. Note that $cc(I) = |P(I)|$, the number of parts of the partition. So, as an intermediate step in our computation of (1.1) we get

$$\chi_G(t) \;=\; \sum_{I \subseteq E} (-1)^{|I|} t^{cc(I)} \;=\; \chi_G(t) \;=\; \sum_{P \in \mathrm{Par}(G)} t^{|P|} \sum_{\substack{I \subseteq E \\ P(I) = P}} (-1)^{|I|}\,.$$

More generally, we write Par_n for the collection of all unordered partitions of $[n]$. This is exactly the unordered partitions that we get for the case $G = K_n$. An unordered partition $P = \{P_1, \ldots, P_k\}$ **refines** the unordered partition $P' = \{P'_1, \ldots, P'_l\}$ if for every $i \in [k]$ there is a $j \in [l]$ such that $P_i \subseteq P'_j$. That is, the unordered partitions refining P' arise by replacing some P'_j by an unordered partition of it. We also say that P' is a **coarsening** of P. 'Refinement' defines a partial order on Par_n. The following picture shows the partition lattice Par_4. The unordered partitions are in black and we read $23 \mid 1 \mid 4$ as $\{\{2, 3\}, \{1\}, \{4\}\}$. The red lines indicate refinements but only the necessary ones!



Observe that $\{\{1, 2, 3\}, \{4\}\} = P(I)$ for three sets I of cardinality 2 and one set I with $|I| = 3$. This explains the blue labels $3 \times 2, 1 \times 3$. As for the orange numbers, this is $\sum_I (-1)^{|I|}$, where the sum is over those I that gives $P(I)$. Thus, in the example above, the sum will be $(-1)^2 + (-1)^2 + (-1)^2 + (-1)^3 = 2$. The chromatic polynomial for K_4 thus gives

$$\chi_{K_4}(t) = 1t^4 - 6t^3 + 11t^2 - 6t\,.$$

The ? is for you to determine. However, we can still compute the orange number. The trick is this: once we know the orange 1 at the bottom, we can go layer by layer. At every partition P the orange number is the negative of all the orange numbers of partitions that refine P!

The collection of unordered partitions $\mathrm{Par}(G) = \{P(I) : I \subseteq E\}$ for a given graph G *partially ordered* by refinement structurally captures what happens when computing the characteristic polynomial. Such partially ordered sets, or **posets** for short give structure and guidance in many situations. As Gian-Carlo Rota wrote

> *It often happens that a set of objects to be counted possesses a natural ordering, in general only a partial order. It may be unnatural to fit the enumeration of such a set into a linear order such as the integers: instead, it turns out in a great many cases that a more effective technique is to work with the natural order of the set. One is led in this way to set up a "difference calculus" relative to an arbitrary partially ordered set.*

He wrote this in the introduction to the seminal paper [3] in which he introduced the concept of a Möbius function. The Möbius function is is related to the number theoretic Möbius function, gives a vast generalization of the principle of inclusion-exclusion, and is responsible for the orange numbers above. We will thoroughly study partially ordered sets, their Möbius functions, and how to compute them. The partially ordered sets that we will be studying are motivated by combinatorics/discrete math and geometry and will show beautiful connections to geometry/topology and algebra. Here are some examples:

- $B_n = (2^{[n]}, \subseteq)$, the collection of subsets of an n-set partially ordered by inclusion;
- Par_n and $\overline{\mathrm{Par}}_n$ the collection of (un)ordered partitions of an n-set ordered by refinement;
- Permutations partially ordered by their inversion sets;
- (normal) subgroups of a given (finite) group ordered by inclusion;
- The collection $B_n(q)$ of linear subspaces of $\mathbb{F}_q^n$, the n-dim vector space over the finite field $\mathbb{F}_q$, ordered by inclusion;
- Cliques or stable sets of a graph G, ordered by inclusion;
- Cycle-free subsets of edges of a graph G ordered by inclusion;
- Subsets of non-crossing diagonals of a convex n-gon ordered by inclusion.

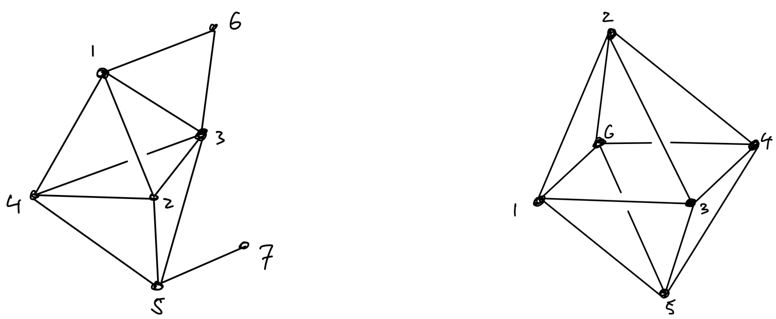
We will see many more examples and, in particular, distill important types of posets.

