ALGEBRAIC AND GEOMETRIC COMBINATORICS

SUMMER TERM 2022

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Contents

↓ 1. Vorlesung, 12.4.2022 ↓

We write $\mathbb{N} = \mathbb{Z}_{\geq 0} = \{0, 1, \dots\}$, $\mathbb{P} = \mathbb{Z}_{>0} = \{1, 2, \dots\}$,
$[n] = \{1, 2, \dots, n\}$ and $[a, b] = \{a, a+1, \dots, b\}$.

1. A motivation

1.1. **Coloring graphs and partially ordered sets.** Let $G = (V, E)$ be an undirected graph. A **k-coloring** is a map $c : V \to [k]$ such that $c(u) \neq c(v)$ for all edges $uv \in E$. Define $\chi_G(k)$ as the number of k-colorings of G. The function $\chi_G : \mathbb{P} \to \mathbb{Z}_{\geq 0}$ was defined by Birkhoff in the hope of proving the 4-color theorem: If G is planar, then $\chi_G(4) > 0$. The reason why this might be a promising approach is that χ_G is a nice function.

Date: June 14, 2022.

Proposition 1.1 (Birkhoff)**.** *Let G be a graph with n nodes. Then there are $w_0, \dots, w_n \in \mathbb{Z}$ such that*

$$\chi_G(k) \;=\; w_n k^n + w_{n-1}k^{n-1} + \cdots + w_1 k + w_0$$

for all $k \in \mathbb{P}$.

We identify χ_G with this polynomial and write

$$\chi_G(t) \;=\; w_n t^n + w_{n-1}t^{n-1} + \cdots + w_1 t + w_0$$

where t is an indeterminate.

In order to prove this result, recall the *Principle of Inclusion-Exclusion.*

Theorem 1.2. *Let S be a finite set and $A_e \subseteq S$ subsets indexed by a finite set E. Then*

$$\Big| S \setminus \bigcup_{e \in E} A_e \Big| \;=\; |S| + \sum_{\varnothing \neq I \subseteq E} (-1)^{|I|} A_I \qquad \textit{where } A_I := \bigcap_{e \in I} A_e \,.$$

This is easily proved by induction on $|E|$ but we will look into conceptual proofs later.

To apply this to our situation, fix $k \in \mathbb{P}$ and let $S = \{c : V \to [k]\}$. For an edge $e = uv \in E$ define $A_e := \{c \in S : c(u) = c(v)\}$. That is, A_e is the set of all labellings $c : V \to [k]$ that fail to be a coloring at the edge $e = uv$. Thus the k-colorings are precisely $S \setminus \bigcup_{e \in E} A_e$.

For a fixed $I \subseteq E$, consider the graph $G[I] := (V, I)$. By construction, A_I is the set of c such that $c(u) = c(v)$ whenever u, v are nodes in the same connected component of (V, I). Define $cc(I)$ to be the number of connected components of $G[I]$. Then $|A_I| = k^{cc(I)}$. Together this shows

$$\chi_G(k) = k^n + \sum_{\varnothing \neq I \subseteq E} (-1)^{|I|} k^{cc(I)} \tag{1.1}$$

For the Petersen P_{10} graph this, for example, gives

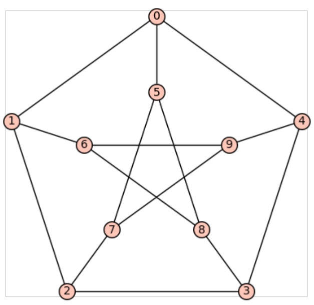
$$\chi_{P_{10}}(t) \;=\; t^{10} - 15t^9 + 105t^8 - 455t^7 + 1353t^6 - 2861t^5 + 4275t^4 - 4305t^3 + 2606t^2 - 704t$$

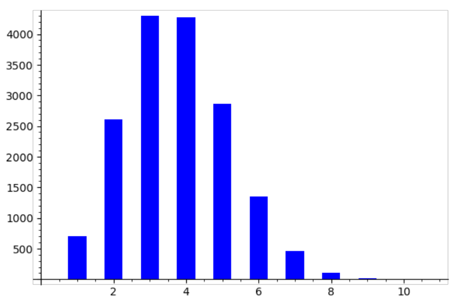
Things to observe:

- the coefficients are integers (clear);
- $\deg \chi_G(t) = |V|$, also clear, as $cc(I) < |V|$ for $I \neq \varnothing$;
- coefficient of $t^{|V|}$ is 1 (clear);
- there is no constant term (think for a second),
- the coefficients alternate in sign (not clear!).

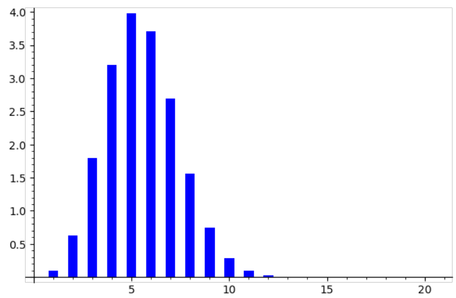
That's a fact that we will prove later. For now, let us write $\chi_G^+(t) = (-1)^{|V|}\chi_G(-t)$ for the polynomial all whose coefficients are positive. What do the coefficients count with respect to G? How are the individual coefficients related?

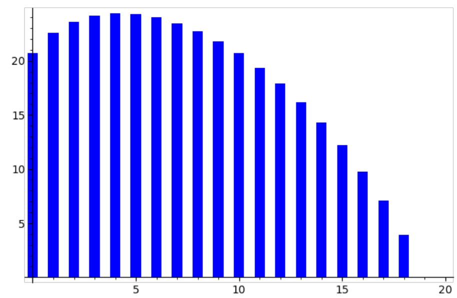
If we plot the coefficients of $\chi^+_{P_{10}}$, we get





This is not a coincidence. For a random graph with 20 nodes, the picture looks like this. Even stronger, if we plot the *logarithms* of the coefficients, we see





A sequence $a = (a_0, a_1, \ldots, a_m)$ of positive integers is **unimodal** if there is an i such that $a_0 \le a_1 \le \cdots \le a_i \ge a_{i+1} \ge \cdots \ge a_m$. The sequence is a is **log concave** if $a_i^2 \ge a_{i-1}a_{i+1}$ for all $0 < i < m$.

Exercise 1.1. If a sequence a of positive integers is log-concave, then a is unimodal.

What we are seeing in the plots is that the sequence of absolute values of coefficients is log-concave and hence unimodal. In particular, the individual entries are not independent of each other!

Example 1.1 (Complete graphs)**.** Let K_n be the complete graph on nodes $[n]$. Coloring the nodes one at a time, we see that the chromatic polynomial is simply $\chi_{K_n}(t) = t(t-1)(t-2)\cdots(t-n+1)$ and $\chi^+_{K_n}(t) = t(t+1)\cdots(t+n-1)$. Recall that every permutation can be uniquely decomposed into cycles. For example the permutation $(\tau(1), \tau(2), \tau(3)) = (3, 2, 1)$ has cycles $(1, 3)$ and (2). The **(unsigned) Stirling number of the first kind** $\bar{s}_{n,k}$ counts the number of permutations with exactly k cycles. For $0 < k \le n$, they satisfy the recurrence relation

$$\bar{s}_{n,k} \;=\; \bar{s}_{n-1,k-1} + (n-1)\bar{s}_{n-1,k}$$

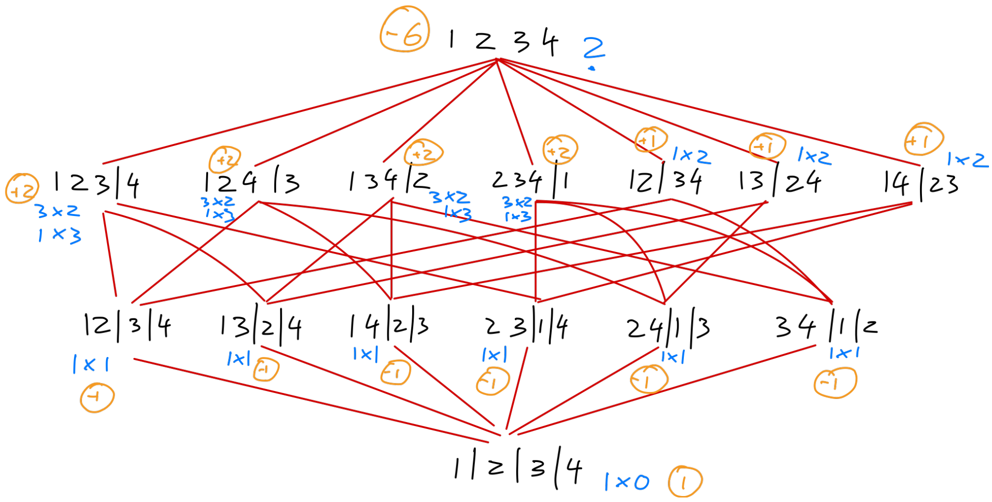
and together with $\bar{s}_{n,1} = (n-1)!$ and $\bar{s}_{n,0} = 0$, ones verifies that

$$\chi^+_{K_n}(t) \;=\; \bar{s}_{n,n}t^n + \cdots + \bar{s}_{n,2}t^2 + \bar{s}_{n,1}t$$

Our derivation of $\chi_G(t)$ had $2^{|E|}$ terms to produce a polynomial with exactly $|V|$ terms. So, there *has* to be a better way to compute $\chi_G(t)$. A more insightful way is to observe that every $I \subseteq E$ defines an equivalence relation on V: call nodes $u, v \in V$ equivalent if they are in the same connected component of $G[I]$. This gives a decomposition $V = V_1 \uplus V_2 \uplus \cdots \uplus V_k$ into equivalence classes. We call $P(I) = \{V_1, \ldots, V_k\}$ an **unordered partition** of V and we write $\mathrm{Par}(G) := \{P(I) : I \subseteq E\}$. For $I = \varnothing$, we obtain $P(\varnothing) = \{\{v\} : v \in V\}$. Note that $cc(I) = |P(I)|$, the number of parts of the partition. So, as an intermediate step in our computation of (1.1) we get

$$\chi_G(t) \;=\; \sum_{I \subseteq E} (-1)^{|I|} t^{cc(I)} \;=\; \chi_G(t) \;=\; \sum_{P \in \mathrm{Par}(G)} t^{|P|} \sum_{\substack{I \subseteq E \\ P(I) = P}} (-1)^{|I|}\,.$$

More generally, we write Par_n for the collection of all unordered partitions of $[n]$. This is exactly the unordered partitions that we get for the case $G = K_n$. An unordered partition $P = \{P_1, \ldots, P_k\}$ **refines** the unordered partition $P' = \{P'_1, \ldots, P'_l\}$ if for every $i = 1, \ldots, k$ there is a $j = 1, \ldots, l$ such that $P_i \subseteq P'_j$. That is, the unordered partitions refining P' arise by replacing some P'_j by an unordered partition of it. We also say that P' is a **coarsening** of P. 'Refinement' defines a partial order on Par_n. The following picture shows the partition lattice Par_4. The unordered partitions are in black and we read $23 \mid 1 \mid 4$ as $\{\{2, 3\}, \{1\}, \{4\}\}$. The red lines indicate refinement but only the necessary!



Observe that $\{\{1,2,3\},\{4\}\} = P(I)$ for three sets I of cardinality 2 and one set I with $|I| = 3$. This explains the blue labels $3\times 2, 1\times 3$. As for the orange numbers, this is $\sum_I (-1)^{|I|}$, where the sum is over those I that gives $P(I)$. Thus, in the example above, the sum will be $(-1)^2+(-1)^2+(-1)^2+(-1)^3 = 2$. The chromatic polynomial for K_4 thus gives

$$\chi_{K_4}(t) = 1t^4 - 6t^3 + 11t^2 - 6t\,.$$

The ? is for you to determine. However, we can still compute the orange number. The trick is this: once we know the orange 1 at the bottom, we can go layer by layer. At every partition P the orange number is the negative of all the orange numbers of partitions that refine P!

The collection of unordered partitions $\mathrm{Par}(G) = \{P(I) : I \subseteq E\}$ for a given graph G *partially ordered* by refinement structurally captures what happens when computing the characteristic polynomial. Such partially ordered sets, or **posets** for short give structure and guidance in many situations. As Gian-Carlo Rota wrote

> *It often happens that a set of objects to be counted possesses a natural ordering, in general only a partial order. It may be unnatural to fit the enumeration of such a set into a linear order such as the integers: instead, it turns out in a great many cases that a more effective technique is to work with the natural order of the set. One is led in this way to set up a "difference calculus" relative to an arbitrary partially ordered set.*

He wrote this in the introduction to the seminal paper [3] in which he introduced the concept of a Möbius function. The Möbius function is is related to the number theoretic Möbius function, gives a vast generalization of the principle of inclusion-exclusion, and is responsible for the orange numbers above. We will thoroughly study partially ordered sets, their Möbius functions, and how to compute them. The partially ordered sets that we will be studying are motivated by combinatorics/discrete math and geometry and will show beautiful connections to geometry/topology and algebra. Here are some examples:

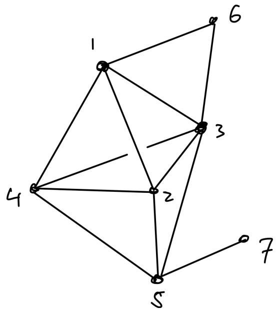
- $B_n = (2^{[n]}, \subseteq)$, the collection of subsets of an n-set partially ordered by inclusion;
- Par_n and $\overline{\mathrm{Par}}_n$ the collection of (un)ordered partitions of an n-set ordered by refinement;
- Permutations partially ordered by their inversions;
- (normal) subgroups of a given (finite) group ordered by inclusion;
- $B_n(q)$, the collection of subspaces of $\mathbb{F}_q^n$, the n-dim vector space over the finite field $\mathbb{F}_q$, ordered by inclusion;
- Cliques or stable sets of a graph G, ordered by inclusion;
- cycle-free subsets of edges of a graph G ordered by inclusion;
- Subsets of non-crossing diagonals of a convex n-gon ordered by inclusion.

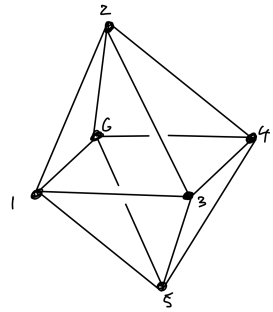
We will see many more examples and, in particular, distill important types of posets.

↓ 2. Vorlesung, 14.4.2022 ↓

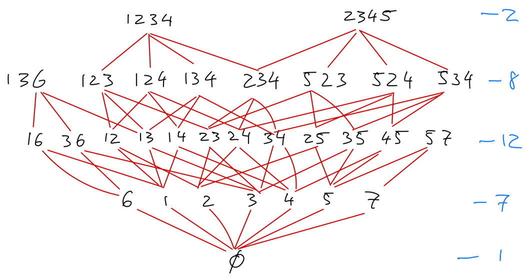
1.2. **Simplicial complexes and the Upper Bound Conjecture.** The last three examples together with B_n they stand out: Let E be a finite set. A collection of subsets $\varnothing \neq \Delta \subseteq 2^E$ is a **hereditary set**

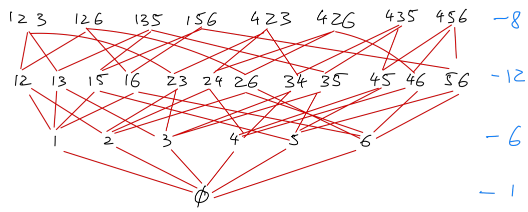
system or a **simplicial complex** if for all $\tau \in \Delta$ and $\sigma \subseteq \tau$ we have $\sigma \in \Delta$. Every subset of a clique (or stable set) in a graph is a clique (or stable set). Every subset of a cycle-free subset is cycle-free. The property of being a clique or cycle-free is *inherited* under taking subsets. The name 'simplicial complex' comes from a geometric context. A geometric simplex is a point, a segment, a triangle, a tetrahedron etc. A simplicial complex is a collection of simplices with the property that if simplices meet, then their intersection is a face of both. We make this more precise later, but for now these two pictures should convey the idea:



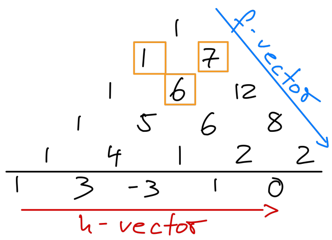


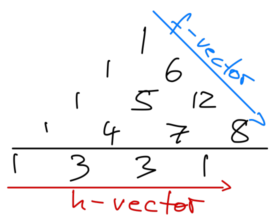
There is a clear IKEA-type gluing description of both simplicial complexes:





A natural complexity measure of a simplicial complex is given by its **face vector** of **f-vector**. For Δ, $f_i(\Delta)$ counts the number of faces of dimension $i \geq -1$. The empty set (at the bottom) is always a face of dimension -1. For the left complex Δ_1 this gives $f(\Delta_1) = (1, 7, 12, 8, 2)$ for the right complex $f(\Delta_2) = (1, 6, 12, 8)$. As we will learn, often a better way to represent the information given by the f-vector is in the form of the h-vector. The following shows how to compute them





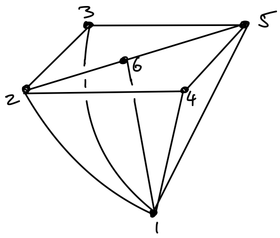
The highlighted 6 is obtained as $7 - 1$ and this gives the complete set of rules to go from the f-vector to the h-vector and back. Whereas the h-vector on the left-hand side does does not reveal more information, the one on the right-hand side looks promising: it is non-negative and symmetric/palindromic.

Exercise 1.2. A simple graph $G = (V, E)$ can be viewed as a simplicial complex $\Delta = \{\varnothing\} \cup V \cup E$. The f-vector is just $f(\Delta) = (1, |V|, |E|)$. Classify when $h(\Delta)$ is non-negative and when it is palindromic.

For geometrically/topologically interesting classes of simplicial complexes the h-vector will always be non-negative and palindromic! The vague answer we give here is that the right complex resembles a sphere whereas the left one does not. In fact, the right complex *is* the unit sphere in the ℓ_1-norm but we mean that it resembles a sphere in a topological sense, independent of how the complex is geometrically realized.

Theorem 1.3. *Let Δ be a simplicial complex with $h(\Delta) = (h_0, \ldots, h_d)$. If Δ is topologically a sphere, then $h_0, \ldots, h_d \geq 0$ and $h_i = h_{d-i}$.*

Thus geometric/topological objects give rise to posets whose invariants (f-vectors, h-vectors) we can combinatorially interpret. Conversely, we will associate to any poset a geometric/topological object whose geometric/topological features will *explain* certain combinatorial information. For example, for the partition lattice, we will see that we can associate to it the following simplicial complex



This is a complex glued from 6 triangles with f-vector $(1, 6, 11, 6)$. These are precisely the coefficients of the chromatic polynomial $\chi^+_{K_4}(t)$. We will see that the entries of the f-vector have to satisfy certain conditions and this automatically gives conditions on the coefficients of $\chi_G(t)$.

A highlight of the course will be a resolution of the *Upper Bound Conjecture* for spheres. Suppose Δ is topologically a sphere of dimension $d-1$. What is the maximal number of i-dimensional faces, that is, what is the maximal $f_i(\Delta)$ for a fixed number of vertices $f_0(\Delta)$? Motzkin constructed geometric spheres, so-called *neighborly* spheres, for which he conjectured that they maximize the number of i-dimensional faces for all i simultaneously among all geometric/convex spheres. McMullen [2] proved Motzkin's Upper Bound Conjecture for convex spheres (that is, simplicial polytopes). There, he introduced the notion of an h-vector. Victor Klee [1] suggested to extend the UBC to all spheres and Richard Stanley [4] combined ideas from combinatorics, topology, and commutative algebra in a spectacular way to resolve the UBC for spheres.

Theorem 1.4. *Let Δ be a $(d-1)$-dimensional simplicial sphere with n vertices and $h(\Delta) = (h_0, \dots, h_d)$. Then $h_i = h_{d-i}$ and*

$$h_i \;\le\; \binom{n-d-1+i}{i}$$

for all i. If equality is attained for $i = \lfloor \frac{d}{2} \rfloor$, then Δ is a neighborly sphere.

1.3. **Polynomials and Hilbert series.** To give an idea of the sort of algebra that we will be using, recall that a polynomial in a single variable t with coefficients in $\mathbb{C}$ is an expression of the form

$$c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0 t^0$$

The collection of all such polynomials is denoted by $\mathbb{C}[t]$. This is a $\mathbb{C}$-vector space and the fact that polynomials can be multiplied along the rules $t^i t^j = t^{i+j}$ turns $\mathbb{C}[t]$ into a or $\mathbb{C}$-algebra.

We may extend this to polynomials in many variables. For $n \ge 1$, let $x_1, \dots, x_n$ indeterminates. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\ge 0}$, we write $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. A multi-variate polynomial is then

$$\sum_{\alpha \in A} c_\alpha \mathbf{x}^\alpha$$

where $A \subset \mathbb{Z}^n_{\ge 0}$ is a finite set and $c_\alpha \in \mathbb{C}$ for all $\alpha \in A$. Again, polynomials form a $\mathbb{C}$-vector space and together with multiplication $\mathbf{x}^\alpha \cdot \mathbf{x}^\beta = \mathbf{x}^{\alpha+\beta}$ give the ring of polynomials $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_n]$. The degree of a monomial is $\deg \mathbf{x}^\alpha = |\alpha| = \alpha_1 + \cdots + \alpha_n$. We write $\mathbb{C}[\mathbf{x}]_d$ for the subvector space spanned by monomials of degree d. Elementary combinatorics shows that the vector space dimension is $\dim_{\mathbb{C}} \mathbb{C}[\mathbf{x}]_d = \binom{n+d-1}{d}$. The Hilbert series is the generating function that incorporates the dimensions:

$$H(\mathbb{C}[\mathbf{x}], z) \;=\; \sum_{d \ge 0} \dim_{\mathbb{C}} \mathbb{C}[\mathbf{x}]_d z^d \;=\; \sum_{d \ge 0} \tbinom{n+d-1}{d} z^d \;=\; \frac{1}{(1-z)^{d+1}}\,.$$

Now let us consider $\mathbb{C}[x_1, \dots, x_6]$ together with the conditions $\mathbf{x}^\alpha = 0$ if $\mathbf{x}^\alpha$ is divisible by x_1x_4, x_2x_5, or x_3x_6. For a monomial $\mathbf{x}^\alpha$, the support is $\operatorname{supp}(\mathbf{x}^\alpha) = \{i : \alpha_i > 0\}$. Thinking back to our example of the *octahedron* on page 4, we can express this condition as $\mathbf{x}^\alpha = 0$ whenever $\operatorname{supp}(\mathbf{x}^\alpha) \notin \Delta$. Thus, we somehow encoded the combinatorics of Δ into polynomials. Our conditions are compatible with multiplication and give a new ring $\mathbb{C}[\Delta]$ whose elements can still be expressed by monomials in $x_1, \dots, x_6$. Moreover, for $d \geq 0$, we can define $\mathbb{C}[\Delta]_d = \mathbb{C}\text{-span}\{\mathbf{x}^\alpha \in \mathbb{C}[\mathbf{x}]_d : \operatorname{supp}(\mathbf{x}^\alpha) \in \Delta\}$. Computing the Hilbert series now yields

$$H(\mathbb{C}[\Delta], z) \;=\; \sum_{d \geq 0} \dim_{\mathbb{C}} \mathbb{C}[\Delta]_d z^d \;=\; \frac{1 + 3z + 3z^2 + z^3}{(1-z)^4}$$

It is not a coincidence that the numerator polynomial is exactly the h-vector of Δ!

