Mathematik für Naturwissenschaftler II

Musterlösung Blatt 7

Aufgabe 7.1

(a)

$$f(x,y) = (y-1)^2 - x(x-3)^2$$

$$grad(f(\vec{a})) = (\frac{\partial}{\partial x}f(\vec{a}), \frac{\partial}{\partial y}f(\vec{a})) = (-(x-3)^2 - x \cdot 2(x-3), 2 \cdot (y-1))$$

$$= (-3x^2 - 9 + 12x, 2 \cdot (y-1)) \overset{!}{=} \vec{0}$$

$$\Rightarrow y = 1$$

$$\Rightarrow -3x^2 - 9 + 12x \overset{!}{=} 0 \Leftrightarrow x^2 - 4x + 3 = 0$$

$$\Rightarrow \text{pq-Formel: } x_{1,2} = \frac{4}{2} \pm \sqrt{(\frac{4}{2})^2 - 3} \Leftrightarrow x_{1,2} = 2 \pm 1$$

$$\Rightarrow \text{potentielle Extremstellen in } \vec{a_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \vec{a_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hessematrix berechnen:

$$H_f(\vec{a}) = \begin{pmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x}f(\vec{a}) & \frac{\partial}{\partial x}\frac{\partial}{\partial y}f(\vec{a}) \\ \frac{\partial}{\partial y}\frac{\partial}{\partial x}f(\vec{a}) & \frac{\partial}{\partial y}\frac{\partial}{\partial y}f(\vec{a}) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}(-3x^2 - 9 + 12x) & \frac{\partial}{\partial x}2 \cdot (y-1) \\ \frac{\partial}{\partial y}(-3x^2 - 9 + 12x) & \frac{\partial}{\partial y}2 \cdot (y-1) \end{pmatrix} = \begin{pmatrix} -6x + 12 & 0 \\ 0 & 2 \end{pmatrix}$$

Eigenwerte berechnen für $\vec{a_1}$:

$$H_f(\begin{pmatrix} 3 \\ 1 \end{pmatrix}) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix}$$

$$det(H_f(a_1) - \lambda \cdot E_2) = det(\begin{pmatrix} -6-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix}) = (-6-\lambda)(2-\lambda) \overset{!}{=} 0$$

$$\Rightarrow \lambda_1 = -6 \text{ , } \lambda_2 = 2$$

$$\Rightarrow H_f \text{ indefinit} \Rightarrow \text{kein lokales Extremum im Punkt } \vec{a_1} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Eigenwerte berechnen für $\vec{a_2}$:

$$H_f(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

$$det(H_f(a_2) - \lambda \cdot E_2) = det(\begin{pmatrix} 6-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix}) = (6-\lambda)(2-\lambda) \overset{!}{=} 0$$

$$\Rightarrow \lambda_1 = 6 \text{ , } \lambda_2 = 2 \Rightarrow \forall \lambda > 0$$

$$\Rightarrow H_f \text{ positiv definitiv} \Rightarrow \text{Minimum in } \vec{a_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(b)

$$f(x,y,z) = x^2 + y^2 + z^2$$

$$grad(f(\vec{a})) = (\frac{\partial}{\partial x}f(\vec{a}), \frac{\partial}{\partial y}f(\vec{a}), \frac{\partial}{\partial z}f(\vec{a})) = (2x, 2y, 2z) \overset{!}{=} \vec{0}$$

$$\Rightarrow \text{potentielle Extremstelle in } \vec{a} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hessematrix berechnen:

$$H_f(\vec{a}) = \begin{pmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x}f(\vec{a}) & \frac{\partial}{\partial x}\frac{\partial}{\partial y}f(\vec{a}) & \frac{\partial}{\partial x}\frac{\partial}{\partial z}f(\vec{a}) \\ \frac{\partial}{\partial y}\frac{\partial}{\partial x}f(\vec{a}) & \frac{\partial}{\partial y}\frac{\partial}{\partial y}f(\vec{a}) & \frac{\partial}{\partial y}\frac{\partial}{\partial z}f(\vec{a}) \\ \frac{\partial}{\partial z}\frac{\partial}{\partial x}f(\vec{a}) & \frac{\partial}{\partial z}\frac{\partial}{\partial y}f(\vec{a}) & \frac{\partial}{\partial z}\frac{\partial}{\partial z}f(\vec{a}) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}2x & \frac{\partial}{\partial x}2y & \frac{\partial}{\partial x}2z \\ \frac{\partial}{\partial y}2x & \frac{\partial}{\partial y}2y & \frac{\partial}{\partial y}2z \\ \frac{\partial}{\partial z}2x & \frac{\partial}{\partial z}2y & \frac{\partial}{\partial z}2z \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\Rightarrow H_f(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Eigenwerte berechnen:

$$det(H_f(\vec{a}) - \lambda \cdot E_3) = det(\begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}) = (2-\lambda)^3 \overset{!}{=} 0$$

$$\Rightarrow \lambda_{1,2,3} = 2 > 0 \Rightarrow H_f \text{ positiv definit} \Rightarrow \text{Minimum in } \vec{a} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Aufgabe 7.2