↓ 2. Vorlesung, 24.4.2025 ↓

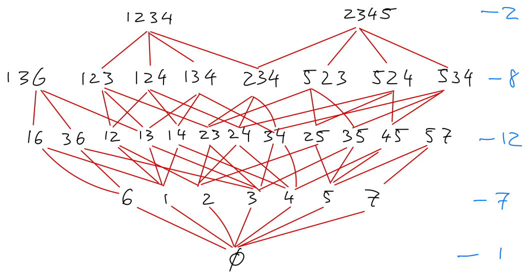
1.2. **Simplicial complexes and the Upper Bound Conjecture.** The last three examples together with B_n they stand out: Let E be a finite set. A collection of subsets $\varnothing \neq \Delta \subseteq 2^E$ is a **hereditary set system** or a **simplicial complex** if for all $\tau \in \Delta$ and $\sigma \subseteq \tau$ we have $\sigma \in \Delta$.

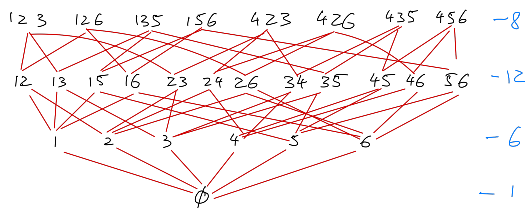
Example 1.2. Let $G = (V, E)$ be a graph. A set $K \subseteq V$ is a **clique** if for any distinct $u, v \in K$, $uv \in E$. A set $S \subseteq V$ is **stable** if for any distinct $u, v \in K$, $uv \notin E$. A subset $F \subseteq F$ is **cycle-free** if $G[F]$ does not contain cycles. Every subset of a clique (or stable set) in a graph is a clique (or stable set). Every subset of a cycle-free subset is cycle-free. ◇

The property of being a clique or cycle-free is *inherited* under taking subsets. The name 'simplicial complex' comes from a geometric/topological context. A geometric simplex is a point, a segment, a triangle, a tetrahedron etc. A simplicial complex is a collection of simplices with the property that if simplices meet, then their intersection is a face of both. We make this more precise later, but for now these two pictures should convey the idea:

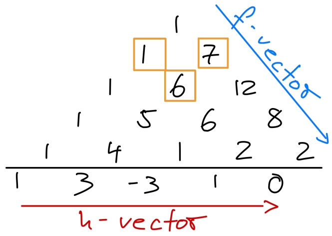


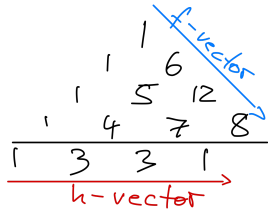
There is a clear IKEA-type gluing description of both simplicial complexes:





A natural complexity measure of a simplicial complex is given by its **face vector** of **f-vector**. For Δ, $f_i(\Delta)$ counts the number of faces of dimension $i \geq -1$. The empty set (at the bottom) is always a face of dimension -1. For the left complex Δ_1 this gives $f(\Delta_1) = (1, 7, 12, 8, 2)$ for the right complex $f(\Delta_2) = (1, 6, 12, 8)$. As we will learn, often a better way to represent the information given by the f-vector is in the form of the h-vector. The following shows how to compute them





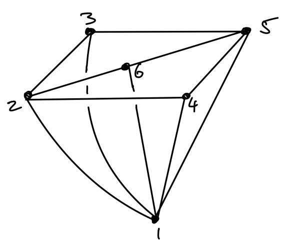
The highlighted 6 is obtained as $7 - 1$ and this gives the complete set of rules to go from the f-vector to the h-vector and back. Whereas the h-vector on the left-hand side does does not reveal more information, the one on the right-hand side looks promising: it is non-negative and symmetric/palindromic.

Exercise 1.2. A simple graph $G = (V, E)$ can be viewed as a simplicial complex $\Delta = \{\varnothing\} \cup V \cup E$. The f-vector is just $f(\Delta) = (1, |V|, |E|)$. Classify when $h(\Delta)$ is non-negative and when it is palindromic.