Let us further add the conditions $x_1 = x_4$, $x_2 = x_5$, $x_3 = x_6$. So, every time we see an x_4, we may replace it by x_1. In particular, the condition $x_1 x_4 = 0$ can be read as $x_1^2 = 0$. This gives us a new ring R obtained from $\mathbb{C}[x_1, x_2, x_3]$ with the conditions $x_i^2 = 0$. This is quite a simple ring. As a $\mathbb{C}$ vector space R has the basis $1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3$ and hence models all subsets of $[3]$. We can do this more general, for $n \geq 1$, let R be the $\mathbb{C}$-vector space with basis $\mathbf{x}^\tau = \prod_{i \in \tau} x_i$ for all subsets $\tau \subseteq [n]$. If $\tau = \varnothing$, then $\mathbf{x}^\tau = 1$. We define a multiplication on R by setting $\mathbf{x}^\tau \cdot \mathbf{x}^\sigma = \mathbf{x}^{\tau \cup \sigma}$ if $\tau \cap \sigma = \varnothing$ and $\mathbf{x}^\tau \cdot \mathbf{x}^\sigma = 0$ if $\tau \cap \sigma \neq \varnothing$. We can write $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n$ where $R_i = \mathbb{C}\text{-span}\{\mathbf{x}^\tau : |\tau| = i\}$. In particular, $\dim R_i = \binom{n}{i}$. Its Hilbert series satisfies

$$H(R, z) \;=\; \sum_{d \geq 0} \dim_{\mathbb{C}} R_i z^i \;=\; \sum_{i=0}^{d} \tbinom{n}{i} z^i \;=\; (1+z)^n \,.$$

We can now algebraically argue that $\binom{n}{i} = \binom{n}{n-i}$. We first note that $R_n \cong \mathbb{C}$. Thus, for any fixed i, multiplication gives a bilinear form $B_i : R_i \times R_{n-i} \to \mathbb{C}$. For fixed $f \in R_i$, the map $R_{n-i} \ni g \mapsto B(f,g)$ is a linear function on R_{n-i}. If we can show that $f \mapsto B(f, \cdot)$ is injective, then we have shown $\dim R_i \leq \dim R_{n-i}$ (why?). Applying the same reasoning to $g \mapsto B(\cdot, g)$, then proves $\dim R_i = \dim R_{n-i}$. That's unnecessarily difficult when it comes to binomial coefficients but in general that's the way to go.

This perspective also suggests a way to show that $\binom{n}{i} \leq \binom{n}{i+1}$ for $i \leq \lfloor \frac{n}{2} \rfloor$. We will show that for $\omega = x_1 + x_2 + \cdots + x_n$, the linear map $R_i \to R_{i+1}$ given by $f \mapsto \omega \cdot f$ is injective whenever $i \leq \lfloor \frac{n}{2} \rfloor$ and surjective otherwise. Injectivity, of course, then implies $\binom{n}{i} = \dim_{\mathbb{C}} R_i \leq \dim_{\mathbb{C}} R_{i+1} = \binom{n}{i+1}$.

↓ 3. Vorlesung, 19.4.2022 ↓

2. Partially ordered sets

We start with partially ordered sets.

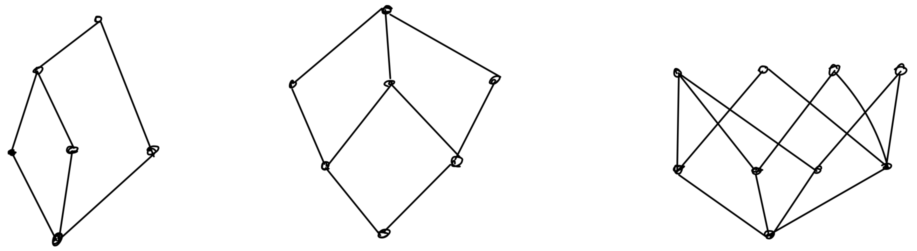
Definition 2.1. A **partially ordered set** (or **poset**) is a pair $(P, \preceq)$ where P is a set and $\preceq$ is a binary relation satisfying

- $a \preceq a$ for all $a \in P$ (Reflexivity)
- $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ for all $a, b, c \in P$ (Transitivity)
- $a \preceq b$ and $b \preceq a$ implies $a = b$ for all $a, b \in P$ (Anti-symmetry)

We usually call P the poset when $\preceq$ is clear from the context and we write $\preceq_P$ to emphasize the relation to the ground set P. We call two elements $a, b \in P$ **comparable** if $a \preceq b$ or $b \preceq a$. We say that b **covers** a or a is **covered** by b if $a \prec b$ and there is no c with $a \prec c \prec b$. In this case we write $a \prec\!\!\cdot\, b$.

We can encode $\preceq$ by a directed graph (digraph) on the node set P with a directed edge (a,b) if $a \prec\!\!\cdot\, b$. This is an acyclic[1] digraph and $a \preceq b$ if there is a path $a = a_0 a_1 \dots a_k = b$ such that (a_{i-1}, a_i) is a directed edge for $i = 1, \dots, k$. We may visualize this digraph by a drawing in the plane for which the edges (a,b) have positive slope. Thus $b \succeq a$ if there is a path from b to a with monotonically decreasing y-coordinate. Such as drawing is called a **Hasse diagram**. Here are three examples:

[1]No directed cycles



An element $a \in P$ is **maximal** if there is no $b \in P$ with $a \prec b$. Minimal elements are defined accordingly. We write $\min(P)$ and $\max(P)$ for the minimal an maximal elements. The poset has a **maximum** if there is $m \in P$ with $a \preceq m$ for all $a \in P$. Note that every maximum is maximal but not necessarily the other way round!

If a maximum exists, it is necessarily unique and denoted by $\hat{1}$. All but the right-most poset in the figure above have a maximum. A minimum, provided it exists, is denoted by $\hat{0}$.

For two elements $a, b \in P$, the **interval** is $[a,b]_P = \{c \in P : a \preceq c \preceq b\}$. This is an *induced* subposet of P by restricting $\preceq$ to $[a,b]_P$. Note that $[a,b]_P = \varnothing$ if $a \not\preceq b$. Otherwise intervals always have a minimum and maximum. Note that our posets are not necessarily finite. We call poset **locally-finite** if $[a,b]_P$ is finite for every $a, b \in P$.

A homomorphism or **order-preserving map** between to posets $(P_1, \preceq_1)$ and $(P_2, \preceq_2)$ and is a map $f : P_1 \to P_2$ such that

$$a \ \preceq_1 \ b \quad \Longrightarrow \quad f(a) \ \preceq_2 \ f(b)$$

is satisfied for all $a, b \in P_1$. If f is a bijection and f^{-1} is also order-preserving, then P_1 and P_2 are **isomorphic**, denoted by $P_1 \cong P_2$.

A subset $C \subseteq P$ is called a **chain** if any two elements in C are comparable. If C is finite then there is a labelling of the elements $C = \{a_0, a_1, \dots, a_k\}$ such that $a_0 \prec a_1 \prec \cdots \prec a_k$. Then **length** of a finite chain is $\ell(C) = |C| - 1$, the number of 'links' in a chain. A chain is **saturated** or **unrefineable** if for any three elements $a \prec b \prec c$ we have that $a, c \in C$ implies $b \in C$. If C is finite, then this is equivalent to $a_{i-1} \prec\!\!\cdot\, a_i$ for $i = 1, \dots, k$. The chain C is **maximal** if there is no chain C' with $C \subset C'$. Thus, maximal chains are saturated but the converse is not true in general.

The **rank** $r(P)$ of a poset P is the maximal length of a chain. For $a, b \in P$, we will write $\ell_P(a,b) = r([a,b]_P)$. We simply write $\ell(a,b)$ if P is clear from the context. A poset P is **graded** if all maximal chains have the same (finite) length $r(P)$. If P is graded, then there is a unique function $r : P \to \mathbb{Z}_{\geq 0}$ called the **rank function** with with $r(a) = 0$ for all $a \in \min(P)$ and $r(b) = r(a) + 1$ for $a \prec\!\!\cdot\, b$. If P is finite, then the distribution of ranks is recorded by the **rank-generating function**

$$F(P,t) = \sum_{a \in P} t^{r(a)} = p_0 + p_1 t^1 + \cdots + p_r t^r \,,$$

where $p_i = \#\{a \in P : r(a) = i\}$ and $r = r(P)$.

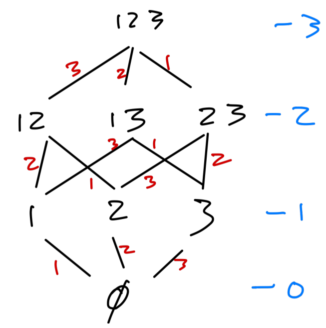
Example 2.1 (Chains and Anti-chains)**.** The prototypical chain of length n is the set $[n] := \{1, \dots, n\}$ together with the natural order. We call $C_n = ([n], \leq)$ **the chain** with n elements. Thus $C \subset P$ is an n-chain if the induced subposet C is isomorphic to $[n]$. Chains are clearly ranked posets with rank-generating function

$$F([n], q) \ = \ 1 + q + q^2 + \cdots + q^{n-1} \ =: (n)_q \,.$$

We call the polynomial $(n)_q$ a 'q-analogue' of the number n.

The conceptual opposite of a chain is a set $A \subseteq P$ such that any two distinct elements in A are incomparable. Such a set is called an **anti-chain**. This is a graded poset with $F(A,t) = |A| t^0$.

Example 2.2 (Boolean lattice)**.** For any set S, the Boolean lattice is the poset on $2^S = \{T : T \subseteq S\}$ partially ordered by inclusion. We write $B_n = (2^{[n]}, \subseteq)$ and note that $(2^S, \subseteq) \cong B_n$ if $|S| = n$. The Boolean lattice has minimum $\hat{0} = \varnothing$ and maximum $\hat{1} = S$. For $A \subseteq B$, we observe $[A,B] \cong (2^{B \setminus A}, \subseteq)$. In particular, $A \prec\!\!\cdot\, B$ if $|B \setminus A| = 1$. Here is B_3:

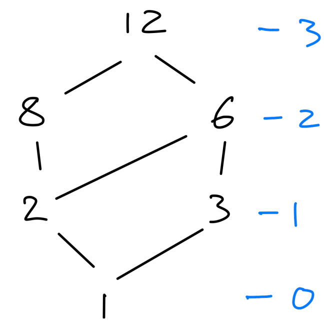


Hence, B_n is graded with $r(A) = |A|$. The rank-generating function satisfies

$$F(B_n, q) \;=\; \sum_{i=0}^{n} \binom{n}{i} t^q \;=\; (1+q)^n \,.$$

There is a close connection between B_n and permutations. A maximal chain in B_n is of the form $\varnothing = S_0 \subset S_1 \subset \cdots S_{n-1} \subset S_n = [n]$. In particular $S_i \setminus S_{i-1} = \{a_i\}$ for a $a_i \in [n]$. Since $a_i \neq a_j$ for $i \neq j$, this defines a permutation $i \mapsto a_i$. Hence, maximal chains in B_n are in one-to-one correspondence with permutations of $[n]$. In the example, the permutations can be read from the red numbers on the cover relations.

Example 2.3 (Divisibility)**.** For $n \in \mathbb{Z}_{>0}$, define the $\mathcal{D}_n$ as the set of $a \in \mathbb{Z}_{>0}$ with a divides n. We partially order $\mathcal{D}_n$ by setting $a \preceq b$ if there is a $k \in \mathbb{Z}_{>0}$ such that $b = ka$. It has a minimum $\hat{0} = 1$ and maximum $\hat{1} = n$. Here is $\mathcal{D}_{12}$:



Note that for $a \prec b$ in $\mathcal{D}_n$, we have $[a,b] \cong \mathcal{D}_{b/a}$. In particular $a \prec\!\!\cdot\, b$ if and only if $\frac{b}{a}$ is prime. It follows from the Fundamental Theorem of Arithmetic that $\mathcal{D}_n$ is graded. The rank of $\mathcal{D}_n$ is $r(\mathcal{D}_n) = k_1 + k_2 + \cdots + k_s$ $n = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$ where $p_1, \dots, p_s$ are the distinct prime factors. To compute the rank-generating function, we observe that any $a \in \mathcal{D}_n$ is of the form $a = p_1^{l_1} p_2^{l_2} \cdots p_s^{l_s}$ for $0 \le l_i \le k_i$ and has rank $r(a) = l_1 + \cdots + l_s$. It is now easy to check that

$$F(\mathcal{D}_n, q) \;=\; \prod_{i=1}^{s} (k_i + 1)_q$$

The above example prompts for a simple construction on posets. For two posets $(P_1, \preceq_1), (P_2, \preceq_2)$ define the **direct/Cartesian product** as the partial order on $P_1 \times P_2$ by

$$(a_1, a_2) \;\preceq\; (b_1, b_2) \quad\Longrightarrow\quad a_1 \;\preceq_1\; b_1 \text{ and } a_2 \;\preceq_2\; b_2 \,.$$

It is straightforward to verify that $(P_1 \times P_2, \preceq)$ is a graded poset whenever P_1 and P_2 are. The rank-generating function satisfies

$$F(P_1 \times P_2, t) \;=\; F(P_1, t) F(P_2, t)$$

If $n = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$, then

$$\mathcal{D}_n \;\cong\; [k_1 + 1] \times [k_2 + 1] \times \cdots \times [k_s + 1] \,.$$

By the same token, we get $B_n \cong ([2], \le)^n$, where we identify subsets $A \subseteq [n]$ with vectors $v_A \in [2]^n$ with $(v_A)_i = 2$ if and only if $i \in A$.

↓ 4. Vorlesung, 21.4.2022 ↓

Example 2.4 (Lattice of subspaces)**.** Let $\mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$, the finite field with q elements. Write $B_n(q)$ for the collection of vector subspaces of $\mathbb{F}_q^n$, partially ordered by inclusion. This is a finite poset with minimum $\hat{0} = \{\mathbf{0}\}$, where $\mathbf{0} \in \mathbb{F}_q^n$ is the zero vector and maximal element $\hat{1} = \mathbb{F}_q^n$. For two subspaces $U \subseteq W$, basic linear algebra tells us that the subspaces $V \subseteq \mathbb{F}_q^n$ with $U \subseteq V \subseteq W$ are precisely the subspaces of the quotient $W/U \cong \mathbb{F}_q^k$ with $k = \dim W - \dim U$. Since the isomorphism retains inclusions, we obtain $[U,W]_{B_n(q)} \cong B_k(q)$. From this, we get that $B_n(q)$ is a graded poset with rank function $r(U) = \dim U$.

The poset $B_n(q)$ is called a 'q-analogue' of $B_n = (2^{[n]}, \subseteq)$ for reasons that are difficult to make precise. In essence, it means that many of its enumerative invariants are polynomials in q whose specializations $q = 1$ yield respective invariants for B_n. For example, a 1-dimensional subspace $L \subset \mathbb{F}_q^n$ is the span of a vector $v \in \mathbb{F}_q^n \setminus \{\mathbf{0}\}$ and $a \cdot v$ spans L for all $a \in \mathbb{F}_q \setminus \{0\}$. It follows that that there are $q^n - 1$ choices of v and

$$\frac{q^n - 1}{q - 1} \ = \ 1 + q + q^2 + \cdots + q^{n-1} \ = (n)_q$$

many distinct lines in $\mathbb{F}_q^n$. Now a subspace U is covered by W if $U \subset W$ and W/U is 1-dimensional. Hence if we want to construct a maximal chain in $B_q(n)$, we start with $V_0 = \{\mathbf{0}\}$. For $i = 1, \dots, n$ we now choose a line $V_i \subset \mathbb{F}_q^n / V_{i-1}$. Since $\dim V_i = i$, we have $(i)_q$ many choice. In total, the number of maximal chains is

$$(n)_q (n-1)_q \cdots (2)_q (1)_q \ =: (n)_q!$$

Of course, there is no finite field with $q = 1$ elements but the above formula still makes sense and yields a q-analogue of the factorial $n! = n \cdot (n-1) \cdots 2 \cdot 1$.

To count the number of elements of $B_n(q)$ of rank k, that is, the number of k-dimensional subspaces of $\mathbb{F}_q^n$, let us count for a fixed k-subspace U the the number of maximal chains V_* with $V_k = U$. This is easy, because we simply need to count the maximal chains in $[U, \hat{1}] \cong B_{n-k}(q)$ and the maximal chains in $[\hat{0}, U] \cong B_k(q)$. This is this gives exactly $(n-k)_q (k)_q$ such chains, independent of the actual choice of U. Thus, the number of k-subspaces is precisely

$$\frac{(n)_q}{(n-k)_q (k)_q} \ =: \ \binom{n}{k}_q ,$$

which is a q-analog of the binomial coefficient, also called a **Gaussian polynomial**. Pleasantly, we obtain $\binom{n}{k}$ for $q = 1$. The rank-generating function for $B_n(q)$ is thus

$$F(B_n(q), t) \ = \ \sum_{k=0}^{n} \binom{n}{k}_q t^k .$$

2.1. **Lattices.** For two elements a, b in a poset P, a **least upper bound** or **supremum** is an element c such that $a \preceq c$ and $b \preceq c$ and for every c' with these properties satisfies $c \preceq c'$. Least upper bounds do not need to exist but if they do, they are unique. We denote them by $a \vee b := c$ and call $a \vee b$ the **join** of a and b. For example if $P = 2^{[n]}$ then the join of $A, B \subseteq [n]$ is clearly $A \cup B$ and the notation derives from there. Dually, if the set $\{c \in P : c \preceq a, c \preceq b\}$ has a unique maximum, it is called the **infimum** of **meet** of a and b and is denoted by $a \wedge b$. This is consistent with $A \wedge B = A \cap B$. If any two elements in P have a meet, then we call $(P, \preceq)$ a **meet-semilattice**. Likewise, we define **join-semilattices** as those posets in which all joins exist. Lastly, if meets and joins exist, we call $(P, \preceq)$ a **lattice**.

(Semi)lattices play an important role and most of the examples in the last section are lattices. For example, in $\mathcal{D}_n$, we have that $a \wedge b$ is the greatest common divisor whereas $a \vee b$ is the least common multiple. For $B_n(q)$ it is obvious that $U \cap V$ is the largest subspace contained in U and V and hence $U \wedge V = U \cap V$. In this case, we can abstractly show that joins have to exist as well.

Lemma 2.2. *Let $(P, \preceq)$ be a meet-semilattice with maximum $\hat{1}$. Then P is a lattice.*

Proof. For $a, b \in P$ consider the set $S = \{p \in P : a \preceq c, b \preceq c\}$. Since $\hat{1} \in S$, S is not empty and one verifies that $c = \bigwedge_{p \in S} p$ is the join of a and b. $\square$

The three Hasse diagrams in the previous sections depict two lattices and one semilattice.

Note that we can recover the partial order relation on a lattice $(L, \preceq)$ from either meets or joins:

$$a \preceq b \implies a = a \wedge b \implies b = a \vee b .$$

On the other hand, meet and join give binary operations $\wedge : L \times L \to L$, $\vee : L \times L \to L$ with certain properties:

Proposition 2.3. *For any a, b, c in a lattice L, the following are satisfied:*

(L1) $a \wedge a = a = a \vee a$, *(Idempotency)*
(L2) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$ *(Commutativity)*
(L3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a \vee b) \vee c = a \vee (b \vee c)$ *(Associativity)*
(L4) $a \wedge (a \vee b) = a$ *and* $b = b \vee (a \wedge b)$ *(Absorption)*

The following result, whose proof we leave as an exercise, states that L1–L4 characterize lattices.

Theorem 2.4. *Let L be a (finite) set with binary operations $\wedge$ and $\vee$ that satisfy (L1)–(L4) above. Then $a \preceq b :\Leftrightarrow a = a \wedge b$ defines a partial order on L for with meets and joins given by $\wedge$ and $\vee$ respectively.*

Exercise 2.1. Proof Theorem 2.4.

↓ 5. Vorlesung, 26.4.2022 ↓

2.2. **Distributive lattices.** Absorption (L4) is the only condition in which meets and joins interact. In particular, it is in general not true that meets and joins 'distribute', that is,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{D1}$$

or

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) . \tag{D2}$$

Proposition 2.5. *Let $(L, \preceq)$ be a lattice. Then (D1) holds if and only if (D2) holds.*

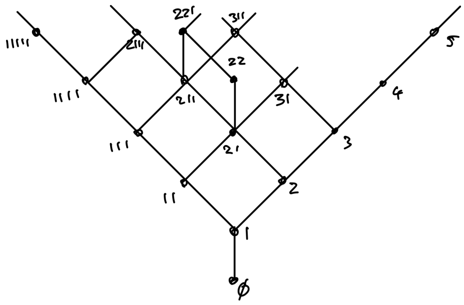
Proof. We only prove (D1) implies (D2): For $a, b, c \in L$

$$\begin{aligned} (a \wedge b) \vee (a \wedge c) &\overset{D1}{=} ((a \wedge b) \vee a) \wedge ((a \wedge b) \vee c) \\ &\overset{L4}{=} a \wedge ((a \wedge b) \vee c) \\ &\overset{D1}{=} a \wedge ((a \vee c) \wedge (b \vee c)) \\ &\overset{L3}{=} (a \wedge ((a \vee c)) \wedge (b \vee c) \\ &\overset{L4}{=} a \wedge (b \vee c) . \quad \square \end{aligned}$$

We call a lattice $(L, \preceq)$ a **distributive lattice** if it satisfies the two equivalent conditions.

Example 2.5 (Young's lattice)**.** A **partition** of $n \in \mathbb{Z}_{\geq 0}$ is a sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of integers with $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$ such that $n = \lambda_1 + \cdots + \lambda_k$. We define $\mathcal{Y}_n$ as the set of partitions of $n \in \mathbb{Z}_{\geq 0}$ and $\mathcal{Y} = \bigcup_{n \geq n} \mathcal{Y}_n$. For $n = 0$, the only admissible partition is the *empty* partition $\lambda = ()$. We turn $\mathcal{Y}$ into a poset by setting $\lambda \leq \mu$ if $\lambda_i \leq \mu_i$ for all i. Since λ and μ can be of different length, we set $\lambda_i := 0$ for all $i > k$. This is a lattice with $(\lambda \vee \mu)_i = \max(\lambda_i, \mu_i)$ and $(\lambda \wedge \mu)_i = \min(\lambda_i, \mu_i)$ – do check that they define valid partitions!