(a)

$$f(x,y) = \sqrt{1 - x^2 - y^2}$$

$$grad(f(\vec{a})) = (\frac{\partial}{\partial x}f(\vec{a}), \frac{\partial}{\partial y}f(\vec{a})) = (\frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}}) \overset{!}{=} \vec{0}$$

$$\Rightarrow \text{potentielle Extremstelle in } \vec{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hessematrix berechnen:

$$H_f(\vec{a}) = \begin{pmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x} f(\vec{a}) & \frac{\partial}{\partial x}\frac{\partial}{\partial y} f(\vec{a}) \\ \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(\vec{a}) & \frac{\partial}{\partial y}\frac{\partial}{\partial y} f(\vec{a}) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}\frac{-x}{\sqrt{1-x^2-y^2}} & \frac{\partial}{\partial x}\frac{-y}{\sqrt{1-x^2-y^2}} \\ \frac{\partial}{\partial y}\frac{-x}{\sqrt{1-x^2-y^2}} & \frac{\partial}{\partial y}\frac{-y}{\sqrt{1-x^2-y^2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{\sqrt{1-x^2-y^2}} - \frac{x^2}{(1-x^2-y^2)^{\frac{3}{2}}} & \frac{-yx}{(1-x^2-y^2)^{\frac{3}{2}}} \\ \frac{-yx}{(1-x^2-y^2)^{\frac{3}{2}}} & \frac{-1}{\sqrt{1-x^2-y^2}} - \frac{y^2}{(1-x^2-y^2)^{\frac{3}{2}}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-(1-x^2-y^2)}{(1-x^2-y^2)^{\frac{3}{2}}} - \frac{x^2}{(1-x^2-y^2)^{\frac{3}{2}}} & \frac{-yx}{(1-x^2-y^2)^{\frac{3}{2}}} \\ \frac{-yx}{(1-x^2-y^2)^{\frac{3}{2}}} & \frac{-(1-x^2-y^2)}{(1-x^2-y^2)^{\frac{3}{2}}} - \frac{y^2}{(1-x^2-y^2)^{\frac{3}{2}}} \end{pmatrix} = \begin{pmatrix} \frac{-(1-x^2-y^2)-x^2}{(1-x^2-y^2)^{\frac{3}{2}}} & \frac{-yx}{(1-x^2-y^2)^{\frac{3}{2}}} \\ \frac{-yx}{(1-x^2-y^2)^{\frac{3}{2}}} & \frac{-(1-x^2-y^2)-y^2}{(1-x^2-y^2)^{\frac{3}{2}}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{y^2-1}{(1-x^2-y^2)^{\frac{3}{2}}} & \frac{-yx}{(1-x^2-y^2)^{\frac{3}{2}}} \\ \frac{-yx}{(1-x^2-y^2)^{\frac{3}{2}}} & \frac{x^2-1}{(1-x^2-y^2)^{\frac{3}{2}}} \end{pmatrix}$$

$$\Rightarrow H_f(\begin{pmatrix} 0 \\ 0 \end{pmatrix}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Eigenwerte berechnen:

$$det(H_f(\vec{a}) - \lambda \cdot E_2) = det(\begin{pmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix}) = (1+\lambda)^2 \stackrel{!}{=} 0$$

$$\Rightarrow \lambda_{1,2} = -1 \Rightarrow H_f \text{ negativ definitiv} \Rightarrow \text{Maximum in } \vec{a} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(b)

$$f(x,y) = e^{-x-y}$$

$$grad(f(\vec{a})) = (\frac{\partial}{\partial x} f(\vec{a}), \frac{\partial}{\partial y} f(\vec{a})) = (-e^{-x-y}, -e^{-x-y}) \stackrel{!}{=} \vec{0}$$

$\Rightarrow -e^{-x-y} \neq 0 \Rightarrow$ Keine Lösung, keine Extremstellen.

Aufgabe 7.3

Zu Berechnen ist das Maximum von $F(\alpha, \beta, \gamma) = \alpha \cdot \beta \cdot \gamma$ mit der Nebenbedingung

$g(\alpha, \beta, \gamma) = \alpha + \beta + \gamma - \pi \overset{!}{=} 0.$

$$grad(g(\alpha, \beta, \gamma)) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \neq \vec{0}$$

Nach Satz 3.5.1 erfüllen die Extremstellen:

$g(\alpha, \beta, \gamma) = \alpha + \beta + \gamma - \pi \overset{!}{=} 0$ und

$$grad(F(\alpha, \beta, \gamma)) = \begin{pmatrix} \frac{\partial}{\partial \alpha} \alpha \cdot \beta \cdot \gamma \\ \frac{\partial}{\partial \beta} \alpha \cdot \beta \cdot \gamma \\ \frac{\partial}{\partial \gamma} \alpha \cdot \beta \cdot \gamma \end{pmatrix} = \begin{pmatrix} \beta \cdot \gamma \\ \alpha \cdot \gamma \\ \alpha \cdot \beta \end{pmatrix} \overset{!}{=} \lambda \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \lambda \cdot grad(g(\alpha, \beta, \gamma))$$

Daraus ergibt sich folgendes Gleichungssystem:

I. $\alpha + \beta + \gamma - \pi = 0$

II. $\beta \cdot \gamma = \lambda$

III. $\alpha \cdot \gamma = \lambda$

IV. $\alpha \cdot \beta = \lambda$

Extremstellen:

$\Rightarrow \alpha = \beta = \gamma = \frac{\pi}{3}$

$\Rightarrow \alpha = \beta = 0$ und $\gamma = 2\pi$ (und alle anderen Kombinationen mit (0,0,2 π))

Das Maximum der Funktion liegt somit offensichtlich bei

$$F_{max} = \frac{\pi^3}{27}.$$