For geometrically/topologically interesting classes of simplicial complexes the h-vector will always be non-negative and palindromic! The vague answer we give here is that the right complex resembles a sphere whereas the left one does not. In fact, the right complex *is* the unit sphere in the ℓ_1-norm but we mean that it resembles a sphere in a topological sense, independent of how the complex is geometrically realized.

Theorem 1.3. *Let Δ be a simplicial complex with $h(\Delta) = (h_0, \ldots, h_d)$. If Δ is topologically a sphere, then $h_0, \ldots, h_d \geq 0$ and $h_i = h_{d-i}$.*

Thus geometric/topological objects give rise to posets whose invariants (f-vectors, h-vectors) we can combinatorially interpret. Conversely, we will associate to any poset a geometric/topological object whose geometric/topological features will *explain* certain combinatorial information. For example, for the partition lattice, we will see that we can associate to it the following simplicial complex



This is a complex glued from 6 triangles with f-vector $(1, 6, 11, 6)$. These are precisely the coefficients of the chromatic polynomial $\chi^+_{K_4}(t)$. We will see that the entries of the f-vector have to satisfy certain conditions and this automatically gives conditions on the coefficients of $\chi_G(t)$.

A highlight of the course will be a resolution of the *Upper Bound Conjecture* for spheres. Suppose Δ is topologically a sphere of dimension $d-1$. What is the maximal number of i-dimensional faces, that is, what is the maximal $f_i(\Delta)$ for a fixed number of vertices $f_0(\Delta)$? Motzkin constructed geometric spheres, so-called *neighborly* spheres, for which he conjectured that they maximize the number of i-dimensional faces for all i simultaneously among all geometric/convex spheres. McMullen [2] proved

Motzkin's Upper Bound Conjecture for convex spheres (that is, simplicial polytopes). There, he introduced the notion of an h-vector. Victor Klee [1] suggested to extend the UBC to all spheres and Richard Stanley [4] combined ideas from combinatorics, topology, and commutative algebra in a spectacular way to resolve the UBC for spheres.

Theorem 1.4. *Let Δ be a $(d-1)$-dimensional simplicial sphere with n vertices and $h(\Delta) = (h_0, \dots, h_d)$. Then $h_i = h_{d-i}$ and*

$$h_i \;\le\; \binom{n-d-1+i}{i}$$

for all i. If equality is attained for $i = \lfloor \frac{d}{2} \rfloor$, then Δ is a neighborly sphere.

1.3. **Polynomials and Hilbert series.** To give an idea of the sort of algebra that we will be using, recall that a polynomial in a single variable t with coefficients in $\mathbb{C}$ is an expression of the form

$$c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0 t^0$$

The collection of all such polynomials is denoted by $\mathbb{C}[t]$. This is a $\mathbb{C}$-vector space and the fact that polynomials can be multiplied along the rules $t^i t^j = t^{i+j}$ turns $\mathbb{C}[t]$ into a or $\mathbb{C}$-algebra.

We may extend this to polynomials in many variables. For $n \ge 1$, let $x_1, \dots, x_n$ indeterminates. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\ge 0}$, we write $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. A multi-variate polynomial is then

$$\sum_{\alpha \in A} c_\alpha \mathbf{x}^\alpha$$

where $A \subset \mathbb{Z}^n_{\ge 0}$ is a finite set and $c_\alpha \in \mathbb{C}$ for all $\alpha \in A$. Again, polynomials form a $\mathbb{C}$-vector space and together with multiplication $\mathbf{x}^\alpha \cdot \mathbf{x}^\beta = \mathbf{x}^{\alpha+\beta}$ give the ring of polynomials $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_n]$. The degree of a monomial is $\deg \mathbf{x}^\alpha = |\alpha| = \alpha_1 + \cdots + \alpha_n$. We write $\mathbb{C}[\mathbf{x}]_d$ for the subvector space spanned by monomials of degree d. Elementary combinatorics shows that the vector space dimension is $\dim_\mathbb{C} \mathbb{C}[\mathbf{x}]_d = \binom{n+d-1}{d}$. The Hilbert series is the generating function that incorporates the dimensions:

$$H(\mathbb{C}[\mathbf{x}], z) \;=\; \sum_{d \ge 0} \dim_\mathbb{C} \mathbb{C}[\mathbf{x}]_d z^d \;=\; \sum_{d \ge 0} \tbinom{n+d-1}{d} z^d \;=\; \frac{1}{(1-z)^{d+1}}\,.$$