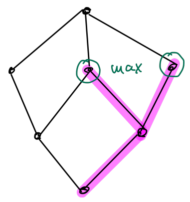
This explicit description also makes it easy to verify that $\mathcal{Y}$ is a distributive lattice. It is less obvious that $\mathcal{Y}$ is graded with $r(\lambda) = \sum_i \lambda_i$. Thus $\mathcal{Y}_n$ are the elements of rank n. The rank-generating function of this infinite lattice is thus $F(\mathcal{Y}, t) = \sum_{n \geq 0} p(n) t^n$, where $p(n)$ is the number of partitions of n.



Of course, the inspiration for distributivity comes from $\cup$ and $\cap$. A collection $\mathcal{C} \subseteq 2^E$ is called a **ring of sets** if for any $A, B \in \mathcal{C}$ we have $A \cup B, A \cap B \in \mathcal{C}$. Every ring of sets $(\mathcal{C}, \subseteq)$ is a distributive lattice. We can manufacture nontrivial rings of sets (and hence distributive lattices) from arbitrary posets. Let $(P, \preceq)$ be a poset and $I \subseteq P$. We call I an **order ideal** (or **down-set**) if for all $b \in I$ and $a \preceq b$ we have $a \in I$. For $S \subseteq P$, we set

$$P_{\preceq S} \ := \ \{a \in P : a \preceq b \text{ for some } b \in S\}$$

for the ideal generated by S. If $S = \{b\}$, then $P_{\preceq b} := P_{\preceq S}$ is called a **principal ideal**. In general $P_{\preceq S} = \bigcup_{b \in S} P_{\preceq b}$. Note that if I is an ideal, then $S = \max(I)$ is the unique smallest set (with respect to inclusion) for which $I = P_{\preceq S}$. Also note that $\max(I)$ is an anti-chain. This gives a one-to-one correspondence between ideals and anti-chains of P. Here is an example of an ideal together with its associated anti-chain:



If $I, J \subseteq P$ are ideals, then so is $I \cup J$ and $I \cap J$. For example, if $b \in I \cap J$, then $b \in I$ and $b \in J$. Hence if $a \preceq b$, then $a \in I \cap J$ since I and J are ideals. Therefore, the collection

$$\mathcal{J}(P) \ := \ \{I \subseteq P : I \text{ ideal}\}$$

is a ring of sets and hence $(\mathcal{J}(P), \subseteq)$ is a distributive lattice, called the **Birkhoff lattice** of P.

Example 2.6 (Boolean lattice)**.** Let A be an anti-chain of size n. Then every subset of A is an order ideal. Hence $\mathcal{J}(A) = (2^A, \subseteq) \cong B_n$.

Example 2.7 (Chains)**.** Let $C = [n]$ be the chain of size n. Every order ideal is of the form $I = \{k \in [n] : k \leq i\}$ for some $i = 0, 1, \ldots, n$. Hence $\mathcal{J}([n]) \cong [n+1]$.

Birkhoff's fundamental insight was that all distributive lattices arise this way. An element c in a lattice L is called **join-irreducible** if $c \neq \hat{0}$ and $c = a \vee b$ implies $a = c$ or $b = c$. An ideal I of a poset P is join-irreducible in $\mathcal{J}(P)$ if and only if I is principal. Join-irreducibles are easy to spot in Hasse diagrams: they are the elements that only cover one other element.

Theorem 2.6. *Let $(L, \preceq)$ be a distributive lattice and let P be the induced subposet of join-irreducible elements of L. Then $L \cong \mathcal{J}(P)$.*

Proof. We define two maps $f : L \to \mathcal{J}(P)$ and $g : \mathcal{J}(P) \to L$ by

$$f(b) \ = \ I_b := \{a \in P : a \preceq b\} \qquad g(I) \ = \ \bigvee_{a \in I} a\,.$$

Note that $f(a) \subseteq f(b)$ whenever $a \preceq b$ and $g(I) \preceq g(I')$ whenever $I \subseteq I'$. Thus both maps are order-preserving and $g(f(b)) = b$ for all $b \in L$. To show that $f(g(I)) = I$ for every ideal $I \subseteq P$, let $t := g(I) = \bigvee_{a \in I} a$. Since $a \preceq t$ for all $a \in I$, we have that $I \subseteq I_t$. To show the converse inclusion, let $u \in I_t$. We compute

$$u \ = \ u \wedge t \ = \ u \wedge \bigvee_{a \in I} a \ \overset{D2}{=} \ \bigvee_{a \in I} (u \wedge a)\,.$$

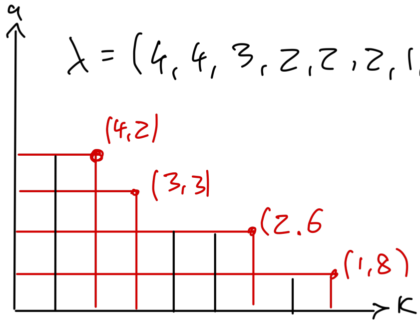
Since u is join-irreducible, we have $u = u \wedge a$ for some $a \in I$. Since this implies $u \preceq a$, we have $u \in I$. Hence $I_t \subseteq I$. $\square$

↓ 6. Vorlesung, 28.4.2022 ↓

Example 2.8 (Chains)**.** The chain $[n+1]$ is a distributive lattice. Since $i \wedge j = \min(i,j)$ and $i \vee j = \max(i,j)$, distributivity is trivially satisfied. This also shows that every element $i > 1$ is join irreducible, which confirms that $[n+1] \cong \mathcal{J}([n])$.

Example 2.9 (Divisor lattice and products of chains)**.** Let $n = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$. It is straightforward to verify that $a \in \mathcal{D}_n$ is join-irreducible if and only if $a = p_i^{l_i}$ for some $i = 1, \dots, s$ and $1 < l_i \le k_i$. This can also be seen from $\mathcal{D}_n \cong [k_1+1] \times \cdots \times [k_s+1]$. In particular, if we write $P \uplus Q$ for the disjoint union of two posets, then $\mathcal{J}(P \uplus Q) = \mathcal{J}(P) \times \mathcal{J}(Q)$ and we see that $\mathcal{D}_n \cong \mathcal{J}(P)$ for $P = [k_1] \uplus [k_2] \uplus \cdots \uplus [k_s]$.

Example 2.10 (Young's lattice – continued)**.** Note that a partition λ is join-irreducible if and only if $\lambda = (a, a, \dots, a)$ for $a \ge 1$ and the number of a is $k \ge 1$. We can identify λ with the pair $(a,k) \in \mathbb{P}^2$. If μ is also join-irreducible with (b,l), then $\lambda \le \mu$ if and only if $a \le b$ and $k \le l$. This implies that $\mathcal{Y} = \mathcal{J}(\mathbb{P} \times \mathbb{P})$. This gives a nice way to visualize partitions:



Such an left-bottom-aligned arrangement of boxes is called a **Young diagram**[2] of λ.

Proposition 2.7. *Let P be a finite poset. Then $\mathcal{J}(P)$ is a graded poset with rank function $r(I) = |I|$. In particular, every finite distributive lattice is graded.*

Proof. Let $I \subseteq J \subseteq P$ be ideals in the poset P. The ideals $K \in [I,J]_{\mathcal{J}(P)}$ are in bijection to the ideals in the induced subposet $J \setminus I$. Since the map $K \mapsto K \setminus I$ is also order-preserving, we see that $[I,J]_{\mathcal{J}(P)} \cong \mathcal{J}(J \setminus I)$ and the chain follows from induction on $|P|$. $\square$

In particular, if $I \prec\!\!\cdot\, J$ in $\mathcal{J}(P)$, then $J \setminus I = \{t\}$ where $t \in \min(P \setminus I)$.

Exercise 2.2. Let $(P, \preceq)$ be a poset. An **order filter** is a set $F \subseteq P$ such that $a \in F$ and $a \preceq b$ implies $b \in F$. Let L be a distributive lattice and let $\mathrm{JI}(L)$ and $\mathrm{MI}(L)$ be the collections of join-irreducibles and meet-irreducibles, respectively.

(a) Show that a distributive lattice L is *anti*-isomorphic to the ring of sets of order filters of the induced poset $\mathrm{MI}(L)$. Anti-isomorphic here means that $a \prec b$ implies $f(a) \succ f(b)$.
(b) Conclude that $|\mathrm{JI}(L)| = |\mathrm{MI}(L)|$.
(c) Let $j \in \mathrm{JI}(L)$. Show that there is a unique meet-irreducible element $m \in L\mathrm{MI}(L)$ with $j \not\preceq m$. Conclude that $j \mapsto m$ gives an injective map $\mathrm{JI}(L) \to \mathrm{MI}(L)$. [Hint: Assume that $L = \mathcal{J}P$ and consider complements of join-irreducible ideals.]

2.3. **Modular lattices.** Let V be a vector space over a (not necessarily finite) field $\mathbb{F}$ and let $A, B \subseteq V$ be two subspaces. The meet and join in the poset with respect to inclusion are $A \wedge B = A \cap B$ and $A \vee B = \mathrm{span}(A \cup B) = A + B$. For a third subspace $Y \subseteq V$, the triple A, B, Y do not in general satisfy the distributive law. For example, take three distinct lines in the plane. However, *if* $A \subseteq B$, then

$$(A+Y) \cap B \; = \; (A \cap B) + (Y \cap B) \; = \; A + (Y \cap B)$$

We call a lattice $(L, \wedge, \vee)$ **modular** if all $a, b, y \in L$ with $a \preceq b$ satisfy

$$(a \vee y) \wedge b \; = \; a \vee (y \wedge b)\,. \qquad \text{(M)}$$

[2]Most of the time Young diagrams are top-left aligned but this just doesn't work here.

We can think about this as follows: for $a \preceq b$, we can define two surjective and order-preserving (projection) maps $L \to [a,b]$, namely $y \mapsto (a \vee y) \wedge b$ and $y \mapsto a \vee (y \wedge b)$. The lattice is modular if the projections coincide.

We can also state this in more general terms: for any $x, y, z \in L$

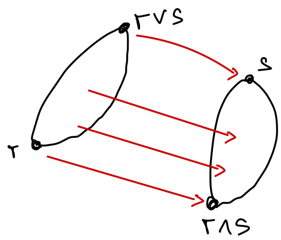
$$(y \vee (x \wedge z)) \wedge z \;=\; (y \wedge z) \vee (x \wedge z)$$

Since $x \preceq z$ if and only if $x = x \wedge z$, we see that the two conditions are equivalent.

Exercise 2.3. Let P be subspaces of a fixed vector space V, normal subgroups of a fixed group G, ideals in a fixed ring R. Verify that in all these cases $(P, \subseteq)$ is a modular lattice.

The exercise shows that modular lattices are important. For now, we show an important application of the modular property:

Theorem 2.8 (Dedekind's transposition principle)**.** *Let L be a modular lattice and $r, s \in L$. Then $[r, r \vee s] \cong [r \wedge s, s]$ with respect to $m(t) := s \wedge t$ and $j(t) := r \vee t$.*



Proof. Let $t \in [r, r \vee s]$. In particular $r \preceq t$ and applying (M) to s yields

$$j(m(t)) \;=\; r \vee (s \wedge t) \vee r \;=\; (s \vee r) \wedge t \;=\; t\,,$$

where the last equality follows from $t \leq s \vee r$. The argument $m(j(t)) = t$ for all $t \in [r \wedge s, s]$ is similar. □

Corollary 2.9. *Let L be modular and $r, s \in L$. Then r and s both cover $r \wedge s$ if and only if both r and s are covered by $r \vee s$.*

Proof. We have that r covers $r \wedge s$ if and only if $\{r \wedge s, r\} = [r \wedge s, r]$. It follows from Theorem 2.8 that $[r \wedge s, r] \cong [s, r \vee s]$ and hence $[s, r \vee s] = \{s, r \vee s\}$. That is, s is covered by $r \vee s$. Exchanging the roles of r and s then proves the claim. □

Corollary 2.10. *If L is a finite modular lattice, then L is graded.*

Proof. Exercise! □

A function $V : L \to \mathbb{C}$ on a lattice L is called a **valuation** or **modular** if for all $a, b \in L$

$$V(a \vee b) \;=\; V(a) + V(b) - V(a \wedge b)\,.$$

Corollary 2.11. *If L is a modular lattice, then the rank function r is modular.*

Proof. The transposition principle implies $[a, a \vee b] \cong [a \wedge b, b]$. In terms of ranks, this implies

$$r(a \vee b) - r(a) \;=\; r(b) - r(a \wedge b)\,,$$

that is, r is modular. □

We will later introduce another class of lattices, the *semimodular lattices*, that are very important from a combinatorial point of view but for now we focus on something more related to inclusion-exclusion.

↓ ——— 7. Vorlesung, 3.5.2022 ——— ↓

3. Möbius inversion

Let $(P, \preceq)$ be a finite poset and consider two functions $f_=, f_\geq : P \to \mathbb{C}$ such that

$$f_\geq(b) \;=\; \sum_{b \preceq_P c} f_=(c)\,.$$

Can we recover $f_=$ from $f_\geq$ (and the knowledge of P)?

We define the **Möbius function** of a P as the unique function $\mu_P : P \times P \to \mathbb{Z}$ as follows. If $a \not\preceq c$, then set $\mu_P(a,c) := 0$. If $a = c$, then set $\mu_P(a,a) := 1$. If $a \prec c$, then we define

$$\mu_P(a,c) \;:=\; -\sum_{a \prec b \preceq c} \mu_P(b,c)\,. \tag{3.1}$$

Note that $\ell(a,b) < \ell(a,c)$ whenever $a \prec b \preceq c$ and hence the recursive definition is legitimate. Also note that the definition of the Möbius function implies that for all $a \preceq c$, we have

$$\sum_{a \preceq b \preceq c} \mu_P(a,b) \;=\; \sum_{a \preceq b \preceq c} \mu_P(b,c) \;=\; \begin{cases} 1 & \text{if } a = c \\ 0 & \text{otherwise} \end{cases}. \tag{3.2}$$

The second equality follows directly from the definition. This also implies that

$$\mu_P(a,c) \;=\; -\sum_{a \preceq b \prec c} \mu_P(a,b)$$

which then implies the first equality in (3.2).

The Möbius function allows us to recover $f_=$ from $f_\geq$. We claim that

$$f_=(a) \;=\; \sum_{a \preceq b} \mu_P(a,b) f_\geq(b)\,. \tag{3.3}$$

Indeed,

$$\sum_{a \preceq b} \mu_P(a,b) f_\geq(b) \;=\; \sum_{a \preceq b} \mu_P(a,b) \sum_{b \preceq_P c} f_=(c) \;=\; \sum_{a \preceq c} f_=(c) \sum_{a \preceq b \preceq c} \mu(a,b) \;=\; f_=(a)\,,$$

where the last equality uses (3.2).

To see how that relates to the principle of inclusion-exclusion, let S and E be finite sets and $A_e \subseteq S$ a subset for each $e \in E$. We can see E as a collection of properties that the objects in S can have. For a property $e \in E$, we think of A_e as those objects that have *at least* the property e and for $I \subseteq E$, we define $f_\geq(I) := |\bigcap_{e \in I} A_e|$ as the number of objects that that have at least all the properties in I. In the introduction, we assumed that E is the set of edges of a finite graph $G = (V, E)$. The set A_e for $e \in E$ was those labellings $c : V \to [k]$ that fail to be a k-coloring at least at the edge e, that is, $c(u) = c(v)$ if $e = uv$. In this language $f_=(I)$ is then the set of labellings that fail to be a k-coloring *precisely* at the edges I and hence $f_=(\varnothing)$ is the number of k-colorings of G.

It turns out that the Möbius function for the Boolean lattice $B_n = (2^{[n]}, \subseteq)$ is given by

$$\mu_{B_n}(I,J) \;=\; (-1)^{|J \setminus I|}$$

for all $I \subseteq J$ and hence

$$f_=(I) \;=\; \sum_{I \subseteq J} (-1)^{|J \setminus I|} f_\geq(J)\,.$$

The name *Möbius function* derives from number theory. The number-theoretic Möbius function is the function $\mu : \mathbb{P} \to \mathbb{Z}$ given by $\mu(p) = -1$ if p is a prime, $\mu(a) = 0$ if $p^2 \mid a$ for some prime p and $\mu(ab) = \mu(a)\mu(b)$ whenever a, b are coprime. The *Möbius inversion* formula then states that for two functions $f_\leq, f_= : \mathbb{P} \to \mathbb{C}$ the following holds

$$f_\leq(n) \;=\; \sum_{d : d \mid n} f_=(d) \qquad \Longleftrightarrow \qquad f_=(n) \;=\; \sum_{d : d \mid n} f_\leq(d)\mu(\tfrac{n}{d})$$

For each fixed $n \in \mathbb{P}$, we can view $f_{\le}$ and $f_{=}$ as functions on $\mathcal{D}_n$. Remembering that $[a,b]_{\mathcal{D}_n} \cong \mathcal{D}_{b/a}$, we define $\mu_{\mathcal{D}_n}(a,b) := \mu(b/a)$. Then number-theoretic Möbius inversion yields

$$f_{\le}(b) \;=\; \sum_{a \preceq b} f_{=}(a) \qquad \Longleftrightarrow \qquad f_{=}(b) \;=\; \sum_{a \preceq b} f_{\le}(a)\mu_{\mathcal{D}_n}(a,b)$$

for all $a, b \in \mathcal{D}_n$.

If $(P, \preceq) = ([n], \le)$ is a chain, then

$$f_{\le}(k) \;=\; \sum_{i \preceq k} f_{=}(i) \;=\; f_{=}(1) + f_{=}(2) + \cdots + f_{=}(k)$$

and hence $f_{=}(1) = f_{\le}(1)$ and for $k > 1$

$$f_{=}(k) \;=\; f_{\le}(k) - f_{\le}(k-1)\,.$$

From this we deduce

$$\mu_{[n]}(i,j) \;=\; \begin{cases} 1 & i = j \\ -1 & i = j-1 \\ 0 & \text{otherwise} \end{cases}\,.$$

To see the connection to B_n and $\mathcal{D}_n$, recall that

$$B_n \;\cong\; \underbrace{[2] \times \cdots \times [2]}_{n} \qquad \mathcal{D}_n \;\cong\; [k_1+1] \times \cdots \times [k_s+1]$$

where $n = p_1^{k_1} \cdots p_s^{k_s}$. The next proposition now explains the Möbius functions in these cases.

Proposition 3.1. *If $P = P_1 \times P_2$ and $(a_1,a_2), (b_1,b_2) \in P$, then*

$$\mu_P((a_1,a_2),(b_1,b_2)) \;=\; \mu_{P_1}(a_1,b_1)\mu_{P_2}(a_2,b_2)\,.$$

Proof. Exercise! □

Let us come back our original motivation and inclusion-exclusion: For finite sets A_e for $e \in E$, define the **intersection poset** as the collection

$$\mathcal{L} \;=\; \mathcal{L}(A_e : e \in E) \;=\; \{A_I : I \subseteq E\}$$

where $A_I := \bigcap_{i \in I} A_i$ and $A_\varnothing := \bigcup_{e \in E} A_e$. We give $\mathcal{L}$ a partial order by *reverse* inclusion. That is, for $A, B \in \mathcal{L}$

$$A \;\preceq_{\mathcal{L}}\; B \quad :\Longleftrightarrow \quad B \subseteq A\,.$$

This way $\hat{0} = A_\varnothing$ and $\hat{1} = A_E$. The reason for doing this is that the map $2^E \to \mathcal{L}$ given by $I \mapsto A_I$ is now order-preserving: $I, J \subseteq E$

$$I \;\subseteq\; J \quad :\Longrightarrow \quad A_I \;\preceq_{\mathcal{L}}\; A_J$$

The reverse inclusion does not hold in general. It does, if we restrict to the *right* sets. For $A \in \mathcal{L}$, define

$$J_A \;:=\; \{e \in E : A \subseteq A_e\}\,.$$

That is, J_A is the inclusion-maximal set J with $A_J = A$. We call J_A a **closed** set. If we restrict to closed sets, then $A_I \preceq_{\mathcal{L}} A_J$ implies $I \subseteq J$.

Corollary 3.2. *Let $Q = \{I \subseteq E : I$ closed$\}$. Then $(Q, \subseteq)$ is isomorphic to $(\mathcal{L}, \preceq)$.*

Theorem 3.3. *Let $\mathcal{L} = \mathcal{L}(A_e : e \in E)$ be an intersection poset and $S \preceq_{\mathcal{L}} T$. Then*

$$\mu_{\mathcal{L}}(S,T) \;=\; \sum_{\substack{J_S \subseteq J \subseteq E \\ A_J = T}} (-1)^{J \setminus J_S}\,.$$

Proof. Given $f_{=}, f_{\ge} : \mathcal{L} \to \mathbb{C}$ as before, we can define $F_{=}, F_{\ge} : 2^E \to \mathbb{C}$ by

$$F_{=}(I) \;:=\; \begin{cases} f_{=}(A_I) & \text{if } I \text{ is closed,} \\ 0 & \text{otherwise.} \end{cases} \qquad \text{and} \qquad F_{\ge}(I) \;:=\; \sum_{I \subseteq J} F_{=}(J)$$

Note that for any $I \subseteq E$, we have

$$F_{\geq}(I) \;=\; \sum_{I \subseteq J} F_{=}(J) \;=\; \sum_{A_I \preceq_{\mathcal{L}} T} f_{=}(T) \;=\; f_{\geq}(A_I)\,.$$

This means that for $S \in \mathcal{L}$, we can compute

$$\begin{aligned} f_{=}(S) \;=\; F_{=}(J_S) \;&=\; \sum_{J_S \subseteq J} (-1)^{|J \setminus J_S|} F_{\geq}(J) \\ &=\; \sum_{S \preceq_{\mathcal{L}} T} f_{\geq}(T) \sum_{\substack{J_S \subseteq J \subseteq E \\ A_J = T}} (-1)^{|J \setminus J_S|} \end{aligned}$$

On the other hand, we can use Möbius inversion (3.3) to conclude that for every $S \in \mathcal{L}$

$$\sum_{S \preceq_{\mathcal{L}} T} \mu_{\mathcal{L}}(S,T)\, f_{\geq}(T) \;=\; \sum_{S \preceq_{\mathcal{L}} T} \Big(\sum_{\substack{J_S \subseteq J \subseteq E \\ A_J = T}} (-1)^{|J \setminus J_S|} \Big)\, f_{\geq}(T)$$

Since $f_{=}$ was chosen arbitrarily and the Möbius function is unique, this yields the claim. □

↓ 8. Vorlesung, 5.5.2022 ↓

3.1. **The incidence algebra.** In order to handle Möbius functions, we introduce an algebraic framework. Let $(P, \preceq)$ be a finite poset. We write $\mathbb{C}^P = \{f : P \to \mathbb{C}\}$ to the collection of all maps from P to $\mathbb{C}$. This is trivially a $\mathbb{C}$ vector space. (After all, $\mathbb{C}^P \cong \mathbb{C}^{|P|}$.)