Now let us consider $\mathbb{C}[x_1, \dots, x_6]$ together with the conditions $\mathbf{x}^\alpha = 0$ if $\mathbf{x}^\alpha$ is divisible by x_1x_4, x_2x_5, or x_3x_6. For a monomial $\mathbf{x}^\alpha$, the support is $\operatorname{supp}(\mathbf{x}^\alpha) = \{i : \alpha_i > 0\}$. Thinking back to our example of the *octahedron* on page 4, we can express this condition as $\mathbf{x}^\alpha = 0$ whenever $\operatorname{supp}(\mathbf{x}^\alpha) \not\in \Delta$. Thus, we somehow encoded the combinatorics of Δ into polynomials. Our conditions are compatible with multiplication and give a new ring $\mathbb{C}[\Delta]$ whose elements can still be expressed by monomials in $x_1, \dots, x_6$. Moreover, for $d \ge 0$, we can define $\mathbb{C}[\Delta]_d = \mathbb{C}\text{-span}\{\mathbf{x}^\alpha \in \mathbb{C}[\mathbf{x}]_d : \operatorname{supp}(\mathbf{x}^\alpha) \in \Delta\}$. Computing the Hilbert series now yields

$$H(\mathbb{C}[\Delta], z) \;=\; \sum_{d \ge 0} \dim_\mathbb{C} \mathbb{C}[\Delta]_d z^d \;=\; \frac{1 + 3z + 3z^2 + z^3}{(1-z)^4}$$

It is not a coincidence that the numerator polynomial is exactly the h-vector of Δ!

Let us further add the conditions $x_1 = x_4$, $x_2 = x_5$, $x_3 = x_6$. So, every time we see an x_4, we may replace it by x_1. In particular, the condition $x_1x_4 = 0$ can be read as $x_1^2 = 0$. This gives us a new ring R obtained from $\mathbb{C}[x_1, x_2, x_3]$ with the conditions $x_i^2 = 0$. This is quite a simple ring. As a $\mathbb{C}$ vector space R has the basis $1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3$ and hence models all subsets of $[3]$. We can do this more general, for $n \ge 1$, let R be the $\mathbb{C}$-vector space with basis $\mathbf{x}^\tau = \prod_{i \in \tau} x_i$ for all subsets $\tau \subseteq [n]$. If $\tau = \varnothing$, then $\mathbf{x}^\tau = 1$. We define a multiplication on R by setting $\mathbf{x}^\tau \cdot \mathbf{x}^\sigma = \mathbf{x}^{\tau \cup \sigma}$ if $\tau \cap \sigma = \varnothing$ and $\mathbf{x}^\tau \cdot \mathbf{x}^\sigma = 0$ if $\tau \cap \sigma \ne \varnothing$. We can write $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n$ where $R_i = \mathbb{C}\text{-span}\{\mathbf{x}^\tau : |\tau| = i\}$. In particular, $\dim R_i = \binom{n}{i}$. Its Hilbert series satisfies

$$H(R, z) \;=\; \sum_{d \ge 0} \dim_\mathbb{C} R_i z^i \;=\; \sum_{i=0}^{d} \tbinom{n}{i} z^i \;=\; (1+z)^n\,.$$

We can now algebraically argue that $\binom{n}{i} = \binom{n}{n-i}$. We first note that $R_n \cong \mathbb{C}$. Thus, for any fixed i, multiplication gives a bilinear form $B_i : R_i \times R_{n-i} \to \mathbb{C}$. For fixed $f \in R_i$, the map $R_{n-i} \ni g \mapsto B(f, g)$

is a linear function on R_{n-i}. If we can show that $f \mapsto B(f, \cdot)$ is injective, then we have shown $\dim R_i \leq \dim R_{n-i}$ (why?). Applying the same reasoning to $g \mapsto B(\cdot, g)$, then proves $\dim R_i = \dim R_{n-i}$. That's unnecessarily difficult when it comes to binomial coefficients but in general that's the way to go.

This perspective also suggests a way to show that $\binom{n}{i} \leq \binom{n}{i+1}$ for $i \leq \lfloor \frac{n}{2} \rfloor$. We will show that for $\omega = x_1 + x_2 + \cdots + x_n$, the linear map $R_i \to R_{i+1}$ given by $f \mapsto \omega \cdot f$ is injective whenever $i \leq \lfloor \frac{n}{2} \rfloor$ and surjective otherwise. Injectivity, of course, then implies $\binom{n}{i} = \dim_{\mathbb{C}} R_i \leq \dim_{\mathbb{C}} R_{i+1} = \binom{n}{i+1}$.

↓ 3. Vorlesung, 29.4.2025 ↓

2. Partially ordered sets

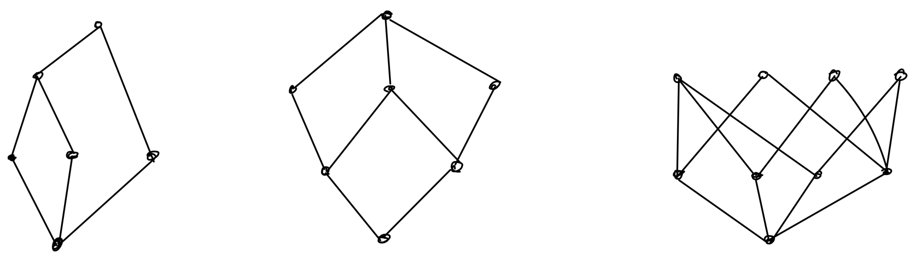
We start with partially ordered sets.

Definition 2.1. A **partially ordered set** (or **poset**) is a pair $(P, \preceq)$ where P is a set and $\preceq$ is a binary relation satisfying

- $a \preceq a$ for all $a \in P$ (Reflexivity)
- $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ for all $a, b, c \in P$ (Transitivity)
- $a \preceq b$ and $b \preceq a$ implies $a = b$ for all $a, b \in P$ (Anti-symmetry)

We usually call P the poset when $\preceq$ is clear from the context and we write $\preceq_P$ to emphasize the relation to the ground set P. We call two elements $a, b \in P$ **comparable** if $a \preceq b$ or $b \preceq a$. We say that b **covers** a or a is **covered** by b if $a \prec b$ and there is no c with $a \prec c \prec b$. In this case we write $a \prec\!\!\cdot\, b$. Note that our posets are not necessarily finite and cover relations need not exist.

For two elements $a, b \in P$, the **interval** is $[a, b]_P = \{c \in P : a \preceq c \preceq b\}$. This is an *induced* subposet of P by restricting $\preceq$ to $[a, b]_P$. Note that $[a, b]_P = \varnothing$ if $a \not\preceq b$. We call poset **locally-finite** if $[a, b]_P$ is finite for every $a, b \in P$. If P is locally-finite, we can encode $\preceq$ by a directed graph (digraph) on the node set P with a directed edge (a, b) if $a \prec\!\!\cdot\, b$. This is an acyclic[1] digraph and $a \preceq b$ if there is a path $a = a_0 a_1 \ldots a_k = b$ such that (a_{i-1}, a_i) is a directed edge for $i = 1, \ldots, k$. We may visualize this digraph by a drawing in the plane for which the edges (a, b) have positive slope. Thus $b \succeq a$ if there is a path from b to a with monotonically decreasing y-coordinate. Such as drawing is called a **Hasse diagram**. Here are three examples:



An element $a \in P$ is **maximal** if there is no $b \in P$ with $a \prec b$. Minimal elements are defined accordingly. We write $\min(P)$ and $\max(P)$ for the minimal an maximal elements. The poset has a **maximum** if there is $m \in P$ with $a \preceq m$ for all $a \in P$. Note that every maximum is maximal but not necessarily the other way round!

If a maximum exists, it is necessarily unique and denoted by $\hat{1}$. All but the right-most poset in the figure above have a maximum. A minimum, provided it exists, is denoted by $\hat{0}$. Note that if $[a, b]_P \neq \varnothing$, then a and b are minimum and maximum, respectively.

A homomorphism or **order-preserving map** between to posets $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ and is a map $f : P_1 \to P_2$ such that

$$a \ \preceq_1 \ b \quad \Longrightarrow \quad f(a) \ \preceq_2 \ f(b)$$

is satisfied for all $a, b \in P_1$. If f is a bijection and f^{-1} is also order-preserving, then P_1 and P_2 are **isomorphic**, denoted by $P_1 \cong P_2$.

[1] No directed cycles

A subset $C \subseteq P$ is called a **totally ordered**, **linearly ordered** or simply a **chain** if any two elements in C are comparable. If C is finite then there is a labelling of the elements $C = \{a_0, a_1, \dots, a_k\}$ such that $a_0 \prec a_1 \prec \cdots \prec a_k$. Then **length** of a finite chain is $\ell(C) = |C| - 1$, the number of 'links' in a chain. A chain is **saturated** or **unrefineable** if for any three elements $a \prec b \prec c$ we have that $a, c \in C$ implies $b \in C$. If C is finite, then this is equivalent to $a_{i-1} \prec\!\!\cdot\, a_i$ for $i = 1, \dots, k$. The chain C is **maximal** if there is no chain C' with $C \subset C'$. Thus, maximal chains are saturated but the converse is not true in general.