We define $\mathrm{I}(P)$ as the collection of all maps $\alpha : P \times P \to \mathbb{C}$ with

$$a \;\not\preceq_P\; b \quad\Longrightarrow\quad \alpha(a,b) \;=\; 0$$

This is also a $\mathbb{C}$-vector space: if $\alpha, \beta \in \mathrm{I}(P)$ and $a, b \in \mathbb{C}$, then clearly $a\alpha + b\beta \in \mathrm{I}(P)$. We can let $\alpha \in \mathrm{I}(P)$ *act* on $\mathbb{C}^P$ by defining for $f \in \mathbb{C}^P$

$$(\alpha f)(b) \;:=\; \sum_{c \in P} \alpha(b,c) f(c) \;=\; \sum_{b \preceq_P c} \alpha(b,c) f(c)\,.$$

This action suggests a multiplicative structure on $\mathrm{I}(P)$: for $\alpha, \beta \in \mathrm{I}(P)$, define $\alpha * \beta : P \times P \to \mathbb{C}$ by

$$(\alpha * \beta)(a,c) \;:=\; \sum_{b \in P} \alpha(a,b)\beta(b,c) \;=\; \sum_{a \preceq b \preceq c} \alpha(a,b)\beta(b,c)\,.$$

We see that $(\alpha * \beta)(a,c) = 0$ whenever $a \not\preceq c$ and hence $\alpha * \beta \in \mathrm{I}(P)$. We leave it to the reader to verify that $*$ is associative and distributive and hence gives $(\mathrm{I}(P), +, *)$ the structure of a $\mathbb{C}$-algebra. The element $\delta \in \mathrm{I}(P)$ with $\delta(a,b) = 1$ if and only if $a = b$ and $= 0$ otherwise serves as the unit in $\mathrm{I}(P)$. Hence, $\mathrm{I}(P)$ is a unital and in general noncommutative algebra.

The **zeta function** is $\zeta_P \in \mathrm{I}(P)$ defined as

$$\zeta_P(a,b) \;=\; \begin{cases} 1 & \text{if } a \preceq b \\ 0 & \text{otherwise.} \end{cases}$$

The zeta function allows us to write

$$f_{\geq}(b) \;=\; (\zeta_P f)(b) \;=\; \sum_{c \in P} \zeta_P(b,c) f(c) \;=\; \sum_{b \preceq c} f(c)\,.$$

Proposition 3.4. *Let $\alpha \in \mathrm{I}(P)$. Then there is $\beta \in \mathrm{I}(P)$ with $\alpha * \beta = \beta * \alpha = \delta_P$ if and only if $\alpha(a,a) \neq 0$ for all $a \in P$.*

Proof. If $\alpha * \beta = \delta_P$, then $1 = \delta(a,a) = (\alpha * \beta)(a,a) = \alpha(a,a)\beta(a,a)$ and hence $\alpha(a,a) \neq 0$ for all $a \in P$. Conversely, we define β by induction on $\ell(a,c) = r([a,c]_P)$. For $\ell(a,c) = 0$ and hence $a = c$, we define $\beta(a,a) = \frac{1}{\alpha(a,a)}$. For $a \prec c$, define

$$\beta(a,c) \;:=\; \frac{-1}{\alpha(a,a)} \sum_{a \prec b \preceq c} \alpha(a,b)\beta(b,c)\,. \qquad \square$$

It follows that β is unique and we write $\alpha^{-1} := \beta$. In particular, ζ_P is invertible and its inverse is the Möbius function $\mu_P = \zeta_P^{-1}$. That is,

$$f_{\geq} \;=\; \zeta_P f_{=} \quad\Longleftrightarrow\quad f_{=} \;=\; \mu_P f_{\geq}\,.$$

The zeta function has many more uses than just to introduce the Möbius function and some interesting counting problems can be modelled with it. Let $(P, \preceq)$ be a finite fixed poset. The **order polynomial** of $(P, \preceq)$ is the function

$$\Omega_P(n) \;:=\; |\{\phi : P \to [n] : \phi \text{ order-preserving}\}|\,.$$

Proposition 3.5. *Let $(P, \preceq)$ be a poset with $d = |P|$ elements. Then $\Omega_P(n)$ agrees with a polynomial of degree d.*

Exercise 3.1 outlines a simple proof of this result but we take the more scenic route.

Exercise 3.1. Let $(P, \preceq)$ be a finite poset.

(a) Show that every order preserving map $\phi : P \to [n]$ factors $P \overset{s}{\twoheadrightarrow} R \overset{i}{\hookrightarrow} [n]$, where R is a subposet of $[n]$, s is surjective and i is injective.
(b) Count the number of k-element subposets of $[n]$.
(c) Conclude that

$$\Omega_P(n) \;=\; \sum_{k=0}^{|P|} c_k \binom{n}{k}$$

and give an interpretation for c_k.

Let $\phi : P \to [2]$ be order-preserving and let $I := \phi^{-1}(1)$. It is straightforward to verify that $I \subseteq P$ is an ideal of P. Indeed, if $b \in I$ and $a \preceq b$, then $\phi(a) \leq \phi(b) = 1$ and hence $a \in I$. (By the same token $\phi^{-1}(2)$ is a filter.) A **multichain** of length k in a poset Q is sequence

$$x_0 \;\preceq\; x_1 \;\preceq\; \cdots \;\preceq\; x_k\,.$$

Proposition 3.6. *There is a bijection between order-preserving maps $\phi : P \to [k]$ and multichains of length $k-2$ in $\mathcal{J}(P)$.*

Proof. For a given $\phi : P \to [k]$ define $I_j^\phi := \{a \in P : \phi(a) \leq j\}$ for $j = 1, \ldots, k-1$. This gives a multichain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{k-1}$ and each I_j is an order ideal.

Conversely, $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_{k-1}$ be a multi-chain of order ideals of length $k-2$ and set $I_k := P$. We define $\phi : P \to [k]$ by setting $\phi(b) := j$ if $j \geq 1$ is the smallest index for which $b \in I_j$. If $a \preceq b$ then $a \in I_j$ and hence $\phi(a) \leq j$. This means ϕ is order preserving and $I_j^\phi = I_j$. □

Thus, counting order-preserving maps into a chain on k elements, we need to count multichains in $\mathcal{J}(P)$. This can be done using the zeta function.

Proposition 3.7. *Let $(P, \preceq)$ be a finite poset with zeta function $\zeta = \zeta_P$. Then for any $a, c \in P$ and $n \geq 0$, $\zeta^n(a, c)$ is the number of multichains*

$$a = a_0 \preceq a_1 \preceq \cdots \preceq a_2 \preceq a_n = c\,.$$

Proof. We prove the claim by induction on n. For $n = 0$ there is only one such multichain if $a = c$ and $\zeta^0(a, c) = \delta(a, c)$. For $n = 1$, the unique multichain is $a = a_0 \preceq a_1 = c$, provided $a \preceq c$ and $\zeta(a, c)^1 = 1$ in this case. For $n + 1$, we compute

$$\zeta^{n+1}(a, c) \;=\; (\zeta^n * \zeta)(a, c) \;=\; \sum_{a \preceq b \preceq c} \zeta^n(a, b)$$

By induction, $\zeta^n(a, b)$ counts the number of multichains of length n ending in b, which we can extend to c with one more element. □

Exercise 3.2. Define $\eta : P \times P \to \mathbb{C}$ by $\eta(a, b) := 1$ if $a \prec b$ and $:= 0$ otherwise. Show that for $a, c \in P$ and $n \geq 0$, $\eta^n(a, c)$ counts the number of chains

$$a = a_0 \prec a_1 \prec \cdots \prec a_2 \prec a_n = c\,.$$

Proposition 3.8. *Let $(P, \preceq)$ be a finite poset and $a \prec_P c$. The function*

$$n \mapsto \zeta_P^n(a, c)$$

agrees with a polynomial of degree $\ell(a, c)$.

Proof. Let $\eta \in \mathrm{I}(P)$ be as defined in Exercise 3.2. We observe that $\zeta = \delta + \eta$ and hence

$$\zeta^n(a, c) \;=\; (\delta + \eta)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} \eta^k(a, b)\,,$$

where the last equation follows from the fact that $\delta * \eta = \eta * \delta$. According to Exercise 3.2, we have for $d := \ell(a, c)$ that $\eta^d \neq 0$ and $\eta^j = 0$ for all $j > d$. That means

$$\zeta^n(a, c) \;=\; \eta^0(a, c)\binom{n}{0} + \eta^1(a, c)\binom{n}{1} + \cdots + \eta^d(a, c)\binom{n}{d}$$

is a finite sum and since $\binom{n}{k} = \frac{1}{k!} n(n-1)\cdots(n-k+1)$ is a polynomial of degree k, this finishes the proof. $\square$

Proof of Proposition 3.5. $\Omega_P(n) = \zeta^n_{\mathcal{J}(P)}(\varnothing, P)$ is a polynomial of degree $\ell(\varnothing, P) = r(\mathcal{J}(P)) = |P|$. $\square$

↓ 9. Vorlesung, 10.5.2022 ↓

We make a definition of what we learnt in the last two propositions: Let $(P, \preceq)$ with $\hat{0}$ and $\hat{1}$. We define the **zeta polynomial**

$$Z_P(n) \;:=\; \zeta_P^n(\hat{0}, \hat{1}) \;=\; \#\{\hat{0} = a_0 \preceq a_1 \preceq \cdots \preceq a_n = \hat{1} \text{ multichain in } P \text{ of length } n\}\,.$$

This is agrees with a polynomial in n of degree $|P|$. Proposition 3.6 now shows that $\Omega_P(n) = Z_{\mathcal{J}(P)}(n)$.

An interesting observation is that, as a polynomial $Z_P(t) \in \mathbb{Z}[t]$, we can also evaluate it at *nonpositive integers*.

Proposition 3.9.

$$Z_P(-n) \;=\; \zeta_P^{-n}(\hat{0}, \hat{1}) \;=\; \mu_P^n(\hat{0}, \hat{1})\,.$$

Proof. Let $d = r(P)$ length of longest chain. Then[3]

$$\mu^n \;=\; \zeta^{-n} \;=\; (\delta + \eta)^{-n} \;=\; (\delta - \eta + \eta^2 - \cdots + (-1)^d \eta^d)^n \;=\; \sum_{k=0}^{d} (-1)^k C_k \eta^k\,.$$

Now C_k is the number of sequences $(c_1, c_2, \ldots, c_n) \in \mathbb{Z}_{\geq 0}$ with $\sum_i c_i = k$. Such a sequence is called a **composition** of k into n parts and is counted by $\binom{n+k-1}{k}$. This is a polynomial in n of degree k.

Now, observe

$$\binom{-n}{k} = \frac{1}{k!}\prod_{i=0}^{k-1}(-n-i) \;=\,= \frac{(-1)^k}{k!}\prod_{i=0}^{k-1}(n+k-1-i) \;=\; (-1)^k\binom{n+k-1}{k}\,. \tag{3.4}$$

This means that the polynomial for $\mu^n(\hat{0}, \hat{1})$ agrees with $Z_P(-n)$. $\square$

Can we interpret $Z_P(-n)$ for $n > 0$?

We record a simple observation that will be important later.

Corollary 3.10 (Philip Hall's theorem)**.** *Let $(P, \preceq)$ be a poset and $a, b \in P$. For $i \geq 0$, let c_i be the number of chains in $(a, b)_P = [a, b]_P \setminus \{a, b\}$ of length $-1 \leq i \leq d := \ell(a, b)$. We set $c_{-1} := 1$ for the unique chain in $(a, a)_P = \varnothing$ with zero elements. Then*

$$\mu_P(a, b) \;=\; c_{-1} - c_0 + \cdots + (-1)^d c_{d-2}\,.$$

[3] If a is an unipotent element of a ring, then $1 - a$ is nilpotent and $(1 - (1-a))\sum_{i=0}^{k}(1-a)^i = 1 - (1-a)^{k+1} = 1$ and hence $a^{-1} = \sum_{i=0}^{k}(1-a)^i$.

Proof. We note that

$$\mu(a,b) \;=\; \zeta^{-1}(a,b) \;=\; (\delta+\eta)^{-1}(a,b) \;=\; \sum_{i=0}^{d}(-1)^i\eta^i(a,b)\,.$$

Exercise 3.2 shows $\eta^i(a,b)$ is the number of chains of $a = a_0 \prec a_1 \prec \cdots \prec a_{i-1} \prec a_i = b$. Every such chain of length i yields a chain in $(a,b)_P$ and vice versa and hence $c_i = \eta^{i+2}(a,b)$. $\square$

3.2. **Lattices and their Möbius algebras.** In this section, we make use of the extra structure a lattice to compute Möbius functions.

A **group** is a set S together with $\circ : S \times S \to S$ such that

(G1) $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a,b,c \in S$; (Associativity)
(G2) There is an element $e \in S$ such that $e \circ a = a \circ e = a$ for all $a \in S$; (Unit)
(G3) For every $a \in S$ there is a $b \in S$ such that $a \circ b = e$. (Inverses)

If we require only (G1) and (G2), then $(S,\circ)$ is a **monoid**. If $(S,\circ)$ only satisfies (G1), then the pair is called a **semigroup**. There is a simple way to turn a semigroup into an algebra. For every $a \in S$, let t^a be a symbol. Let $\mathbb{C}[S]$ be the vector space of finite formal linear combinations of $(t^a)_{a\in S}$, that is, elements look like

$$\sum_{a\in S} c_a t^a\,,$$

where $c_a \in \mathbb{C}$ and $\{a \in S : c_a \neq 0\}$ is finite. The t^a form a basis. A multiplication on $\mathbb{C}[S]$ is a bilinear map $\cdot : \mathbb{C}[S] \times \mathbb{C}[S] \to \mathbb{C}[S]$ that we only need to define on a basis: $t^a \cdot t^b := t^{a\circ b}$. More concretely,

$$\Big(\sum_{a\in S} c_a t^a\Big)\cdot\Big(\sum_{b\in S} d_b t^b\Big) \;:=\; \sum_{z\in S} g_z t^z \quad \text{for} \quad g_z \;:=\; \sum_{\substack{a,b\in S\\ a\circ b = z}} c_a d_b$$

It follows from (G1) that the bilinear map $\cdot : \mathbb{C}[S] \times \mathbb{C}[S] \to \mathbb{C}[S]$ is associative and distributive. If S is a monoid with neutral element e, then t^e is the multiplicatively neutral element in $\mathbb{C}[S]$. If $(S,\circ)$ is commutative/abelian, then $\mathbb{C}[S]$ is commutative.

Example 3.1. The natural numbers $(\mathbb{N},+)$ form a monoid under addition. Thus $\mathbb{C}[\mathbb{N}]$ is the set of linear combinations $c_0t^0 + c_1t^1 + \cdots + c_dt^d$ and multiplication $t^it^j = t^{i+j}$. Thus $\mathbb{C}[\mathbb{N}]$ is the polynomial ring with variable t over $\mathbb{C}$.

Let $(L,\preceq)$ be a meet-semilattice. Then $\wedge : L \times L \to L$ is associative and hence $(L,\wedge)$ is a semigroup. We call $\mathbb{C}[L]$ the **Möbius algebra** of L. The Möbius algebra is commutative and if L has a maximal element $\hat{1}$ (and hence is a lattice), then $\mathbb{C}[L]$ has a unit $t^{\hat{0}}$.

Note that as a vector space $\mathbb{C}[L] \cong \mathbb{C}^L$. The standard basis of $\mathbb{C}^L$ is given by e_a for $a \in L$. We can give $\mathbb{C}^L$ a simple algebra structure by defining

$$e_ae_b \;:=\; \begin{cases} e_a & \text{if } a = b,\\ 0 & \text{otherwise.}\end{cases}$$

In particular $e_a^2 = e_a$ for all $a \in L$. Two $\mathbb{C}$-algebras A, B are isomorphic if there is a vector space isomorphism $f : A \to B$ such that $f(aa') = f(a)f(a')$ for all $a,a' \in A$[4].

Theorem 3.11 (Solomon)**.** *Let $(L,\preceq)$ be a finite lattice, then $\mathbb{C}[L] \cong \mathbb{C}^L$ as $\mathbb{C}$-algebras.*

For the proof, define $f : \mathbb{C}[L] \to \mathbb{C}^L$ by

$$f(t^b) \;=\; f_b \;:=\; \sum_{a\preceq b} e_a \;\in\; \mathbb{C}^L$$

on basis elements and extend it to $\mathbb{C}[L]$ by linearity. Note that for $a,b \in L$

$$e_af_b \;=\; \zeta(a,b)e_a\,.$$

[4]Check that for all $b,b' \in B$ it holds that $f^{-1}(bb') = f^{-1}(b)f^{-1}(b')$!

Proof. For $b, c \in L$, we compute

$$f(t^b)f(t^c) \;=\; f_b \cdot f_c \;=\; \Big(\sum_{x \preceq b} e_x\Big)\Big(\sum_{y \preceq c} e_y\Big) \;=\; \sum_{\substack{x \preceq b \\ y \preceq c}} e_x e_y \;=\; \sum_{z \preceq b \wedge c} e_z \;=\; f_{b \wedge c} \;=\; f(t^a \cdot t^b)\,,$$

where the third equality follows from $e_a e_b = \delta(a,b)e_a$ and the fourth from the definition of meets. This means that f is a linear map that respects products.

To see that f is an isomorphism, use that $\dim \mathbb{C}[L] = |L| = \dim \mathbb{C}^L$ and only check that f is injective. Let $k \geq 2$ be minimal such that there are $a_1, \dots, a_k \in L$ and $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ such that $0 = \alpha_1 f_{a_1} + \alpha_2 f_{a_2} + \cdots + \alpha_k f_{a_k}$. Minimality implies that $\alpha_1, \dots, \alpha_k \neq 0$ and $a_i \neq a_j$ for $i \neq j$. Assume that a_k is maximal among $a_1, \dots, a_k$, that is $a_k \not\preceq a_i$ for $i \neq k$. Then

$$0 \;=\; e_{a_1}(\alpha_1 f_{a_1} + \alpha_2 f_{a_2} + \cdots + \alpha_k f_{a_k}) \;=\; \alpha_1 e_{a_1} f_{a_1} + \alpha_2 e_{a_1} f_{a_2} + \cdots + \alpha_k e_{a_1} f_{a_k} \;=\; \alpha_k e_{a_k}$$

This means $\alpha_k = 0$, which contradicts the assumption that k was minimal. □

↓ 10. Vorlesung, 12.5.2022 ↓

We can also provide an explicit description of $f^{-1}(e_a)$ for all $a \in L$. For $c \in L$ define

$$m_c \;:=\; \sum_{b \preceq c} \mu_P(b,c) t^b$$

Then

$$f(m_c) \;=\; \sum_{b \preceq c} f_b \mu_P(b,c) \;=\; \sum_{a \preceq b} \mu_P(b,c) \sum_{a \preceq b} e_a \;=\; \sum_{a \preceq c}\Big(\sum_{a \preceq b \preceq c} \mu_P(b,c)\Big)e_a \;=\; e_c\,.$$

In particular, by Möbius inversion

$$t^b \;=\; \sum_{a \preceq b} m_b\,. \tag{3.5}$$

Since f is an algebra isomorphism, we get

$$m_a \cdot m_b \;=\; \begin{cases} m_a & \text{if } a = b \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

A basis of an algebra with these properties under multiplication is called a complete system of **orthogonal idempotents**.

We can use this algebra to prove Möbius function identities such as

Proposition 3.12 (Weisner's theorem)**.** *Let L be a finite lattice and $b \prec \hat{1}$. Then*

$$\sum_{s : s \wedge b = \hat{0}} \mu(s, \hat{1}) \;=\; 0\,.$$

The result implies

$$\mu(\hat{0}, \hat{1}) \;=\; -\sum_{\substack{\hat{0} \prec s \\ s \wedge b = \hat{0}}} \mu(s, \hat{1})$$

This is similar to (3.1) but has in general fewer terms.

Proof. Let us compute $t^b m_{\hat{1}}$ in two different ways. By (3.5)

$$t^b m_{\hat{1}} \;=\; \Big(\sum_{a \preceq b} m_a\Big) m_{\hat{1}} \;=\; \sum_{a \preceq b} (m_a m_{\hat{1}}) \;=\; 0\,,$$

where the last equality follows from (3.6) and the fact that $b \prec \hat{1}$. On the other hand,

$$t^b m_{\hat{1}} \;=\; t^b \Big(\sum_{s \preceq \hat{1}} \mu(s, \hat{1}) t^s\Big) \;=\; \sum_{s \preceq \hat{1}} \mu(s, \hat{1}) t^{b \wedge s} \;=\; \sum_{r \in L} c_r t^r$$

where $c_r = \sum_{s : s \wedge b = r} \mu(s, \hat{1})$. Hence $c_{\hat{0}} = 0$. □

A **crosscut** in a finite lattice L is a set $K \subseteq L$ such that $\hat{1} \notin K$ and for every $a \in L \setminus \hat{1}$, there is $b \in K$ with $a \preceq b$.

Theorem 3.13 (Crosscut theorem)**.** *Let $(L, \preceq)$ be a finite lattice and K a crosscut. Then*

$$\mu_L(\hat{0}, \hat{1}) \;=\; \sum_{k \geq 0} (-1)^k N_k \,,$$

where N_k is the number of k-element subsets $K' \subseteq K$ with $\bigwedge K' = \hat{0}$.

Proof. Consider the expression $\prod_{b \in K}(t^{\hat{1}} - t^b) = \sum_{x \in L} c_x t^x$. It is not hard to see that

$$c_{\hat{0}} \;=\; \sum_{k \geq 0} (-1)^k N_k \,.$$

On the other hand, we have for all $b \in L$

$$t^{\hat{1}} - t^b \;=\; \sum_{a \preceq \hat{1}} m_a - \sum_{a \preceq b} m_a \;=\; \sum_{a \not\preceq b} m_a \,.$$

It follows then from (3.6) that

$$\prod_{b \in K} (t^{\hat{1}} - t^b) \;=\; \prod_{b \in K} \sum_{a \not\preceq b, \forall b \in K} m_a \;=\; m_{\hat{1}} \;=\; \sum_{a \preceq \hat{1}} \mu(a, \hat{1}) \,.$$

Comparing coefficients we see that $c_{\hat{0}} = \mu(\hat{0}, \hat{1})$. ☐

An element $a \in P$ is an **atom** if it covers $\hat{0}$. Likewise, a is a **coatom** if a is covered by $\hat{1}$.

Corollary 3.14. *Let L be a lattice. If $\hat{0}$ is not the meet of coatoms, then $\mu(\hat{0}, \hat{1}) = 0$. Dually, if $\hat{1}$ is not the join of atoms, then $\mu(\hat{0}, \hat{1}) = 0$.*

For the second statement, we use the following if $(P, \preceq)$ is a poset, we define $(P, \preceq')$ the **dual** poset with $a \preceq' b$ if and only if $b \preceq a$. It follows that $(P, \preceq')$ is a (distributive) lattice if and only if $(P, \preceq)$ is. Meets and joins switch places as do atoms and coatoms.