The **rank** $r(P)$ of a poset P is the maximal length of a chain. For $a, b \in P$, we will write $\ell_P(a, b) = r([a, b]_P)$. We simply write $\ell(a, b)$ if P is clear from the context. A poset P is **graded** if all maximal chains have the same (finite) length $r(P)$. If P is graded, then there is a unique function $r : P \to \mathbb{Z}_{\geq 0}$ called the **rank function** with with $r(a) = 0$ for all $a \in \min(P)$ and $r(b) = r(a) + 1$ for $a \prec\!\!\cdot\, b$. If P is finite, then the distribution of ranks is recorded by the **rank-generating function**

$$F(P, t) = \sum_{a \in P} t^{r(a)} = p_0 + p_1 t^1 + \cdots + p_r t^r ,$$

where $p_i = \#\{a \in P : r(a) = i\}$ and $r = r(P)$.

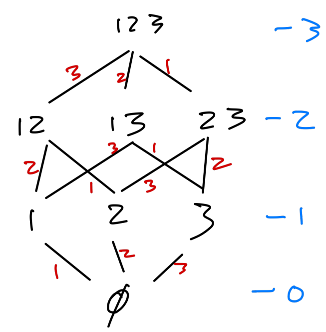
Example 2.1 (Chains and Anti-chains)**.** The prototypical chain of length n is the set $[n] := \{1, \dots, n\}$ together with the natural order. We call $C_n = ([n], \leq)$ **the chain** with n elements. Thus $C \subset P$ is an n-chain if the induced subposet C is isomorphic to $[n]$. Chains are clearly ranked posets with rank-generating function

$$F([n], q) \;=\; 1 + q + q^2 + \cdots + q^{n-1} \;=: (n)_q \,.$$

We call the polynomial $(n)_q$ a 'q-analogue' of the number n.

The conceptual opposite of a chain is a set $A \subseteq P$ such that any two distinct elements in A are incomparable. Such a set is called an **anti-chain**. This is a graded poset with $F(A, t) = |A| t^0$. ⋄

Example 2.2 (Boolean lattice)**.** For any set S, the Boolean lattice is the poset on $2^S = \{T : T \subseteq S\}$ partially ordered by inclusion. We write $B_n = (2^{[n]}, \subseteq)$ and note that $(2^S, \subseteq) \cong B_n$ if $|S| = n$. The Boolean lattice has minimum $\hat{0} = \varnothing$ and maximum $\hat{1} = S$. For $A \subseteq B$, we observe $[A, B] \cong (2^{B \setminus A}, \subseteq)$. In particular, $A \prec\!\!\cdot\, B$ if $|B \setminus A| = 1$. Here is B_3:

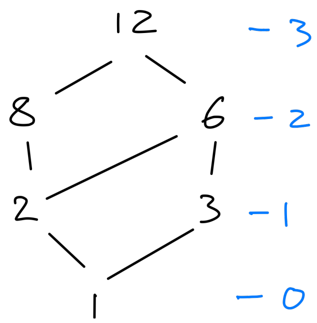


Hence, B_n is graded with $r(A) = |A|$. The rank-generating function satisfies

$$F(B_n, q) \;=\; \sum_{i=0}^{n} \binom{n}{i} t^q \;=\; (1 + q)^n \,.$$

There is a close connection between B_n and permutations. A maximal chain in B_n is of the form $\varnothing = S_0 \subset S_1 \subset \cdots S_{n-1} \subset S_n = [n]$. In particular $S_i \setminus S_{i-1} = \{a_i\}$ for a $a_i \in [n]$. Since $a_i \neq a_j$ for $i \neq j$, this defines a permutation $i \mapsto a_i$. Hence, maximal chains in B_n are in one-to-one correspondence with permutations of $[n]$. In the example, the permutations can be read from the red numbers on the cover relations. ⋄

Example 2.3 (Divisibility)**.** For $n \in \mathbb{Z}_{>0}$, define the $\mathcal{D}_n$ as the set of $a \in \mathbb{Z}_{>0}$ with a divides n. We partially order $\mathcal{D}_n$ by setting $a \preceq b$ if there is a $k \in \mathbb{Z}_{>0}$ such that $b = ka$. It has a minimum $\hat{0} = 1$ and maximum $\hat{1} = n$. Here is $\mathcal{D}_{12}$:



Note that for $a \prec b$ in $\mathcal{D}_n$, we have $[a, b] \cong \mathcal{D}_{b/a}$. In particular $a \prec\!\!\cdot\, b$ if and only if $\frac{b}{a}$ is prime. It follows from the Fundamental Theorem of Arithmetic that $\mathcal{D}_n$ is graded. The rank of $\mathcal{D}_n$ is $r(\mathcal{D}_n) = k_1 + k_2 + \cdots + k_s$ $n = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$ where $p_1, \dots, p_s$ are the distinct prime factors. To compute the rank-generating function, we observe that any $a \in \mathcal{D}_n$ is of the form $a = p_1^{l_1} p_2^{l_2} \cdots p_s^{l_s}$ for $0 \le l_i \le k_i$ and has rank $r(a) = l_1 + \cdots + l_s$. It is now easy to check that

$$F(\mathcal{D}_n, q) \;=\; \prod_{i=1}^{s} (k_i + 1)_q \,. \qquad \diamond$$

The above example prompts for a simple construction on posets. For two posets $(P_1, \preceq_1), (P_2, \preceq_2)$ define the **direct/Cartesian product** as the partial order on $P_1 \times P_2$ by

$$(a_1, a_2) \;\preceq\; (b_1, b_2) \quad :\Longleftrightarrow \quad a_1 \;\preceq_1\; b_1 \text{ and } a_2 \;\preceq_2\; b_2 \,.$$

It is straightforward to verify that $(P_1 \times P_2, \preceq)$ is a graded poset whenever P_1 and P_2 are. The rank-generating function satisfies

$$F(P_1 \times P_2, t) \;=\; F(P_1, t) F(P_2, t)$$

If $n = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$, then

$$\mathcal{D}_n \;\cong\; [k_1 + 1] \times [k_2 + 1] \times \cdots \times [k_s + 1] \,.$$

By the same token, we get $B_n \cong ([2], \le)^n$, where we identify subsets $A \subseteq [n]$ with vectors $v_A \in [2]^n$ with $(v_A)_i = 2$ if and only if $i \in A$.

Example 2.4 (Lattice of subspaces)**.** Let $\mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$, the finite field with q elements. Write $B_n(q)$ for the collection of vector subspaces of $\mathbb{F}_q^n$, partially ordered by inclusion. This is a finite poset with minimum $\hat{0} = \{\mathbf{0}\}$, where $\mathbf{0} \in \mathbb{F}_q^n$ is the zero vector and maximal element $\hat{1} = \mathbb{F}_q^n$. For two subspaces $U \subseteq W$, basic linear algebra tells us that the subspaces $V \subseteq \mathbb{F}_q^n$ with $U \subseteq V \subseteq W$ are precisely the subspaces of the quotient $W/U \cong \mathbb{F}_q^k$ with $k = \dim W - \dim U$. Since the isomorphism retains inclusions, we obtain $[U, W]_{B_n(q)} \cong B_k(q)$. From this, we get that $B_n(q)$ is a graded poset with rank function $r(U) = \dim U$.

The poset $B_n(q)$ is called a 'q-analogue' of $B_n = (2^{[n]}, \subseteq)$ for reasons that are difficult to make precise. In essence, it means that many of its enumerative invariants are polynomials in q whose specializations $q = 1$ yield respective invariants for B_n. For example, a 1-dimensional subspace $L \subset \mathbb{F}_q^n$ is the span of a vector $v \in \mathbb{F}_q^n \setminus \{\mathbf{0}\}$ and $a \cdot v$ spans L for all $a \in \mathbb{F}_q \setminus \{0\}$. It follows that that there are $q^n - 1$ choices of v and

$$\frac{q^n - 1}{q - 1} \;=\; 1 + q + q^2 + \cdots + q^{n-1} \;= (n)_q$$

many distinct lines in $\mathbb{F}_q^n$. Now a subspace U is covered by W if $U \subset W$ and W/U is 1-dimensional. Hence if we want to construct a maximal chain in $B_q(n)$, we start with $V_0 = \{\mathbf{0}\}$. For $i = 1, \dots, n$ we now choose a line $V_i \subset \mathbb{F}_q^n / V_{i-1}$. Since $\dim V_i = i$, we have $(i)_q$ many choice. In total, the number of maximal chains is

$$(n)_q (n-1)_q \cdots (2)_q (1)_q \;=: (n)_q!$$

Of course, there is no finite field with $q = 1$ elements but the above formula still makes sense and yields a q-analogue of the factorial $n! = n \cdot (n-1) \cdots 2 \cdot 1$.