Proof. We simply observe that the collection of coatoms is a crosscut for L and the result follows from the Crosscut theorem. ☐

Let us go back to distributive lattices and the Birkhoff lattice.

Corollary 3.15. *Let $(L, \preceq)$ be a distributive lattice and $a \preceq b$. Then*

$$\mu(a, b) \;=\; \begin{cases} (-1)^{\ell(a,b)} & \text{if } [a,b] \text{ is isomorphic to a Boolean lattice,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall that if $(L, \preceq)$ is a distributive lattice, then $(L, \preceq) \cong (\mathcal{J}(P), \subseteq)$ for some poset P. We argued that if $I \subseteq I'$ are ideals, then $[I, I'] \cong \mathcal{J}(I' \setminus I)$. Hence $[I, I']$ is isomorphic to a Boolean lattice if and only if $I' \setminus I$ is an anti-chain. It follows that $\mu_{\mathcal{J}(P)}(I, I') = (-1)^{|I' \setminus I|}$.

So, we only need to argue the second case. By $[I, I'] \cong \mathcal{J}(I' \setminus I)$, it suffices to show that $\mu_{\mathcal{J}(P)}(\hat{0}, \hat{1}) = 0$ if P is not an anti-chain. Let M be the minimal elements of P. Then $I \subset P$ is an atom in $\mathcal{J}(P)$ if and only if $I = \{m\}$ for some $m \in M$. Thus, a join of atoms is a subset of M. If P is not an anti-chain that M is a proper subset of $P = \hat{1}_{\mathcal{J}(P)}$ and hence $\mu(\hat{0}, \hat{1}) = 0$ by the Corollary above. ☐

↓ 11. Vorlesung, 17.5.2022 ↓

This allows us to interpret $Z_L(-n)$ when L is a distributive lattice.

Corollary 3.16. *Let $(L, \preceq)$ be a distributive lattice of rank d. Then $(-1)^d Z_L(-n)$ is the number of multichains*

$$\hat{0} \;=\; a_0 \;\preceq\; a_1 \;\preceq\; \cdots \;\preceq\; a_n \;=\; \hat{1}$$

such that $[a_{i-1}, a_i]$ is a Boolean lattice for all $i = 1, \ldots, n$.

Proof. We compute

$$Z_L(-n) \;=\; \mu_L^n(\hat{0},\hat{1}) \;=\; \sum_{\hat{0}=a_0\preceq\cdots\preceq a_n=\hat{1}} \mu_L(a_0,a_1)\mu_L(a_1,a_2)\cdots\mu_L(a_{n-1},a_n)$$

By Corollary 3.15, we know that the product of Möbius function evaluations is $=0$ unless $[a_{i-1},a_i]$ is Boolean for all $i=1,\ldots,n$. In this case, we know that $\mu(a_{i-1},a_i)=(-1)^{\ell(a_{i-1},a_i)}$. Since every distributive lattice is graded, this means $\ell(a_{i-1},a_i)=r(a_i)-r(a_{i-1})$ and hence

$$(-1)^{r(a_1)-r(a_0)}(-1)^{r(a_2)-r(a_1)}\cdots(-1)^{r(a_n)-r(a_{n-1})} \;=\; (-1)^{r(a_n)-r(a_0)} \;=\; (-1)^d\,. \qquad \square$$

A map $\phi : P \to [n]$ is **strictly order preserving** if $\phi(a) < \phi(b)$ for all $a \preceq_P b$ and $a \neq b$.

Theorem 3.17. *Let $(P,\preceq)$ be a finite poset with d elements. Then $(-1)^d\Omega_P(-n)$ is the number of strictly order preserving maps $\phi : P \to [n]$.*

If $P=[d]$ is a chain on d elements, then

$$\Omega_P(n) \;=\; \binom{n+d-1}{d}$$

is the number of d-multisubsets of $[n]$. Now (3.4) yields that

$$(-1)^d\Omega_P(n) \;=\; (-1)^d\binom{n+d-1}{d} \;=\; \binom{n}{d}$$

and every d-subset of $[n]$ naturally corresponds to a strictly order preserving map $[d]\hookrightarrow[n]$.

Proof. We recall that

$$\Omega_P(n) \;=\; \zeta^n_{\mathcal{J}(P)}(\varnothing,P) \;=\; Z_{\mathcal{J}(P)}(n)$$

and hence by Corollary 3.16 that $(-1)^d\Omega_P(-n)$ is the number of multichains of ideals $\varnothing = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = P$ such that $I_i \setminus I_{i-1}$ is an anti-chain. From Proposition 3.6 we get that these are precisely the order-preserving maps $\phi : P \to [n]$ such that $\phi^{-1}(j) \subseteq P$ is an anti-chain for all j. That is, if $\phi(a)=\phi(b)$ if and only if $a=b$ or a and b are incomparable. $\square$

3.3. **Interlude: Valuations on distributive lattices.** For a general poset P, for which we don't have meets, we cannot define Möbius algebras from the semigroup perspective but we can mimic quite a bit. Let us define $\mathbb{C}[P]$ as the $\mathbb{C}$-vector space spanned by symbols t^b for $b\in P$. As before, define the map $f:\mathbb{C}[P]\to\mathbb{C}^P$ by

$$f(b) \;:=\; f_b \;=\; \sum_{a\preceq b} e_a$$

The map f is still an isomorphism of vector spaces. To make it an algebra map, we need to define a suitable multiplication on $\mathbb{C}[P]$ so that it fits. Observe that

$$f_bf_c \;=\; \sum_{a\preceq b, a\preceq c} e_a$$

but in the absence of meets, this is all we can do. Using Möbius inversion, we can write this expression in terms of the f-basis:

$$f_bf_c \;=\; \sum_{x\in P}\Big(\sum_{a\preceq b,a\preceq c}\mu_P(x,a)\Big)f_x$$

Thus, if we define a multiplication on $\mathbb{C}[P]$ by

$$t^bt^c \;=\; \sum_{x\in P}\Big(\sum_{a\preceq b,a\preceq c}\mu_P(x,a)\Big)t^x$$

then $f:\mathbb{C}[P]\to\mathbb{C}^P$ is an isomorphism of algebras. In particular, the elements

$$m_c \;:=\; \sum_{b\preceq c}\mu_P(b,c)t^b$$

for $c\in P$ still form a complete set of orthogonal idempotents. We can use this for studying valuations on (distributive) lattices.

Recall that a valuation on a lattice is a map $V : L \to \mathbb{C}$ such that

$$V(a \vee b) \;=\; V(a) + V(b) - V(a \wedge b) \tag{3.7}$$

for all $a, b \in L$. Valuations originate from a geometric context: Let $\mathcal{C}$ be a ring of sets, for example of $\mathbb{R}^d$. A valuation now is a function $V : \mathcal{C} \to \mathbb{C}$ such that $V(A \cup B) = V(A) + V(B) - V(A \cap B)$. For example the volume is a valuation and so are all measures and integrals. But valuations differ from general measures as we only require *finite additivity*. In this subsection, we want to make the combinatorics of valuations on lattices more transparent.

For a fixed lattice L, let $\mathsf{Vals}(L) \subseteq \mathbb{C}^L$ be vector subspace of valuations, that is, all functions $V : L \to \mathbb{C}$ satisfying (3.7). We can view this subspace from a different perspective. Let $\mathbb{C}[L]$ be the Möbius algebra of L and let $I_{\mathsf{Vals}} \subseteq \mathbb{C}[L]$ be the vector subspace spanned by all

$$t^{a \vee b} - t^a - t^b + t^{a \wedge b}$$

for $a, b \in L$. Then a function $V : L \to \mathbb{C}$ is a valuation if and only if $V(r) = 0$ for all $r \in I_{\mathsf{Vals}}$.

Proposition 3.18. *Let $V : L \to \mathbb{C}$ be a valuation, then $\overline{V} : \mathbb{C}[L]/I_{\mathsf{Vals}} \to \mathbb{C}$ defined on generators t^a by $\overline{V}(t^a) := V(a)$ is a linear function. Conversely, if $\overline{V} : \mathbb{C}[L]/I_{\mathsf{Vals}} \to \mathbb{C}$ is a linear function, then $V(a) := \overline{V}(t^a)$ is a valuation. This shows that $\mathsf{Vals}(L)$ is isomorphic to the vector space dual to $\mathbb{C}[L]/I_{\mathsf{Vals}}$.*

It turns out that many lattices do not have many valuations. For example,

Exercise 3.3. Let L be a modular lattice and $V : L \to \mathbb{C}$ a valuation. Show that there are $\alpha, \beta \in \mathbb{C}$ such that $V(a) = \alpha r(a) + \beta$ for all $a \in L$.

However, if L is distributive, then there is more structure.

Let $(R, +, \cdot)$ be a commutative ring. A subset $\varnothing \neq I \subseteq R$ is an **ideal** if $(I, +)$ is a subgroup and $rf \in I$ for all $f \in I$ and $r \in R$. That is, for all $f, g \in I$ and $r \in R$, we have $f - g \in I$ and $rf \in I$. If $T : R \to S$ is a ring map, then $\ker T = \{r \in R : T(r) = 0\}$ is an ideal. If $I \subseteq R$ is an ideal, then the collection of cosets $R/I := \{r + I : r \in R\}$ inherits a ring structure

$$(r + I)(r' + I) \;:=\; rr' + I\,.$$

In particular, if $T : R \to S$ is surjective, then $R/\ker(T) \cong S$.

Proposition 3.19. *If L is distributive, then $I_{\mathsf{Vals}} \subseteq \mathbb{C}[L]$ is an ideal.*

Proof. We only need to verify that $t^c(t^{a \vee b} - t^a - t^b + t^{a \wedge b}) \in I_{\mathsf{Vals}}$.

$$t^c(t^{a \vee b} - t^a - t^b + t^{a \wedge b}) \;=\; t^{c \wedge (a \vee b)} - t^{c \wedge a} - t^{c \wedge b} + t^{c \wedge (a \wedge b)} \;=\; t^{a' \vee b'} - t^{a'} - t^{b'} + t^{a' \wedge b'} \;\in\; I_{\mathsf{Vals}}\,,$$

where the last equality follows from distributivity (plus associativity and idempotency) and $a' := c \wedge a$ and $b' := c \wedge b$. $\square$

Note that in $\mathbb{C}[L]/I_{\mathsf{Vals}}$, we have

$$t^{a \vee b} \;=\; t^a + t^b - t^{a \wedge b} \;=\; 1 - (1 - t^a)(1 - t^b)\,.$$

If we iterate this, we obtain a version of inclusion-exclusion:

$$t^{a_1 \vee \cdots \vee a_k} \;=\; 1 - (1 - t^{a_1}) \cdots (1 - t^{a_k}) \;=\; \sum_{\varnothing \neq I \subseteq [k]} (-1)^{|I|} \prod_{i \in I} t^{a_i}\,. \tag{3.8}$$

Theorem 3.20. *Let L be a finite distributive lattice and $P \subseteq L$ its poset of join irreducibles, then*

$$\mathbb{C}[L]/I_{\mathsf{Vals}} \;\cong\; \mathbb{C}[P \cup \hat{0}] \;\cong\; \mathbb{C}[P] \oplus \mathbb{C}$$

Proof. As usual, we identify $L = \mathcal{J}(P)$ and hence every element of L is an order ideal $J \subseteq P$. Define the map $V : \mathcal{J}(P) \to \mathbb{C}[P]$ by $V(J) := \sum_{a \in P} t^a$. It is then clear that V is a valuation:

$$V(J \cup J') \;=\; \sum_{a \in J \cup J'} t^a \;=\; \sum_{a \in J} t^a + \sum_{a \in J'} t^a - \sum_{a \in J \cap J'} t^a\,.$$

It follows from Proposition 3.18 that V extends to a surjective linear map $\overline{V} : \mathbb{C}[L]/I_{\mathsf{Vals}} \to \mathbb{C}[P]$. Using, for example (3.8), one sees that $\overline{V}$ is a map of algebras. Since $\mathbb{C}[L]/I_{\mathsf{Vals}}$ is generated by join-irreducibles and hence at most $|P|$-dimensional, this shows that the map is an isomorphism. □

↓ 12. Vorlesung, 19.5.2022 ↓

4. Semimodular lattices and matroids

In this section, we will consider a very important class of posets that occur in many many different guises (many of them not as posets) in combinatorics.

To motivate, let us come back to the partition lattice from our first week: For $n \geq 1$, the partition lattice Par_n is the collection of all unordered partition of $[n]$ into non-empty sets, partially ordered by refinement: If $P = \{P_1, \dots, P_k\}$ and $P' = \{P'_1, \dots, P'_l\}$ are partitions, then $P \preceq P'$ if for every P_i there is P'_j such that $P_i \subseteq P'_j$. This is a poset with minimal element $\hat{0} = \{\{i\} : i = 1, \dots, n\}$ and maximal element $\hat{1} = \{[n]\}$. It's not hard to see that Par_n is graded with rank function $r(P) = n - k$, where k is the number of parts of the partition.

What is also not hard to see is that Par_n is a lattice. For partitions P, P' define P'' as the collection of all non-empty and distinct $P_i \cap P'_j$. This is a partition of $[n]$ that refines both P and P' and, in fact, the coarsest common refinement. This means that Par_n is a meet-semilattice with maximum $\hat{1}$ and hence Par_n is a lattice (Lemma 2.2). The actual join of two partitions is more cumbersome to describe. It's easier to remember that we defined partitions in terms of subsets of edges of the complete graph. Joins there are easy.

We wish to compute the Möbius function. For that, let us first look at intervals. Let $P \preceq P'$ with $P' = \{P'_1, \dots, P'_l\}$. For $1 \leq j \leq l$, let $L_j := \{i : P_i \subseteq P'_j\}$. Every partition $P'' \in [P, P'']$ arises from merging some P_i with i contained in the same L_j. In fact, every $P'' \in [P, P']$ corresponds to a unique sequence $(S^1, \dots, S^l)$, where S^i is a partition of L_j. The following figure illustrates this fact.

We record:

Proposition 4.1. *Let $P \preceq P'$ be partitions of $[n]$ and let L_j be as above. Define $n_j = |L_j|$. Then*

$$[P, P'] \;\cong\; \mathrm{Par}_{n_1} \times \cdots \times \mathrm{Par}_{n_l}\,.$$

It follows from Proposition 3.1 that is suffices to understand understand $\mu_{\mathrm{Par}_n}(\hat{0}, \hat{1})$.

Theorem 4.2. *Let $n \geq 1$. Then*

$$\mu_{\mathrm{Par}_n}(\hat{0}, \hat{1}) \;=\; (-1)^{n-1}(n-1)!$$

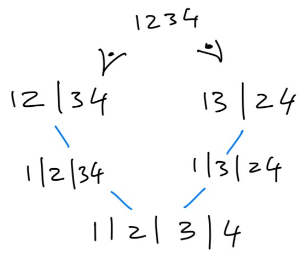
Proof. We prove the result by induction on n. For $n = 1$, we have that Par_1 is a single element and the claim holds. For $n > 1$, we compute $\mu_{\mathrm{Par}_n}(\hat{0}, \hat{1})$ using Weisner's theorem (Proposition 3.12). We pick the coatom $P' = \{[n-1], \{n\}\}$. Let $P = \{P_1, \dots, P_k\} \neq \hat{0}$ be another partition. In order to have $P \wedge P' = \hat{0} = \{\{i\} : i \in [n]\}$, we need for all $s, t \in [n-1]$ with $s \neq t$ that s, t are not contained in the same part of P. But then $P = \{\{1\}, \dots, \{i, n\}, \dots, \{n\}\}$. For every such P, we have that $[P, \hat{1}] \cong \mathrm{Par}_{n-1}$. From Proposition 3.12, we obtain

$$\mu_{\mathrm{Par}_n}(\hat{0}, \hat{1}) \;=\; -(n-1)\mu_{\mathrm{Par}_{n-1}}(\hat{0}, \hat{1})$$

which proves the claim. □

The proof gives and idea how to compute the Möbius function of $\mathrm{Par}(G)$, the collection of partitions of V induced by a graph G. However, the interesting to notice is that $(-1)^{n-1}(n-1)!$ has many interpretations and we will explore what this is for general graphs and, in fact, an important class of lattices.

Notice that Par_n is a type of lattice that we have not encountered yet. For starters, it's not modular:



However, the other implication works:

Proposition 4.3. *Let $P, P' \in \mathrm{Par}_n$ be partitions such that P, P' cover $P \wedge P'$, then P and P' is covered by $P \vee P'$*

Proof. In the end, this reduces to $P = \{A' \cup A'', B', B''\}$ and $P' = \{A', A'', B' \cup B''\}$ so that $P \wedge P' = \{A', A'', B', B''\}$. But then $P \vee P' = \{A' \cup A'', B' \cup B''\}$. $\square$

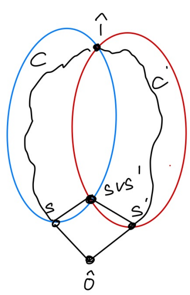
We call a finite lattice L **upper semimodular** if for all $s, t \in L$ such that s and t cover $s \wedge t$, we have that $s \vee t$ covers both s and t.

Proposition 4.4. *Let L be a finite lattice. Then L is upper semimodular if and only if L is graded and for all $s, t \in L$*

$$r(s \vee t) \ \leq \ r(s) + r(t) - r(s \wedge t)\,.$$

Proof. Assume that L is graded and the rank function satisfies the stated submodular inequality. Assume that s, t both cover $s \wedge t$, then $r(t) = r(s \wedge t) + 1$ and $r(s) < r(s \vee t) \leq r(s) + r(t) - r(s \wedge t) = r(s) + 1$. Hence s is covered by $s \vee t$. The same argument shows that t is covered by $s \vee t$.

If not every semimodular lattice is graded, then there is one with $r(L)$ minimal. In particular $[t, \hat{1}]$ is a graded semimodular lattice for all $\hat{0} \prec t$. Assume that C, C' are maximal chains of different length and let s, s' be the atoms contained in C and C', respectively. Then s and s' both cover $\hat{0}$ and hence $t := s \vee s'$ covers both s and s'.

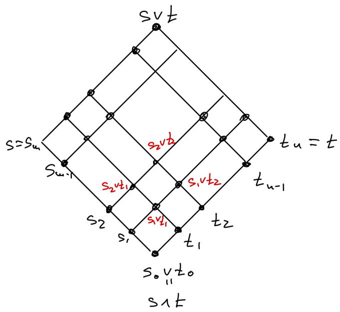


Now every chain in $[s, \hat{1}]$ is of the same length $|C| - 1$ and $|C| - 1 = 1 + \ell(t, \hat{1})$. But the same argument also shows that $|C'| = 1 + \ell(t, \hat{1})$. A contradiction. Thus every semimodular lattice is graded.

Let $s, t \in L$ and pick saturated chains $s \wedge t = s_0 \prec\!\!\cdot\, s_1 \prec\!\!\cdot\, \cdots \prec\!\!\cdot\, s_m = s$ and $s \wedge t = t_0 \prec\!\!\cdot\, t_1 \prec\!\!\cdot\, \cdots \prec\!\!\cdot\, t_n = t$. Consider the elements $s_i \vee t_j$. By induction on $i + j - 1$, submodularity implies that

$$s_i \vee t_j \ = \ (s_{i-1} \vee t_j) \vee (s_i \vee t_{j-1})$$

is either equal to $s_{i-1} \vee t_j$ or covers it and hence $r(s_i \vee t_j) - r(s_{i-1} \vee t_j) \leq 1$.



For fixed j this implies

$$r(s \vee t_j) - r(t_j) \;=\; r(s \vee t_j) - r((s \wedge t) \vee t_j) \;=\; \sum_{i=1}^{m} r(s_i \vee t_j) - r(s_{i-1} \vee t_j) \;\leq\; m$$

and for $j = n$, this becomes

$$r(s \vee t) - r(t) \;\leq\; m \;=\; r(s) - r(s \wedge t)\,. \qquad \square$$

Note that modular lattices are semimodular and the result above yields a proof of Corollary 2.10. Conversely, call a lattice L **lower semimodular** if the dual lattice L' is upper semimodular. Then L is modular if and only if L is lower and upper semimodular.

Corollary 4.5. *A graded lattice L is modular if and only if $r(s \vee t) = r(s) + r(t) - r(s \wedge t)$ for all $s, t \in L$.*

We also notice that Par_n is atomic: Every unordered partition of $[n]$ is a join of partitions of the form $\{\{1\}, \ldots, \{i, n\}, \ldots, \{n\}\}$. This is best viewed from the graph perspective: The atoms correspond to edges of K_n.

↓ ——— 13. Vorlesung, 24.5.2022 ——— ↓

Definition 4.6 (Geometric lattice)**.** A **geometric lattice** is a finite, atomic, and (upper) semimodular lattice.

A lattice L is **complemented** if for every $a \in L$ there is a $b \in L$ such that $a \wedge b = \hat{0}$ and $a \vee b = \hat{1}$. If b is unique for every a, then L is **uniquely complemented**. If every interval of L is complemented, then L is called **relatively complemented**.

Proposition 4.7. *Let L be a finite semimodular lattice. Then L is relatively complemented if and only if L is **atomic** (every element is a join of atoms).*

Proof. Homework! $\square$

4.1. **Matroids.** Geometric lattices are a *cryptomorphism*[5] for matroids. That is, matroids viewed from the perspective of lattice theory. A **matroid** is a pair $M = (E, \mathcal{I})$ where E is a finite set and $\mathcal{I} \subseteq 2^E$ is a collection of subsets such that

(I1) $\varnothing \in \mathcal{I}$, (non-empty)
(I2) if $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$, (hereditary)
(I3) if $I, J \in \mathcal{I}$ and $|I| < |J|$, then there is $e \in J \setminus I$ with $I \cup e \in \mathcal{I}$. (independence augmentation)

We call $\mathcal{I}$ the **independent sets** of M.

Example 4.1 (Linear/Vector matroids)**.** Let $A \in \mathbb{C}^{d \times n}$ be a matrix with columns $a_1, \ldots, a_n$. Define

$$\mathcal{I} \;:=\; \{\{i_1, \ldots, i_k\} \subseteq [n] : a_{i_1}, \ldots, a_{i_k} \text{ linearly independent}\}$$

Then $M(A) = ([n], \mathcal{I})$ is a matroid, called a **linear matroid** or **vector matroid**. Clearly $\varnothing \in \mathcal{I}$ and $\mathcal{I}$ is hereditary. For $I, J \in \mathcal{I}$ with $|I| < |J|$ and define $U = \mathrm{span}\{a_i : i \in I\}$ and $V = \{a_i : i \in J\}$. Then $\dim U = |I| < |J| = \dim V$ and, since $\{a_i : i \in J\}$ is a basis for V, there is $e \in J$ with $a_e \notin U$. Thus $\{a_i : i \in I \cup e\}$ is linearly independent and $I \cup e \in \mathcal{I}$. Thus $\mathcal{I}$ satisfies (I1)–(I3).

[5]A cool word used a lot by Rota and, supposedly, coined originally by Birkhoff.

Matroids are a combinatorial abstraction of linear (in)dependence. This example also reinforces the following terminology. An inclusion-maximal set $B \in \mathcal{I}$ is called a **basis** of M. The **rank** of a set $X \subseteq E$ is

$$r_M(X) \ := \ \max\{|I| : I \in \mathcal{I}, I \subseteq X\}\,. \tag{4.1}$$

The rank of the matroid is $r(M) = r(E)$. The rank of M is thus the cardinality of some basis. The augmentation property directly implies that every basis has the same number of elements.

Proposition 4.8. *Let $B, B' \in \mathcal{I}$ be two bases, then $|B| = |B'|$.*

We write $\mathcal{B} \subseteq \mathcal{I}$ for the collection of bases of M. A first glimpse at the many facets of matroids is a characterization in terms of bases.

Proposition 4.9. *Let $\mathcal{B} \subseteq 2^E$ a non-empty collection of subsets. Then $\mathcal{B}$ is the set of basis of a matroid M if and only if*

(B1) *for $B, B' \in \mathcal{B}$ and $e \in B \setminus B'$ there is $f \in B' \setminus B$ such that $(B \setminus e) \cup f \in \mathcal{B}$.*

Proof. Try yourself (or look it up). □

Let $G = (V, E)$ be a graph and define $\mathcal{I} := \{I \subseteq E : (V, I) \text{ cycle-free}\}$. Then $\mathcal{I}$ trivially satisfies (I1) and (I2). The last condition is less obvious. We get around it by using bases. The collection $\mathcal{B} \subseteq \mathcal{I}$ of inclusion-maximal sets are precisely the edge-sets of *spanning forests*, that is, every connected component of (V, B) is a tree.

Proposition 4.10. *Let $B, B' \subseteq E$ be a spanning forests and $e \in B \setminus B'$. Then there is $f \in B' \setminus B$ with $(B \setminus e) \cup f$ is a spanning forest.*

Proof. Let us assume that G is connected and B, B' are trees. Now $(V, B \setminus e)$ has two connected components with nodes V_1 and V_2. Since (V, B') is a spanning tree, there is an edge $f = v_1v_2 \in B'$ with $v_1 \in V_1$ and $v_2 \in V_2$. It follows that $(V, (B \setminus e) \cup f)$ is connected and hence a spanning tree. □

A set $C \subseteq E$ is a **circuit** of M if C is minimally dependent, that is, dependent but $C \setminus e$ is independent for all $e \in C$. Let $\mathcal{C}$ be the collection of circuits of M.

Proposition 4.11. *$\mathcal{C}$ is the collection of circuits of a matroid if and only if*

(C1) *$\varnothing \not\in \mathcal{C}$,*
(C2) *if $C_1 \subseteq C_2$ for $C_1, C_2 \in \mathcal{C}$, then $C_1 = C_2$;*
(C3) *if $C_1, C_2 \in \mathcal{C}$ are distinct and $e \in C_1 \cap C_2$, then there is a circuit C_3 with $C_3 \subseteq (C_1 \cup C_2) \setminus e$.*

Condition (C3) is quite easily seen for graphical matroids and linear matroids. The independent sets can be easily recovered from $\mathcal{C}$:

$$\mathcal{I} \ = \ \{I \subseteq E : C \not\subseteq I \text{ for all } C \in \mathcal{C}\}\,.$$

Again it's doable but fiddly to prove that (C1)–(C3) characterizes the circuits of a matroid.

Note that for our graphical matroids, we do not need that the graph G is simple. A loop of G is an edge of the form uu. Two edges are parallel if they connect the same nodes. This terminology extends to matroids. An element $e \in E$ is a **loop** if $\{e\}$ is a circuit. Two distinct elements $e, f \in E$ are called **parallel** $\{e, f\}$ is a circuit. Being parallel is an equivalence relation on E. A matroid M is called **simple** if it has no loops and parallel elements. Note that we can always *simplify* a matroid by removing loops and picking an element from each equivalence class of parallel elements. An element $e \in E$ is a **coloop** if e is contained in every basis of M. For graphic matroids, such edges are also called bridges – removing a bridge increases the number of connected components.

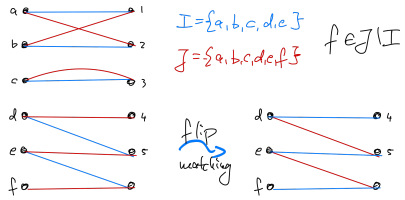
We want to introduce one more important class of matroids. Let S be a finite set and $A_1, \dots, A_m \subseteq S$. A **transversal** or **system of distinct representatives** is a choice of distinct elements $s_1, \dots, s_m \in S$ such that $s_i \in A_i$. Alternatively, $T \subseteq S$ is a transversal if there is a bijection $f : [m] \to T$ such that $f(i) \in A_i$ for $i = 1, \dots, m$. A subset $P \subseteq S$ is a **partial** transversal if there is $J \subseteq [m]$ such that P is a transversal for $(A_i)_{i \in J}$.

Proposition 4.12. *Let $A_1, \dots, A_m \subseteq S$. The collection of partial transversals are the independent sets of a matroid.*

Such matroids are called **transversal matroids**.

For the proof, another perspective is useful: Consider the bipartite graph G on nodes $[m] \uplus S$ and edges $(i, s) \in E$ if $s \in A_i$. A **matching** in G is a collection of pairwise disjoint edges. A transversal is then a matching (i, s_i) that covers all nodes in $[m]$. A partial transveral is a set T that can be matched.

Proof. Let T, T' be two partial transversals represented as matchings, that is $T, T' \subseteq E$ and assume that $|T| < |T'|$. Think about the elements in T and T' as red and blue edges in G. Note that $(T \setminus T') \cup (T' \setminus T)$ is a disjoint union of cycles and paths in which the edges alternate in color. Since $|T| < |T'|$ there are more blue edges than red ones and hence there is a path of odd length with one more blue edge than red edge. There is one more element $s \in S$ covered by the red edges than by the blue edges. We claim that $T \cup e$ is also a partial transversal. To see this, simply flip the colors of this path:



This proves the claim. ☐

↓ 14. Vorlesung, 31.5.2022 ↓

Matroids can also be characterized via their rank function and this is the path that leads us back to posets. The rank function of M is a function $r : 2^E \to \mathbb{Z}$ such that

(R1) $0 \leq r(X) \leq |X|$; (nonnegative+subcardinal)
(R2) $r(X) \leq r(Y)$ whenever $X \subseteq Y$; (monotone)
(R3) $r(X \cup Y) \leq r(X) + r(Y) - r(X \cap Y)$. (submodular)

The first two properties follow directly from (4.1). Only the last property needs a justification: Let $I \subseteq X \cap Y$ be an inclusion-maximal independent set so that $r(X \cap Y) = |I|$. Using the augmentation property, we can extend I to a maximal independent $J \subseteq X \cup Y$. We compute

$$r(X \cup Y) \;=\; |J| \;=\; |J \cap X| + |J \cap Y| - |J \cap (X \cap Y)| \;\leq\; r(X) + r(Y) - r(X \cap Y) \,.$$

(R3) is reminiscent of the dimension formula from linear algebra: If U, V are vector subspaces of some fixed vector space, then

$$\dim(U + V) \;=\; \dim(U) + \dim(V) - \dim(U \cap V)$$

For vector matroids, we deal with vectors and not subspaces:

$$r(X) \;=\; \dim \operatorname{span}\{a_e : e \in X\}$$

Hence, even if X and Y span the same subspace, they might be disjoint and hence $r(X \cup Y) = r(X) = r(Y) < r(X) + r(Y) - r(X \cap Y)$.

We can recover M from its rank function as the independent sets are precisely those $I \subseteq E$ for which $r(I) = |I|$.

Theorem 4.13. *A function $r : 2^E \to \mathbb{Z}$ is the rank function of a matroid if and only if (R1)–(R3) are satisfied.*

For the proof, we need the following lemma.

Lemma 4.14. *Let r be a function that satisfies (R2) and (R3). Let $X, Y \subseteq E$ be disjoint such that $r(X \cup e) = r(X)$ for all $e \in Y$. Then $r(X \cup Y) = r(X)$.*

Proof. Induction on $|Y| = k$. For $k \le 1$, this is clear. For $k > 1$ fix $f \in Y$ and set $A := X \cup (Y \setminus f)$ and $B := X \cup f$. By induction and assumption $r(A) = r(X) = r(B)$. Now

$$r(X) \;\le\; r(X \cup Y) \;=\; r(A \cup B) \;\le\; r(A) + r(B) - r(A \cap B) \;=\; r(X) + r(X) - r(X)\,,$$

which proves the claim. □

Proof of Theorem 4.13. Let r be a function satisfying (R1)–(R3) and define $\mathcal{I} := \{I \subseteq E : r(I) = |I|\}$. We need to show that $M = (E, \mathcal{I})$ is a matroid with $r = r_M$.

Since $0 \le r(\varnothing) \le |\varnothing|$, we have $\varnothing \in \mathcal{I}$. Let $I \in \mathcal{I}$. For $e \in I$ define $I' := I \setminus e$. Now

$$|I| \;=\; r(I) \;=\; r(I' \cup e) \;\le\; r(I') + r(e) - r(\varnothing) \;\le\; r(I') + 1\,.$$

This implies $|I'| = |I| - 1 \le r(I') \le |I'|$ and thus $I' \in \mathcal{I}$.

Let $I, J \in \mathcal{I}$ with $|I| < |J|$ and assume that $I \cup e \notin \mathcal{I}$ for all $e \in J \setminus I$. That is, $r(I \cup e) = r(I)$ for all $e \in J \setminus I$. It follows from the previous lemma that $r(I \cup J) = r(I)$. But $|J| = r(J) \le r(I \cup J) = r(I) = |I|$.

Hence M is a matroid. Let $X \subseteq E$ and $I \subseteq X$ inclusion-maximal with $I \in \mathcal{I}$. Then $r_M(X) = |I| = r(I) \le r(X)$. Since I is inclusion-maximal, this means that $r(I \cup e) = r(I)$ for all $e \in X \setminus I$. By the previous lemma, this implies $r(I) = r(X)$ and hence $r_M(X) = r(X)$ for all X. □

We call $F \subseteq E$ a **flat** if $r_M(F) < r_M(F \cup e)$ for all $e \in E \setminus F$. This also defines as **closure operator** $\mathrm{cl}_M : 2^E \to 2^E$ by $\mathrm{cl}_M(X) := \{e \in E : r_M(X \cup e) = r_M(X)\}$. For a linear matroid M the closure of $S \subseteq E$ is

$$\mathrm{cl}(S) \;=\; \{e \in E : a_e \in \mathrm{span}(a_f : f \in S)\}\,.$$

In general, a closure operator is a map $\mathrm{cl} : 2^E \to 2^E$ such that for all $A \subseteq B \subseteq E$

(C1) $A \subseteq \mathrm{cl}(A)$,
(C2) $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$,
(C3) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

We state (but don't proof) that closure operators give yet another characterization of matroids.

Theorem 4.15. *A closure operator* cl *satisfies* $\mathrm{cl} = \mathrm{cl}_M$ *for some matroid if and only if for all* $A \subseteq E$ *and distinct* $x, y \notin \mathrm{cl}(A)$

(C3) $x \in \mathrm{cl}(A \cup y)$ *if and only if* $y \in \mathrm{cl}(A \cup x)$.

The **lattice of flats** $\mathcal{L}(M) \subseteq 2^E$ is the collection of flats of M partially ordered by inclusion. This is a partially ordered set with maximum $\hat{1} = E$ and minimum $\hat{0} = \mathrm{cl}(\varnothing) = \{e \in E : e \text{ loop}\}$.

Example 4.2 (Graphical matroids)**.** **??** Let $G = (V, E)$ be a graph and $M(G)$ the graphical matroid with rank function r. The rank of $X \subseteq E$ is the maximal number of edges of a cycle-free subset. Let $G' = G[X]$ be the edge-induced subgraph. If G' has node set V' and c' connected components, then $r(X) = |V'| - c'$. If $e = uv \in E$ is an edge, then $e \in \mathrm{cl}(X)$ if and only if there is a path from u to v in G'.

The previous example also gives intuition to the following useful characterization of closures.

Proposition 4.16. *Let* $X \subseteq E$ *and* $e \in E \setminus X$. *Then* $e \in \mathrm{cl}(X)$ *if and only if there is a circuit* $C \subseteq X \cup e$ *with* $e \in C$.

Proof. If $r(X \cup e) = r(X)$, then for every independent subset $I \subseteq X$ with $|I| = r(X)$, we have that $I \cup e \notin \mathcal{I}$ and hence contains a circuit $C \subseteq I \cup e \subseteq X \cup e$ with $e \in C$. Conversely, suppose that there is such a circuit. Then, by definition of circuit, $r(C) = |C| - 1$ and since $C \setminus e \subseteq X$, we get from (C2) $e \in \mathrm{cl}(C \setminus e) \subseteq \mathrm{cl}(X)$. □

Proposition 4.17. $\mathcal{L}(M)$ *is a geometric lattice.*

Proof. Let $F, F' \in \mathcal{L}(M)$ be flats. We claim that $F \cap F'$ is a flat, which is then automatically the meet of F and F'. If $F \cap F'$ is not a flat, then there is $e \in \mathrm{cl}(F \cap F') \setminus (F \cap F')$. By Proposition 4.16, this means that there is a circuit $e \in C \subseteq (F \cap F') \cup e$. But then $C \subseteq F \cup e$ and hence $e \in \mathrm{cl}(F) = F$. The same holds for F', which then means $e \in F \cap F'$. A contradiction.

As for the join, we claim $F \vee F' = \mathrm{cl}(F \cup F')$. Indeed, $\mathrm{cl}(F \cup F')$ is closed and if F'' is any closed set containing F and F', then $\mathrm{cl}(F \cup F') \subseteq \mathrm{cl}(F'') = F''$.

Let $F \in \mathcal{L}(M)$ be a flat and let $I = \{b_1, \dots, b_k\} \subseteq F$ be an inclusion-maximal independent set. In particular, every b_i is not a loop and $B_i := \mathrm{cl}(\{b_i\})$ is an atom of $\mathcal{L}(M)$. Since

$$B_1 \vee B_2 \vee \cdots \vee B_k \;=\; \mathrm{cl}(\{b_1, \dots, b_k\}) \;=\; F$$

this shows that $\mathcal{L}(M)$ is atomic.

To show that $\mathcal{L}(M)$ is semimodular, it suffices to show that r_M is the rank function of $\mathcal{L}(M)$. But this is easy: for $\hat{0} = \mathrm{cl}(\varnothing)$, we have $r(\hat{0}) = r_M(\varnothing) = 0$. If $F \subseteq F'$ are flats such that F is covered by F', then necessarily $\mathrm{cl}(F \cup e) = F'$ for all $e \in F' \setminus F$ and

$$r(F) \;<\; r(F') \;=\; r(F \cup \{e\}) \;\leq\; r(F) + 1 \,. \qquad \square$$

The proof already hints at the fact that every geometric lattice gives rise to a matroid: Let E be the set of atoms of a geometric lattice $\mathcal{L}$ and define for $A \subseteq E$

$$f(A) \;:=\; r_{\mathcal{L}}\Big(\bigvee_{a \in A} a\Big)$$

This definition implies that $0 \leq f(A) \leq f(A \cup B)$ for all $A \subseteq B \subseteq E$ and hence f satisfies (R1). Submodularity implies $f(A) \leq |A|$ as well as (R3). Hence f is the rank function of a matroid M. One further checks that $\mathcal{L}$ and $\mathcal{L}(M)$ are isomorphic. This establishes the following.

Theorem 4.18. *There is a one-to-one correspondence between simple matroids and geometric lattices.*

↓ ——— 15. Vorlesung, 2.6.2022 ——— ↓

This let's us see the partition lattices associated to a graph G in a different light.

Proposition 4.19. *Let $G = (V, E)$ be a graph. The partition lattice* $\mathrm{Par}(G)$ *associated to G is isomorphic to the lattice of flats of the graphic matroid $M(G)$.*

Proof. For a set $X \subseteq E$, we defined $P(X) = \{V_1, \dots, V_k\}$ where $V_1, \dots, V_k$ are the node sets of the connected components of $G[X]$. By our characterization of flats in Example **??**, it follows that $P(X) = P(\mathrm{cl}(X))$. Thus, P is a surjective map from $\mathcal{L} = \mathcal{L}(M(G))$ to $\mathrm{Par}(G)$. To see that it is also injective, we note that the flat that maps to $\{V_1, \dots, V_k\} \in \mathrm{Par}(G)$ is given by $F = \bigcup_{i=1}^k (\binom{V_i}{2} \cap E)$. Both maps are order preserving and hence give the isomorphism. $\square$

4.2. **Characteristic polynomials.** There are two basic operations on matroids that turn out to be quite important. Let $M = (E, \mathcal{I})$ be a matroid. The first one is a little underwhelming at first sight. For $S \subseteq E$, we define the **restriction** of M to S as $M|_S = (S, \mathcal{I}')$ with

$$\mathcal{I}' \;:=\; \{I \in \mathcal{I} : I \subseteq S\} \,.$$

It is trivial to check that $M|_S$ is a matroid. For $T \subseteq E$, we define the **deletion** of T from M as $M \setminus T = M|_{E \setminus T}$. In particular, if $T = \{e\}$, then $M \setminus e$ is the matroid on $E \setminus e$ with independent sets $I \in \mathcal{I}$ with $e \notin I$. If e is not a coloop, then $M \setminus e$ is a matroid of rank $r(M)$.

If M is the graphical matroid for the graph $G = (V, E)$, then $M \setminus e$ is the graphical matroid for the graph $G \setminus e$. If M is the linear matroid for $A = (a_f)_{f \in E}$, then $M \setminus e$ is the linear matroid for $A \setminus e = (a_f)_{f \in E \setminus e}$.

For $e \in E$ not a loop, we define the **contraction** of M at e as $M/e = (E \setminus e, \mathcal{I}'')$ with

$$\mathcal{I}'' \;:=\; \{I \setminus e : e \in I \in \mathcal{I}\} \,.$$

If e is a loop, then M/e is simply the removal of e from the ground set. For $T \subseteq E$, we can define the contraction M/T recursively as $M/T = (M/e)/T'$ for $e \in T$ and $T' := T \setminus e$. The rank function is then $r_{M/T}(X) = r_M(X \cup T) - r_M(T)$. Thus, if e is not a loop, then M/e is a matroid of rank $r(M) - 1$.

If $M(G)$ is a graphical matroid, then $M(G)/e = M(G/e)$, where G/e is the *usual* contraction of an edge. If M is the linear matroid associated to $A = (a_f)_{f\in E} \in \mathbb{R}^d$, then a subset $I \subseteq E\backslash e$ is independent if and only if the vectors a_f with $f \in I \cup e$ are linearly independent. Let π be the orthogonal projection of $\mathbb{R}^d$ onto the hyperplane $\{x \in \mathbb{R}^d : \langle a_e, x\rangle = 0\} \cong \mathbb{R}^{d-1}$. It is not difficult to check that M/e is the linear matroid for $A' = (a'_f = \pi(a_f))_{f\in E\backslash e}$.

The operations of deletion and contraction facilitate inductive arguments on matroids. For example:

Proposition 4.20. *For a matroid M let $b(M)$ be the number of bases. For $e \in E$, we have*

$$b(M) \;=\; b(M \setminus e) + b(M/e)\,.$$

Proof. Every basis of M either contains e or not. The bases that contain e are precisely the bases of M/e. The bases that do not contain e are precisely the bases of $M \setminus e$. □

Recall that for a simple graph, we wrote $\chi_G(k)$ for the number of proper k-colorings. For any edge $e \in E$, we then have

$$\chi_G(k) \;=\; \chi_{G\backslash e}(k) - \chi_{G/e}(k)\,.$$

Indeed, if $e = uv$, then $\chi_{G\backslash e}(k)$ counts all proper k-colorings of G plus those that give u and v the same color. But those are precisely the proper k-colorings of G/e. Note that we can delete any parallel edges of G/e without changing the chromatic polynomial. This way, we will never encounter loops.

Also notice that the recursion completely determines $\chi_G(k)$. In every step, we loose (at least) an edge or an edge and a node. Therefore, we end up either with a graph on d nodes and no edges, in which case $\chi_G(k) = k^d$.

We take the opportunity to extend the definition of chromatic polynomials from graphs to matroids. For a matroid $M = (E, \mathcal{I})$ we define the **characteristic polynomial** $\chi_M(t) \in \mathbb{Z}[t]$ recursively as follows: If M has loops, then $\chi_M(t) := 0$. If M has no loops, then set $\chi_M(t) = 1$ if $r(M) = 0$ and $\chi_M(t) = t - 1$ if $r(M) = 1$. For higher rank and number of elements, we let $e \in E$ not be a coloop and define

$$\chi_M(t) \;=\; \chi_{M\backslash e}(t) - \chi_{M/e}(t)\,.$$

If e, f are parallel elements, then f is a loop in M/e and hence $\chi_M(t) = \chi_{M\backslash e}(t)$. Thus, we can always assume that M is simple and, in particular, pass to the simplification of M/e in each step.

A matroid $M = (E, \mathcal{I})$ is a **direct sum** of matroids $M_i = (E_i, \mathcal{I}_i)$ for $i = 1, 2$ if $E = E_1 \uplus E_2$ and

$$\mathcal{I} \;=\; \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}\,.$$

We write $M = M_1 \oplus M_2$. We call M **connected** if M is not the direct sum of two matroids. That is, if for every two elements $e, f \in E$ there is a circuit C containing both. It is easy to see that $\chi_{M_1\oplus M_2}(t) = \chi_{M_1}(t)\chi_{M_2}(t)$. Note that a graphical matroid $M(G)$ is connected if and only if G is 2-*connected*, that is, every two edges lie on a common cycle.

Proposition 4.21. *Let $M = (E, \mathcal{I})$ be a simple matroid. Then*

$$\chi_M(t) \;=\; \sum_{X\subseteq E} (-1)^{|X|} t^{r(M)-r_M(X)}\,.$$

Proof. We can prove the claim by induction on $n = |E|$. If $n = 0$ then $r(M) = 0$ and $\chi_M(t) = 1$. For the right-hand side the only summand is $X = \varnothing$ that gives $(-1)^0 t^{0-0} = 1$.

For $n > 0$, let $e \in E$ not a coloop. Then the right-hand side splits into

$$\sum_{e\notin X} (-1)^{|X|} t^{r(M)-r_M(X)} + \sum_{e\in X} (-1)^{|X|} t^{r(M)-r_M(X)} \;=\; \sum_{X\subseteq E\backslash e} (-1)^{|X|} t^{r(M\backslash e)-r_{M\backslash e}(X)} - \sum_{X\subseteq E\backslash e} (-1)^{|X|} t^{r(M)-r(X\cup e)}\,.$$

By induction and the fact that there is a basis without e, the first sum is $\chi_{M\backslash e}(t)$. For the second summand, we note that since e is not a loop, we have $r(M/e) = r(M) - 1$ and $r_{M/e}(X) = r_M(X \cup e)$ for $X \subseteq E \setminus e$. Hence, the second summand is $\chi_{M/e}(t)$ by induction. □

The connection to chromatic polynomials is simple: If $G = (V, E)$ is a connected graph, then $\chi_G(t) = t\chi_{M(G)}(t)$. Indeed, if $r(M) = |V| - 1$ and hence the starting conditions for the recursion differ by a factor t.