To count the number of elements of $B_n(q)$ of rank k, that is, the number of k-dimensional subspaces of $\mathbb{F}_q^n$, let us count for a fixed k-subspace U the number of maximal chains V_* with $V_k = U$. This is easy, because we simply need to count the maximal chains in $[U, \hat{1}] \cong B_{n-k}(q)$ and the maximal chains in $[\hat{0}, U] \cong B_k(q)$. This is this gives exactly $(n-k)_q(k)_q$ such chains, independent of the actual choice of U. Thus, the number of k-subspaces is precisely

$$\frac{(n)_q}{(n-k)_q(k)_q} \;=:\; \binom{n}{k}_q,$$

which is a q-analog of the binomial coefficient, also called a **Gaussian polynomial**. Pleasantly, we obtain $\binom{n}{k}$ for $q = 1$. The rank-generating function for $B_n(q)$ is thus

$$F(B_n(q), t) \;=\; \sum_{k=0}^{n} \binom{n}{k}_q t^k\,. \qquad \diamond$$

2.1. **Lattices.** For two elements a, b in a poset P, a **least upper bound** or **supremum** is an element c such that $a \preceq c$ and $b \preceq c$ and for every c' with these properties satisfies $c \preceq c'$. Least upper bounds do not need to exist but if they do, they are unique. We denote them by $a \vee b := c$ and call $a \vee b$ the **join** of a and b. For example if $P = 2^{[n]}$ then the join of $A, B \subseteq [n]$ is clearly $A \cup B$ and the notation derives from there. Dually, if the set $\{c \in P : c \preceq a, c \preceq b\}$ has a unique maximum, it is called the **infimum** of **meet** of a and b and is denoted by $a \wedge b$. This is consistent with $A \wedge B = A \cap B$. If any two elements in P have a meet, then we call $(P, \preceq)$ a **meet-semilattice**. Likewise, we define **join-semilattices** as those posets in which all joins exist. Lastly, if meets and joins exist, we call $(P, \preceq)$ a **lattice**.

(Semi)lattices play an important role and most of the examples in the last section are lattices. For example, in $\mathcal{D}_n$, we have that $a \wedge b$ is the greatest common divisor whereas $a \vee b$ is the least common multiple. For $B_n(q)$ it is obvious that $U \cap V$ is the largest subspace contained in U and V and hence $U \wedge V = U \cap V$. In this case, we can abstractly show that joins have to exist as well.

Lemma 2.2. *Let $(P, \preceq)$ be a meet-semilattice with maximum $\hat{1}$. Then P is a lattice.*

Proof. For $a, b \in P$ consider the set $S = \{p \in P : a \preceq c, b \preceq c\}$. Since $\hat{1} \in S$, S is not empty and one verifies that $c = \bigwedge_{p \in S} p$ is the join of a and b. □

The three Hasse diagrams in the previous sections depict two lattices and one semilattice.

Note that we can recover the partial order relation on a lattice $(L, \preceq)$ from either meets or joins:

$$a \;\preceq\; b \quad\Longrightarrow\quad a \;=\; a \wedge b \quad\Longrightarrow\quad b \;=\; a \vee b\,.$$

On the other hand, meet and join give binary operations $\wedge : L \times L \to L$, $\vee : L \times L \to L$ with certain properties:

Proposition 2.3. *For any a, b, c in a lattice L, the following are satisfied:*

(L1) $a \wedge a = a = a \vee a$, *(Idempotency)*
(L2) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$ *(Commutativity)*
(L3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a \vee b) \vee c = a \vee (b \vee c)$ *(Associativity)*
(L4) $a \wedge (a \vee b) = a$ *and* $b = b \vee (a \wedge b)$ *(Absorption)*

The following result, whose proof we leave as an exercise, states that L1–L4 characterize lattices.

Theorem 2.4. *Let L be a (finite) set with binary operations $\wedge$ and $\vee$ that satisfy (L1)–(L4) above. Then $a \preceq b :\Leftrightarrow a = a \wedge b$ defines a partial order on L for with meets and joins given by $\wedge$ and $\vee$ respectively.*

Exercise 2.1. Proof Theorem 2.4.

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