In the same way as we computed the Moebius function of the intersection lattice in Theorem 3.3, we can use it to relate the characteristic polynomial of a matroid to the characteristic polynomial of the lattice of flats. Let $(P, \preceq)$ be a graded poset of rank d with $\hat{0}$. We define the **characteristic polynomial** of P as

$$\chi_P(t) \ := \ \sum_{a \in P} \mu_P(\hat{0}, a) t^{d - r_P(a)}$$

Proposition 4.22. *Let M be a simple matroid with lattice of flats $\mathcal{L} = \mathcal{L}(M)$. Then $\chi_M(t) = \chi_{\mathcal{L}}(t)$.*

Proof. Since M is simple, we have that $\hat{0} = \varnothing$ is a flat and by Exercise 4.1 below, we have for $F \in \mathcal{L}$

$$\mu_{\mathcal{L}}(\hat{0}, F) \ = \sum_{X \subseteq E, \mathrm{cl}(X) = F} (-1)^{|X|}$$

Together with Proposition 4.21 and the fact that the rank function of $\mathcal{L}$ is the rank function of M, this proves the claim. ☐

Exercise 4.1. Let $\mathrm{cl} : 2^E \to 2^E$ a closure operator, that is, a set map that satisfies (C1)–(C3) above. Let $\mathcal{L} = \{\mathrm{cl}(X) : X \subseteq E\} \subseteq 2^E$ be the subposet of closed sets. Adapt the ideas in the proof of Theorem 3.3 to show that for $A, B \in \mathcal{L}$ with $A \subset B$

$$\mu_{\mathcal{L}}(A, B) \ = \sum_{A \subset X, \mathrm{cl}(X) = B} (-1)^{|X \setminus A|}$$

4.3. **Hyperplane arrangements.** There is no general interpretation for $\chi_M(k)$ for (positive) integers k. In a very nice geometric setting, there is a neat interpretation. Let E be a finite set and $a_e \in \mathbb{C}^d$ be nonzero vectors. For each $e \in E$ define the linear hyperplane

$$H_e \ := \ \{x \in \mathbb{C}^d : \langle a_e, x \rangle = 0\}.$$

This is a linear subspace of dimension $d - 1$ and $\mathcal{H} = \{H_e : e \in E\}$ is called a **hyperplane arrangement**. Let us consider the intersection poset of the arrangement: Recall that this is the set

$$\mathcal{L}(\mathcal{H}) \ := \ \Big\{ \bigcap_{e \in I} H_e : I \subseteq E \Big\}$$

of *distinct* intersection of the hyperplanes in $\mathcal{H}$, partially ordered by reverse inclusion. The maximal element is $\hat{1} = \bigcap_{e \in E} H_e$ and minimal element $\hat{0} = \mathbb{C}^d$. We call $\mathcal{H}$ **essential** if $\hat{1} = \{0\}$.

In the case of hyperplane arrangements, $\mathcal{L}(\mathcal{H})$ is a graded lattice of rank $d - \dim \hat{0}$ with rank function $r_{\mathcal{L}}(U) = d - \dim U$ for $U \in \mathcal{L}(\mathcal{H})$. In fact, we know this lattice already.

Proposition 4.23. *Let $A = (a_e)_{e \in E} \in (\mathbb{C}^d \setminus 0)$ and let $\mathcal{H}$ be the associated hyperplane arrangement. Then $\mathcal{L}(\mathcal{H}) \cong \mathcal{L}(M(A))$.*

Proof. For $X \subseteq E$, let $H_X := \bigcap_{e \in X} H_e$. Then $X \mapsto \phi(X) := H_X$ is an order-preserving surjective map $\phi : 2^E \to \mathcal{L}(\mathcal{H})$. Moreover, $\phi(X \cup e) = \phi(X)$ if and only if $a_e \in \mathrm{span}(a_f : f \in X)$ and hence $\phi(X) = \phi(\mathrm{cl}(X))$. This means that ϕ is a bijective and order-preserving map. The inverse $\phi(U) = \{e \in E : U \subseteq H_e\}$ for $U \in \mathcal{L}(\mathcal{H})$. ☐

For $\mathcal{L}(\mathcal{H})$, the characteristic polynomial takes the form

$$\chi_{\mathcal{L}(\mathcal{H})}(t) \ = \sum_{U \in \mathcal{L}(\mathcal{H})} \mu_{\mathcal{L}(\mathcal{H})}(\hat{0}, U) t^{\dim U - \dim \hat{0}}.$$

We define the characteristic polynomial of the arrangement $\mathcal{H}$ by

$$\chi_{\mathcal{H}}(t) \ = \ t^{\dim \hat{0}} \chi_{\mathcal{L}(\mathcal{H})}(t) \ = \sum_{U \in \mathcal{L}(\mathcal{H})} \mu_{\mathcal{L}(\mathcal{H})}(\hat{0}, U) t^{\dim U}.$$

If $\mathcal{H}$ is essential, then $\chi_{\mathcal{H}}(t) = \chi_{\mathcal{L}(\mathcal{H})}(t) = \chi_{\mathcal{L}(M(A))}(t) = \chi_{M(A)}(t)$.

Now let us assume that $a_e \in \mathbb{Q}^d$ are rational vectors. In fact, since we only worry about linear (in)dependence, we might scale and assume $a_e \in \mathbb{Z}^d$. If q is a prime such that $a_e \notin q\mathbb{Z}^d$ for all $e \in E$, then H_e defines a hyperplane in $\mathbb{F}_q^d$ and we denote the arragement in $\mathbb{F}_q^d$ by $\mathcal{H}_q$. Note that $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}_q)$ might differ for certain primes. This comes from the fact that for $X \subseteq E$, the rank of the collection $(a_e : e \in X)$ might be different over different $\mathbb{F}_q$. However, one can prove that there are only finitely many primes q for which $\mathcal{L}(\mathcal{H}) \not\cong \mathcal{L}(\mathcal{H}_q)$. We say that $\mathcal{H}$ has **good reduction** over $\mathbb{F}_q$ if $\mathcal{L}(\mathcal{H}) \cong \mathcal{L}(\mathcal{H}_q)$.

Proposition 4.24 (Finite field trick)**.** *Let $\mathcal{H}$ be an arrangement with good reduction over $\mathbb{F}_q$. Then*

$$\chi_{\mathcal{H}}(q) \;=\; \left| \mathbb{F}_q^d \setminus \bigcup_{e \in E} H_e \right| .$$

Proof. We can compute the number on the right by simple inclusion-exclusion over $\mathcal{H}$. For this we observe that $U \in \mathcal{L}(\mathcal{H})$ with $\dim U = k$ satisfies $U \cong \mathbb{F}_q^k$ and $|U| = q^k$. Let $M = M(A)$ be the associated vector matroid. Hence the right-hand side is computed by

$$\sum_{X \subseteq E} (-1)^{|X|} q^{\dim H_X} \;=\; \sum_{X \subseteq E} (-1)^{|X|} q^{d - r_M(X)} \;=\;=\; q^{d-r(M)} \sum_{X \subseteq E} (-1)^{|X|} q^{r(M) - r_M(X)} \;=\; q^{d-r(M)} \chi_M(q) .$$

□

Let $G = (V, E)$ be a graph with $V = [d]$. For every edge $e = ij$ associate the hyperplane $H_e = \{x \in \mathbb{R}^d : x_i = x_j\}$. The **graphical hyperplane arrangement** is then $\mathcal{H}_G = \{H_e : e \in E\}$. For sufficiently large[6] primes q, the finite field trick then yields

$$\chi_{\mathcal{H}_G}(q) \;=\; |\{x \in \mathbb{F}_q^d : x_i \neq x_j \text{ for all } ij \in E\}| .$$

This is exactly the number of colorings of G with q colors. That is $\chi_{\mathcal{H}_G}(q) = \chi_G(q)$. If G is connected, then $\bigcap_{e \in E} H_e$ is 1-dimensional and hence $\chi_G(q) = q\chi_{M(G)}(q)$ as we knew before.

↓ 16. Vorlesung, 7.6.2022 ↓

4.4. **R-labellings and broken circuits.** Consider what deletion-contraction is on the level of geometric lattices: Let $(L, \preceq)$ be a geometric lattice and $a \in L$ an atom. Then $L' = L_{\succeq a}$ is the geometric lattice corresponding to the contraction and $L'' := (L \setminus L_{\succeq a}) \cup \hat{1}$ is the geometric lattice corresponding to the deletion. It follows that

$$\chi_L(t) \;=\; \chi_{L''}(t) - \chi_{L'}(t)$$

From this one can prove that the coefficients alternate in sign. However, we look at in from the perspective of partition lattices (Theorem 4.2).

Proposition 4.25. *Let $(L, \preceq)$ be a semimodular lattice and $a \in L$ an atom. Then*

$$\mu_L(\hat{0}, \hat{1}) \;=\; -\sum_t \mu(\hat{0}, t)$$

where the sum is over all coatoms $t \in L$ with $t \notin L_{\succeq a}$.

Proof. We want to use the dualized version of Weisner's theorem (Proposition 3.12): For $a \succ \hat{0}$

$$\mu_L(\hat{0}, \hat{1}) \;=\; - \sum_{\substack{t \neq \hat{1} \\ a \vee t = \hat{1}}} \mu(\hat{0}, t) .$$

Let $a \in L$ be an atom and $t \neq \hat{0}$ with $a \vee t = \hat{1}$. From semimodularity, we get

$$r(\hat{1}) \;=\; r(a \vee t) \;\leq\; r(t) + r(a) - r(a \wedge t) \;=\; r(t) + 1 - 0$$

using the fact that a is an atom. Thus, $r(t) \geq r(\hat{1}) - 1$ and t is a coatom of L. For a cotom t we have $a \vee t = \hat{1}$ if and only if $a \not\preceq t$, which proves the claim. □

Corollary 4.26. *Let L be a semimodular lattice and $x \preceq y$, then $(-1)^{r(y)-r(x)} \mu_L(x, y) > 0$.*

[6]It is easily proved that $\mathcal{H}_G$ has good reduction over all primes!

Proof. Since every interval of a semimodular lattice is semimodular, we only need to prove it for $x = \hat{0}$ and $y = \hat{1}$. If $r(\hat{1}) = r(\hat{0}) + 1$, then $L = \{\hat{0}, \hat{1}\}$ and $\mu(\hat{0}, \hat{1}) = -1$. We can use the previous result to get

$$(-1)^{r(\hat{1})}\mu_L(\hat{0}, \hat{1}) \;=\; \sum_t (-1)^{r(\hat{1})-1}\mu(\hat{0}, t)$$

Since t is a coatom and hence of rank $r(t) = r(\hat{1}) - 1$ and, by induction, $(-1)^{r(t)}\mu(\hat{0}, t) > 0$. $\square$

Let P be a graded poset and let $\operatorname{Cov}(P) \subset P \times P$ be the set of cover relations of P. An **R-labelling** is a map $\lambda : \operatorname{Cov}(P) \to \mathbb{Z}$ such that for every $a \prec b$ there is a unique saturated chain $a = a_0 \prec\!\!\cdot\; a_1 \prec\!\!\cdot \cdots \prec\!\!\cdot\; a_k = b$ such that $\lambda(a_0, a_1) \le \lambda(a_1, a_2) \le \cdots \le \lambda(a_{k-1}, a_k)$. We call such a saturated chain an **increasing chain**.

Corollary 4.27. *Let P be a graded poset with R-labelling and $a \preceq b$. Then $(-1)^{r(b)-r(a)}\mu_P(a, b)$ is the number of saturated chains $a = a_0 \prec\!\!\cdot\; a_1 \prec\!\!\cdot \cdots \prec\!\!\cdot\; a_k = b$ such that*

$$\lambda(a_0, a_1) \;>\; \lambda(a_1, a_2) \;>\; \cdots \;>\; \lambda(a_{k-1}, a_k)\,.$$

In preparation of the proof, assume that P is of rank n and let $S = \{s_1 < s_2 < \cdots < s_k\} \subseteq [n-1]$. Define $\alpha_P(S)$ to the be number of chains $a_1 \prec a_2 \prec \cdots \prec a_k$ such that $r_P(a_i) = s_i$. That is, $\alpha_P(S)$ counts the number of *rank-selected* chains in P. The collection $\alpha_P = (\alpha_P(S))_S$ is called the **flag f-vector** of P.

Define the **order complex**

$$\Delta(P) \;=\; \{\{x_1, \ldots, x_k\} : \hat{0} \prec x_1 \prec \cdots \prec x_k \prec \hat{1}\}$$

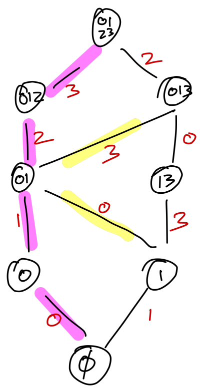
Philip Hall's theorem (Corollary 3.10) states

$$\mu_P(\hat{0}, \hat{1}) \;=\; \sum_{C \in \Delta(P)} (-1)^{|C|}$$

For a chain $C \in \Delta(P)$, let $r(C) = \{r(c) : c \in C\} \subseteq [n-1]$ be the rank set of C. By grouping chains according to $r(C)$, we infer that

$$\mu_P(\hat{0}, \hat{1}) \;=\; \sum_S (-1)^{|S|}\alpha_P(S)\,.$$

For example, consider the poset



we get

S	$\varnothing$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\alpha_P(S)$	1	2	2	2	3	4	3	5

This presentation is quite reminiscent of inclusion-exclusion and we define

$$\beta_P(S) \;:=\; \sum_{T \subseteq S} (-1)^{|S \setminus T|}\alpha_P(T)$$

so that

$$\alpha_P(S) \;:=\; \sum_{T \subseteq S} \beta_P(T)\,.$$

We call $\beta_P = (\beta_P(S))_S$ the **flag h-vector** of P. In particular
$$\mu_P(\hat{0},\hat{1}) \;=\; (-1)^{n-1}\beta_P([n-1])$$
Again, for our example, we get we get

S	$\varnothing$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\beta_P(S)$	1	1	1	1	0	1	0	0

For a maximal chain $C = \{\hat{0} = a_0 \prec\!\!\cdot\; a_1 \prec\!\!\cdot \cdots \prec\!\!\cdot\; a_n = \hat{1}\}$ let
$$\lambda(C) = (\lambda(a_0,a_1), \lambda(a_1,a_2), \dots, \lambda(a_{n-1},a_n))$$
We say that C has a **descent** at $s \in [n-1]$ if $\lambda(a_{s-1},a_s) > \lambda(a_s,a_{s+1})$. We set $\mathrm{Des}(C) \subseteq [n-1]$ the descents of C.

Theorem 4.28. *Let $(P,\preceq)$ be a graded poset of rank n with R-labelling λ. Then $\beta_P(S)$ is the number of maximal chains of P with descent set S.*

For our example, here are the maximal chains with their labellings and descents:
$$\begin{aligned}
S = \{1,3\}:&\quad \varnothing \xrightarrow{1} \{\mathbf{1}\} \xrightarrow{0} \{0,1\} \xrightarrow{3} \{\mathbf{0},\mathbf{1},\mathbf{3}\} \xrightarrow{2} \{0,1,2,3\}\\
S = \{1\}:&\quad \varnothing \xrightarrow{1} \{\mathbf{1}\} \xrightarrow{0} \{0,1\} \xrightarrow{2} \{0,1,2\} \xrightarrow{3} \{0,1,2,3\}\\
S = \{2\}:&\quad \varnothing \xrightarrow{1} \{1\} \xrightarrow{3} \{\mathbf{1},\mathbf{3}\} \xrightarrow{0} \{0,1,3\} \xrightarrow{2} \{0,1,2,3\}\\
S = \{3\}:&\quad \varnothing \xrightarrow{0} \{0\} \xrightarrow{1} \{0,1\} \xrightarrow{3} \{\mathbf{0},\mathbf{1},\mathbf{3}\} \xrightarrow{2} \{0,1,2,3\}\\
S = \varnothing:&\quad \varnothing \xrightarrow{0} \{0\} \xrightarrow{1} \{0,1\} \xrightarrow{2} \{0,1,2\} \xrightarrow{3} \{0,1,2,3\}
\end{aligned}$$

Proof. We claim that $\alpha_P(S)$ is the number of maximal chains with $\mathrm{Des}(C) \subseteq S$. If this is true, then by inclusion-exclusion
$$\sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_P(T)$$
is the number of chains with descent set precisely S. But this is $\beta_P(S)$ and will then prove the claim.

Let $\hat{0} = a_0 \prec a_1 \prec a_2 \prec \cdots \prec a_k \prec a_{k+1} = \hat{1}$ be a chain with $S = \{r(a_1), r(a_2), \dots, r(a_k)\}$. For every $i = 0,\dots,k$, there is a unique maximal chain $C_i \subseteq [a_i, a_{i+1}]$ that is increasing with respect to λ. We define the chain $C = C_0 a_1 C_1 a_2 C_2 \dots a_{k-1} C_{k-1} a_k C_{k+1}$. This is a maximal chain and since we padded with increasing chains, the only descents can occur at the elements a_i. Hence $\mathrm{Des}(C) \subseteq S$. ☐

Let us consider some examples of posets with natural R-labellings.

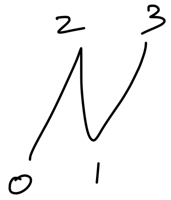
Example 4.3 (Boolean lattices)**.** Let $P = 2^{[n]}$ for some $n \geq 1$. Now, $I \prec\!\!\cdot\; I'$ if $I' \setminus I = \{i\}$. Define $\lambda(I,I') := i$. It is clear that λ is an R-labelling: for any $A \subset B$ with $B \setminus A = \{i_1 < i_2 < \cdots < i_k\}$ define $A_j := A \cup \{i_1,\dots,i_j\}$ for $j = 0,\dots,k$.

In this case, the maximal chains are precisely all permutations π of $[n]$. A descent of a maximal chain is a descent of π. In particular, $\pi = n\, n-1\, \dots\, 2\, 1$ is the unique permutation with $\mathrm{Des}(\pi) = [n-1]$. This again shows $\mu_{B_n}(\varnothing,[n]) = (-1)^n$. In particular, $\beta_P(S)$ is the number of permutations with descents precisely at S. There is no closed formula known but for $S = \{1 \leq s_1 < s_2 < \cdots < s_k < n\}$
$$\alpha_P(S) \;=\; \binom{n}{s_1}\binom{n-s_1}{s_2-s_1}\cdots\binom{n-s_2}{s_3-s_2}\cdots\binom{n-s_k}{n-s_k} \;=\; \binom{n}{s_1, s_2-s_1, \dots, n-s_k}$$
This also means
$$A(n,k) = \sum_{|S|=k} \beta_P(S)$$
is the number of permutations of $[n]$ with exactly k descents. These are the famous Eulerian numbers.

Example 4.4 (Distributive lattices)**.** Let $L = \mathcal{J}(P)$ be a distributive lattice. Let $\ell : P \to [n]$ be a linear extension of P, that is, a bijection with $\ell(a) < \ell(b)$ if $a \prec_P b$.

The elements in L are ideals of P and $I \prec\!\!\cdot\; I'$ in L if $I' \setminus I = \{a\}$ for some minimal element $P \setminus I$. Define $\lambda(I,I') := \ell(a)$. It is straightforward to verify that λ is an R-labelling. For every order ideal $I \subsetneq P$, there is a unique $a \in P \setminus I$ with $\ell(a)$ minimal and $I \cup a$ an ideal. Since ℓ is strictly order-preserving, we have $\ell(a) > \ell(b)$ for all $b \in I$.

This is precisely what we did in the example above: The underlying poset P is



Example 4.5 (Modular lattices)**.** Let $(P, \preceq)$ be a modular lattice of rank n. Pick a maximal chain $\hat{0} = t_0 \prec\!\!\cdot\, t_1 \prec\!\!\cdot \cdots \prec\!\!\cdot\, t_n = \hat{1}$. For any cover relation $s \prec\!\!\cdot\, t$, define

$$\lambda(s,t) \ := \ \min\{i : s \vee t_i = t \vee t_i\}\,.$$

Since $t_n = \hat{1}$, $\lambda(s,t) \le n$ is well-defined. We leave it as an exercise to verify that λ is an R-labelling and give a concrete example instead.

Example 4.6 (Lattice of subspaces)**.** Let $B_n(q)$ be the lattice of subspaces of $V := \mathbb{F}_q^n$. Let $e_1, \dots, e_n$ be the standard basis of V and for $i = 0, \dots, n$ define $V_i := \langle e_1, \dots, e_i \rangle$ as the subspace spanned by $e_1, \dots, e_i$. Since $B_n(q)$ is a modular lattice, we can use the previous example to get an R-labelling but we can be more explicit.

Any subspace $U \subseteq V$ is given by the rowspan of a $k \times n$ matrix $M_U \in \mathbb{F}_q^{k \times n}$. Let us consider M_U in *reduced row echelon form*. There are $1 \le j_1 < j_2 < \cdots < j_k \le n$ such that j_i is the first nonzero entry in the i-th row. The entry (i, j_i) is equal to 1 and is the only nonzero entry in the column j_i. For example

$$\begin{bmatrix} 0 & \mathbf{1} & 5 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

If U' is a subspace such that $U \prec\!\!\cdot\, U'$, then there is a vector $u \in V \setminus U$ such that $U' = U + \langle u \rangle$. The vector u is not unique but we can make it unique given M_U by assuming that $u_{j_i} = 0$ and the first nonzero entry of u is 1. For example

$$\left[\begin{array}{ccccccc} 0 & 0 & 1 & 8 & 0 & 0 & 0 \\ \hline 0 & \mathbf{1} & 5 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{array}\right]$$

If the first nonzero entry of u is at position k, then we define $\lambda(U, U') := k$.

To see that this defines an R-labelling, let $U \subset U'$ be to general subspaces. Pick a basis $u^1, \dots, u^l$ for U and complete to a basis $u^{l+1}, \dots, u^m$ for U'. Use $u^1, \dots, u^l$ to compute the canonical form M_U and assume that $u^{l+1}, \dots, u^m$ are also in canonical form with respect to M_U. Now compute the reduced row echelon form of $M' = (u^{l+1}, \dots, u^m)$. We can assume that $u^{l+1}, \dots, u^m$ are the rows of M' in the right order with pivot $h_1 < h_2 < \cdots < h_{m-l+1}$. The unique increasing chain is then $U_i := U + \langle u^{l+1}, \dots, u^j \rangle$ for $j = l, \dots, m$. For example,

$$\left[\begin{array}{ccccccc} 0 & 0 & \mathbf{1} & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ \hline 0 & \mathbf{1} & 5 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{array}\right]$$

A maximal chain U_i of $B_q(n)$ is uniquely determined by an invertible $n \times n$-matrix M. The subspace U_i is spanned by the last i rows. We can put M into canonical form by performing row operations but we are not allowed to permute rows. The pivots determine a permutation matrix from which the descents can be directly read off. This also allows us to determine $\beta_P(S)$ directly. Just count the number of entries above the pivots. Here are the six possibilities for $n = 3$:

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & * \end{bmatrix} \begin{bmatrix} & 1 & \\ 1 & * & \\ & & 1 \end{bmatrix}$$

$$\begin{bmatrix} & 1 & \\ & & 1 \\ 1 & * & * \end{bmatrix} \begin{bmatrix} & & 1 \\ 1 & & * \\ & 1 & * \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & * \\ 1 & * & * \end{bmatrix}$$

The number of such matrices with all descents is thus $q^{\binom{n}{2}}$, which fits with our previous computation of $\mu_{B_n(q)}(\hat{0}, \hat{1}) = (-1)^n q^{\binom{n}{2}}$.

Example 4.7 (Semimodular lattices)**.** For a semimodular lattice $(L, \preceq)$, let P be the poset of join-irreducibles and pick a linear extension $\ell : P \to [k]$. For a cover relation $s \prec\!\!\cdot\, t$ in L, define

$$\lambda(s,t) \ := \ \min\{\ell(a) : a \in P, s \vee a = t\}\,.$$

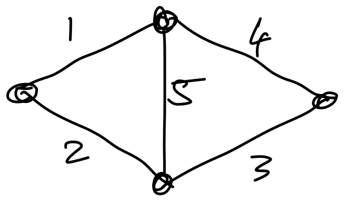
Since every element in L is a join of elements in P, this is well-defined. One can check that this gives an R-labelling on L.

In the case that $(L, \preceq)$ is a geometric lattice, the R-labelling becomes much simpler. We do the construction on the level of matroids and closed sets. Let M be a simple matroid of rank r on ground set E. Let $\mathcal{L} \subseteq 2^E$ be the lattice of flats. Let $F \subset F'$ be two flats with $r(F') = r(F) + 1$. Then $F' = \mathrm{cl}(F \cup e)$ for all $e \in F' \setminus F$. Let us fix any bijection $\ell : E \to [n]$ for $n = |E|$. Hence, we can write $E = \{e_1, \dots, e_n\}$ so that $\ell(e_i) = i$. For a cover relation $F \prec\!\!\cdot\, F'$, we define

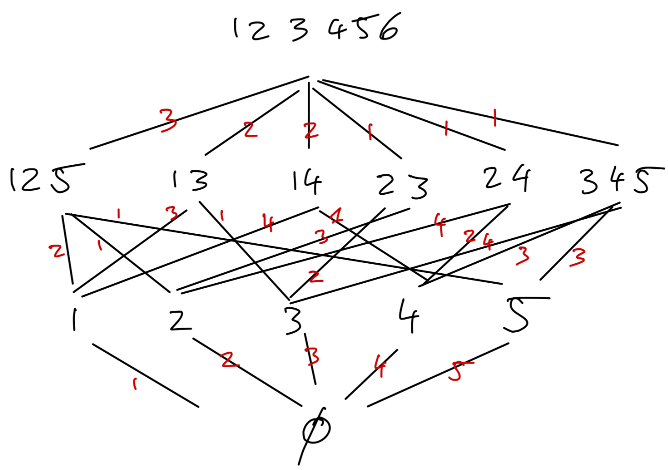
$$\lambda(F, F') \ := \ \min(F' \setminus F)\,,$$

that is $e = \lambda(F, F')$ is the smallest (in the fixed total order) element in F' that brings F to F'.

Here is our running example:



with lattice of flats and cover relations labelled by λ



Proposition 4.29. *λ is an R-labelling.*

Proof. Let $F \subset F'$ be two flats. We show the existence of a unique increasing chain by induction on $k = r(F') - r(F)$. If $k = 1$, $F \subset F'$ is a cover relation and we are good.

For $k > 1$, pick $e = \min(F' \setminus F)$ and let $F_1 = \mathrm{cl}(F \cup e)$. Then $F \prec\!\!\cdot\, F_1$ is a cover relation and $\lambda(F, F_1) = e$. By induction, there is a unique increasing chain $F_1 \prec\!\!\cdot\, F_2 \prec\!\!\cdot\, \cdots \prec\!\!\cdot\, F_k = F'$. Note that $\lambda(F_i, F_{i+1}) \in F' \setminus F_1 = F' \setminus (F \cup e)$ and by choice of e, we have $\lambda(F_i, F_{i+1}) > e$ for all $i = 1, \dots, k-1$. Thus setting $F_0 = F$, we obtain that $F_0 \prec\!\!\cdot\, F_1 \prec\!\!\cdot\, \cdots \prec\!\!\cdot\, F_k$ is an increasing chain.

To see that the chain is unique, let $F'_0, \dots, F'_k$ be another increasing chain. Then $F_1 \neq F'_1$ and hence $\lambda(F, F'_1) = f \neq e$. But $e \in F' \setminus F'_1$. Hence, there is a maximal j with $e \notin F'_j$. But then $e \in F_{j+1}$ and $\lambda(F'_j, F'_{j+1}) = e < f$ and the chain is not increasing. $\square$

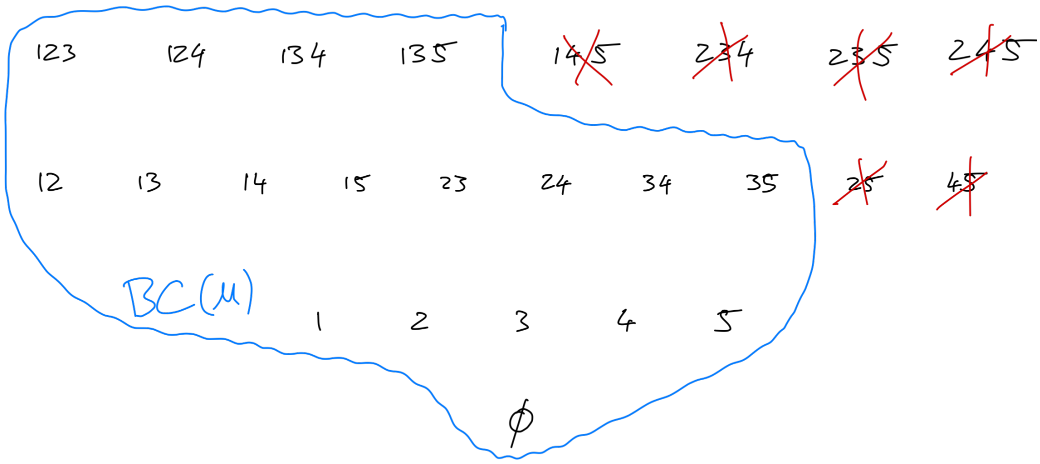
The R-labelling for a matroid with totally ordered ground set is quite natural: Every flat is the closure of *some* set of elements. The R-labelling suggests a canonical representation: For a flat $F \in \mathcal{L}(M)$ of rank k define the saturated chain $C = \{\varnothing = F_0 \prec\!\!\cdot\, F_1 \prec\!\!\cdot\, \cdots \prec\!\!\cdot\, F_k = F\}$ by $F_i := \mathrm{cl}(F_{i-1} \cup a_i)$, where $a_i = \min(F \setminus F_{i-1})$. The sequence $(a_1, \dots, a_k)$ is precisely the R-labelling $\lambda(C)$. This is the unique increasing chain that leads from $\varnothing$ to F.

Conversely, if $\varnothing = F_0 \prec\!\!\cdot\, F_1 \prec\!\!\cdot\, \cdots \prec\!\!\cdot\, F_k$ is any saturated chain in $\mathcal{L}(M)$ with $\lambda(C) = (a_1, a_2, \dots, a_k)$. Then $F_i = \mathrm{cl}(\{a_1, \dots, a_i\})$. This implies that $\{a_1, \dots, a_k\}$ is an independent set and in particular $a_i \neq a_j$ for $i \neq j$. But not every independent set can occur.

To see what kind of sequences can occur, let $\mathcal{C}$ be the circuits of M and $C \in \mathcal{C}$. We call $C \setminus \min(C)$ a **broken circuit**. The **broken circuit complex** of M is the collection $\mathrm{BC}(M) \subseteq 2^E$ of sets not containing a broken circuit

$$\mathrm{BC}(M) \ := \ \{S \subseteq E : C \setminus \min(C) \not\subseteq S \text{ for all } C \in \mathcal{C}\}\,.$$

Note that $\mathrm{BC}(M)$ is hereditary: if $S \in \mathrm{BC}(M)$ and $S' \subseteq S$, then $S' \in \mathrm{BC}(M)$. In particular, if S does not contain a broken circuit, it also does not contain a circuit. Hence $\mathrm{BC}(M) \subseteq \mathcal{I}(M)$.



Proposition 4.30. *Let $C = \{\varnothing = F_0 \prec\!\!\cdot\, F_1 \prec\!\!\cdot\, \cdots \prec\!\!\cdot\, F_k\}$ be a saturated chain with $\lambda(C) = (a_1, a_2, \dots, a_k)$. Then $\{a_1, \dots, a_k\} \in \mathrm{BC}(M)$.*

Proof. Assume that $S = \{a_1, \dots, a_k\}$ contains the broken circuit $C \setminus \min(C)$ and set $b = \min(C)$. Since $C \subseteq \mathrm{cl}(S) \subseteq F_k$, there is minimal j such that $C \subseteq F_j$. Thus, $C \setminus b$ was not a subset of F_{j-1} and hence $a_j = \lambda(F_{j-1}, F_j)$ is the last element to complete the broken circuit in F_j. But $\lambda(F_{j-1}, F_j) = \min(F_j \setminus F_{j-1})$ but $b < a_j$. $\square$

The broken circuit complex allows us to give an interpretation for the coefficients of the characteristic polynomial of a simple matroid or, equivalently, of a geometric lattice. If M is of rank r, we write

$$\chi_M(t) \ = \ w_0 t^r + w_1 t^{r-1} + \cdots + w_r\,.$$

The coefficients $w_i = \sum_{a \in \mathcal{L}, r(a) = i} \mu_{\mathcal{L}}(\hat{0}, a)$ are called the **Whitney numbers of the first kind**. From Corollary 4.26 we know that $(-1)^i w_i \geq 0$. The next result gives a combinatorial interpretation.

Theorem 4.31. *Let M be a simple matroid of rank r with characteristic polynomial $\chi_M(t) = \sum_i w_i t^{r-i}$. Then*

$$(-1)^i w_i \ = \ \#\{S \in \mathrm{BC}(M) : |S| = i\}\,.$$

Proof. We already did most of the leg work. Fix a rank k. Then $w_k = \sum_F (-1)^k \mu(\varnothing, F)$, where the sum is over all flats F or rank k. From Corollary 4.27, we know that for a fixed F, $(-1)^k \mu(\varnothing, F)$ is the number of saturated chains $\lambda(C) = \{\varnothing = F_0 \prec\!\!\cdot\, F_1 \prec\!\!\cdot\, \cdots \prec\!\!\cdot\, F_k = F\}$ for which $\lambda(C) = (a_1 > a_2 > \cdots > a_k)$ is decreasing. The ordered sequence uniquely determines the chain and Proposition 4.30 shows that $\{a_1, a_2, \cdots, a_k\} \in \mathrm{BC}(M)$. What is left to show is that any $S \in \mathrm{BC}(M)$ comes from a decreasing chain.

Let $S \in \mathrm{BC}(M)$ and assume that $S = \{a_1 > a_2 > \cdots > a_k\}$. Define $F_i = \mathrm{cl}(\{a_1, \dots, a_i\})$ for $i = 0, \dots, k$. Since S is an independent set, $F_0, F_1, \dots, F_k$ is a saturated chain. Let $\lambda(F_i, F_{i+1}) =$

$\min(F_{i+1} \setminus F_i) = b$. If $b \neq a_{i+1}$, then, since $a_1, \ldots, a_{i+1}$ is independent, there is a circuit $C \subseteq \{a_1, \ldots, a_{i+1}, b\}$. Since $b < a_{i+1}$, we have $\min(C) = b$ and hence $C \setminus \min(C) \subseteq \{a_1, \ldots, a_{i+1}, b\} \subseteq S$, which contradicts that $S \in \mathrm{BC}(M)$. □

For our running example, we can use deletion-contraction to compute

$$\chi_M(t) \;=\; t^3 - 5t^2 + 8t - 4\,.$$

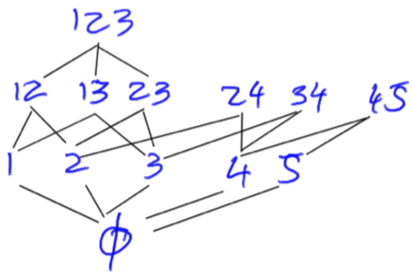
This is confirmed by looking at the broken circuit complex.

5. Simplicial complexes and some topology

Let V be a finite set. An **abtract simplicial complex** is a non-empty collection $\Delta \subseteq 2^V$ such that

$$\sigma \in \Delta, \tau \subseteq \sigma \quad \Longrightarrow \quad \tau \in \Delta$$

That is, Δ is a non-empty hereditary set system. We'll see in a second why we call them now simplicial complexes. Here is an example



If $\{v\} \in \Delta$ for all $v \in V$, then we call V the vertices of Δ. The **dimension** of $\sigma \in \Delta$ is $\dim \sigma := |\sigma| - 1$. In particular, every simplicial complex has a face of dimension $-1 = \dim \varnothing$. The dimension of Δ is $\dim \Delta := \max\{\dim \sigma : \sigma \in \Delta\}$ and any $\sigma \in \Delta$ with $\dim \sigma = \dim \Delta$ is called a **facet** of Δ. We call Δ **pure** if every inclusion-maximal $\sigma \in \Delta$ is of the same dimension.

Example 5.1 (Graphs)**.** Every simple undirected graph $G = (V, E)$ with $E \subseteq \binom{V}{2}$ can be viewed as a simplicial complex $\{\varnothing\} \cup V \cup E$. The complex is pure if and only if G has no edges or no isolated nodes.

For every simplicial complex Δ of dimension d and $k \leq d$, we define the k**-skeleton** as the subcomplex

$$\mathrm{skel}_k(\Delta) := \;\{\sigma \in \Delta : \dim \sigma \leq k\}$$

In particular $\mathrm{skel}_0(\Delta)$ are the vertices of Δ and $G(\Delta) = \mathrm{skel}_1(\Delta)$ is the **graph** of Δ. We call Δ **connected** if $G(\Delta)$ is connected.

Example 5.2 (Order complexes)**.** Let $(P, \preceq)$ be a poset. The collection $\Delta(P)$ of all chains not containing $\hat{0}$ and $\hat{1}$ (provided P has them) is a simplicial complex. The complex is pure if and only if every maximal chain has the same length. That is, if P is graded.

Example 5.3 (Independence and broken circuit complexes)**.** Let $M = (E, \mathcal{I})$ be a matroid. Then $\mathcal{I} \subseteq 2^E$ is a simplicial complex of dimension $r(M) - 1$. The basis exchange property guarantees that $\mathcal{I}$ is pure.

The broken circuit complex $\mathrm{BC}(M) \subseteq \mathcal{I}$ is a subcomplex.

Exercise 5.1. Show that the broken circuit complex is pure.

A **convex polytope** $P \subset \mathbb{R}^d$ is the convex hull of finitely many points $V \subset \mathbb{R}^d$, $P = \mathrm{conv}(V)$. P is a k-dimensional **simplex** if $V = \{v_0, \ldots, v_k\}$ are affinely independent. A 0-simplex is a point, a 1-simplex is a segment, a 2-simplex is a tetrahedron, etc. Note that for $V' \subseteq V$, we have that $\mathrm{conv}(V')$ is a simplex of dimension $|V'| - 1$ and is called a face of P. A collection $\mathcal{T}$ of simplices in some $\mathbb{R}^n$ is called a **geometric simplicial complex** if for every $P \in \mathcal{T}$ and $F \subseteq P$ face, we have $F \in \mathcal{T}$.

Moreover, we require that for $P = \operatorname{conv}(V), P' = \operatorname{conv}(V') \in \mathcal{T}$, we have $P \cap P' = \operatorname{conv}(V \cap V')$. The **support** of $\mathcal{T}$ is the underlying pointset

$$|\mathcal{T}| := \bigcup_{P \in \mathcal{T}} P .$$

Proposition 5.1. *If $\mathcal{T}$ is a geometric simplicial complex, then $\{V : \operatorname{conv}(V) \in \mathcal{T}\}$ is an abstract simplicial complex.*

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. The standard $(n-1)$-simensional simplex is $\Delta_{n-1} = \operatorname{conv}(e_1, \dots, e_n)$. For any set $I \subseteq [n]$, define $\Delta_I := \operatorname{conv}(e_i : i \in I)$.

Proposition 5.2. *Let $\Delta \subseteq 2^{[n]}$ be an abstract simplicial complex. Then*

$$\{\Delta_\sigma : \sigma \in \Delta\}$$

is a geometric simplicial complex.

Note that the two propositions give an equivalence between geometric and abstract simplicial complexes in the sense that the abstract simplicial complex obtained from the geometric simplicial complex associated to Δ is Δ up to relabelling vertices. From now we will essentially focus on abstract simplicial complexes, keeping the idea in mind that we can make them geometric whenever necessary.

Two subsets $X_i \subset \mathbb{R}^{n_i}$, $i = 1, 2$ are **homeomorphic** if there is a continuous and bijective function $f : X_1 \to X_2$ such that f^{-1} is also continuous. In this case we write $X_1 \cong X_2$.

Proposition 5.3. *If $\mathcal{T}, \mathcal{T}'$ are two geometric simplicial complexes such that their underlying abstract simplicial complexes are isomorphic, then $\mathcal{T} \cong \mathcal{T}'$.*

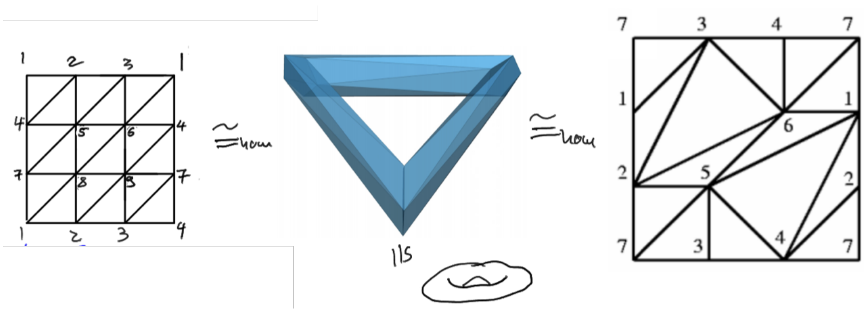
Proof. For a geometric simplicial complex $\mathcal{T}$ with underlying abstract simplicial complex $\Delta \subseteq 2^{[n]}$, we can construct a homeomorphism from the canonical realization as a subcomplex of Δ_{n-1}. To that end, we note that for two simplices P, P' of the same dimension, there is an affine homeomorphism $f : P \to P'$. This local homeomorphism can be extended to the whole complex Δ. □

We will need the following a number of times.

Proposition 5.4. *Let Δ be a simplicial complex, viewed as a poset under inclusion. Then the order complex $\Delta(\Delta)$ is homeomorphic to Δ.*

Proof. We only need to show that if $\Delta = 2^{[n]}$ is a simplex, then $\Delta(\Delta)$ is homeomorphic to Δ. **Picture of barycentric subdivision.** □

We can use simplicial complexes to combinatorially model complex topological spaces:



A particularly important topological space is the sphere $S^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$. We call Δ a **simplicial sphere** if $|\Delta| \cong S^{d-1}$. If $P = \operatorname{conv}(V)$ is any polytope (not necessarily a simplex), then ∂P is homeomorphic to a sphere. In fact, in dimensions $d \le 2$, one can prove that all simplicial d-spheres are boundaries of (simplicial) polytopes. This drastically changes for $d \ge 3$.

Theorem 5.5. *For $d \ge 3$ is is undecidable to check if a d-dimensional complex Δ is a simplicial d-sphere.*

5.1. **Simplicial homology.** Simplicial homology are (effectively) computable invariants of a simplicial complex that help us distinguish topologically different complexes.

Let V be a totally ordered set, such that $V = [n]$. This allows us to write every k-simplex $\sigma \in \Delta$ uniquely as $[i_0, i_2, \dots, i_k]$, where $i_1 < i_2 < \cdots < i_k$.

Let A be an abelian group, f r example, $A \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{F}_q\}$. For $k \in \mathbb{Z}$, we define the k-th **chain group** $C_k(\Delta; A)$ the abelian group of expressions of the form

$$\sum_{\sigma \in \Delta, \dim \sigma = k} a_\sigma \sigma \,,$$

that is, $C_k(\Delta; A) \cong A^{f_k}$ where f_k is the number of k-simplices of Δ. If $A = \mathbb{F}$ is a field, then $C_k(\Delta; A)$ is simply an $\mathbb{F}$-vector space with basis σ for every k-simplex $\sigma \in \Delta$. Note that $C_k(\Delta; A) = 0$ for $k < -1$ and $k > \dim \Delta$. For $k = -1$, we have $C_{-1}(\Delta) = A$ with basis element $\varnothing$.

We define the k-th **boundary map** $\partial_k : C_k(\Delta; A) \to C_{k-1}(\Delta; A)$ on basis elements

$$\partial_k [i_0, i_1, \dots, i_k] \;:=\; \sum_{j=0}^{k} (-1)^j [i_0, \dots, i_{j-1}, i_{j+1}, \dots, i_k]$$

EXAMPLE

Lemma 5.6. *$\partial_{k-1} \circ \partial_k = 0$ for all k.*

Proof. We simply compute

$$\begin{aligned} \partial_{k-1}\partial_k [i_0, i_1, \dots, i_k] \;&=\; \sum_{j=0}^{k} (-1)^j \partial_{k-1} [i_0, \dots, i_{j-1}, i_{j+1}, \dots, i_k] \\ &=\; \sum_{r<s} (-1)^r (-1)^s [i_0, \dots, \widehat{i_r}, \dots, \widehat{i_s}, \dots, i_k] \\ &+\; \sum_{r>s} (-1)^{r-1} (-1)^s [i_0, \dots, \widehat{i_s}, \dots, \widehat{i_r}, \dots, i_k] \;=\; 0 \,. \qquad \square \end{aligned}$$

The **chain complex** of Δ over A is $(C_k, \partial_k)_{k \in \mathbb{Z}}$. We call $Z_k = \ker \partial_k \subseteq C_k$ the k-**cycles** of Δ and $B_k = \mathrm{im} \partial_{k+1}$ the k-**boundaries** of Δ. The k-**th (reduced) homology group** of Δ is

$$\tilde{H}_k(\Delta) \;:=\; Z_k(\Delta; A) / B_k(\Delta; A)$$

Theorem 5.7 (Fundamental theorem of algebraic topology)**.** *Let Δ, Δ' be simplicial complexes. If $\Delta \cong \Delta'$ then $\tilde{H}_k(\Delta; A) \cong \tilde{H}_k(\Delta'; A)$ for all $k \in \mathbb{Z}$.*

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