Mathematik für Naturwissenschaftler I

Musterlösung Blatt 8

Aufgabe 8.1

(a)

$$t(x) = 2 - x^2$$
$$t'(x) = \frac{dt}{dx} = -2x \Leftrightarrow dx = \frac{-dt}{2x}$$
$$x_a = 0 \Rightarrow t_a = t(x_a) = 2$$
$$x_b = 1 \Rightarrow t_b = t(x_b) = 1$$
$$\int_0^1 \frac{x}{\sqrt{2-x^2}}\,dx = \int_2^1 \frac{x}{\sqrt{t}} \cdot \frac{-1}{2x}\,dt = \int_1^2 \frac{1}{2\sqrt{t}}\,dt = \left[\sqrt{t}\right]_1^2 = \sqrt{2} - 1$$

(b)

$$t(x) = \ln(x)$$
$$t'(x) = \frac{dt}{dx} = \frac{1}{x} \Leftrightarrow dx = dt \cdot x$$
$$\int \frac{\ln(x)}{x}\,dx = \int \frac{t}{x} \cdot x\,dt = \int t\,dt = \frac{t^2}{2} + C \overset{\text{Rücksub.}}{=} \frac{\ln(x)^2}{2} + C$$

(c)

$$t(x) = 1 + 2x$$
$$t'(x) = \frac{dt}{dx} = 2 \Leftrightarrow dx = dt \cdot \frac{1}{2}$$
$$x_a = 0 \Rightarrow t_a = t(x_a) = 1$$
$$x_b = 1 \Rightarrow t_b = t(x_b) = 3$$
$$\int_0^1 \frac{1}{\sqrt{1+2x}}\,dx = \int_1^3 \frac{1}{2 \cdot \sqrt{t}}\,dt = \left[\sqrt{t}\right]_1^3 = \sqrt{3} - 1$$

(d)

$$x(t) = \tan(t)$$

$$x'(t) = \frac{dx}{dt} = \frac{1}{\cos^2(t)} \Leftrightarrow dx = dt \cdot \frac{1}{\cos^2(t)}$$

$$x_a = 0 \Rightarrow t_a = \tan^{-1}(x_a) = 0$$

$$x_b = 1 \Rightarrow t_b = \tan^{-1}(x_b) = \frac{\pi}{4}$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{1+\tan^2 t} \cdot \frac{1}{\cos^2(t)}\,dt = \int_0^{\frac{\pi}{4}} \frac{1}{1+\frac{\sin^2(t)}{\cos^2(t)}} \cdot \frac{1}{\cos^2(t)}\,dt$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{(1+\frac{\sin^2(t)}{\cos^2(t)}) \cdot \cos^2(t)}\,dt = \int_0^{\frac{\pi}{4}} \frac{1}{\cos^2(t)+\sin^2(t)}\,dt = \int_0^{\frac{\pi}{4}} \frac{1}{1}\,dt = \Big[t\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$$

Aufgabe 8.2

(a)

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{b\to 0} \int_b^1 \frac{1}{x^2}\,dx = \lim_{b\to 0}\Big[-\frac{1}{x}\Big]_b^1 = \lim_{b\to 0}(-1+\frac{1}{b}) = \infty$$

$\Rightarrow$ Nicht uneigentlich integrierbar, da divergiert. Integral existiert nicht.

(b)

$$\int_0^{\frac{\pi}{2}} \frac{1}{\tan(x)}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)}\,dx = \lim_{b\to 0} \int_b^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)}\,dx$$

Wir nutzen die Integrationsregel aus dem Skript:

$$\int_a^b \frac{f'(x)}{f(x)}\,dx = \Big[\ln(|f(x)|)\Big]_a^b$$

$$\Rightarrow f(x) = \sin(x)\ \&\ f'(x) = \cos(x)$$

$$\Rightarrow \lim_{b\to 0} \int_b^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)}\,dx = \lim_{b\to 0}\Big[\ln(|\sin(x)|)\Big]_b^{\frac{\pi}{2}} = \lim_{b\to 0}(\ln(1) - ln(|\sin(b)|)) = \infty$$

$\Rightarrow$ Nicht uneigentlich integrierbar, da divergiert. Integral existiert nicht.

Aufgabe 8.3

(a)
$$\int \frac{x^3-2x^2+6}{x^2-3x+2}dx$$

1.Polynomdivision:

$$(x^3-2x^2+6):(x^2-3x+2) = x+1+\frac{x+4}{x^2-3x+2}$$

$$\begin{array}{l} x^3-3x^2+2x \\ \hline \quad x^2-2x+6 \\ \quad x^2-3x+2 \\ \hline \qquad x+4 \end{array}$$

$$\Rightarrow \int (x+1+\frac{x+4}{x^2-3x+2})dx = \int (x+1)dx + \int \frac{x+4}{x^2-3x+2}dx$$

Betrachte den letzten Term:

$$\int \frac{x+4}{x^2-3x+2}dx$$

2.Linearfaktorzerlegung des Nenners(Nullstellen erraten oder durch pq-Formel bestimmen):

$$x^2-3x+2 = (x-1)(x-2)$$

3.Partialbruchzerlegung:

$$\frac{x+4}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{x^2-3x+2}$$

$$\Rightarrow A(x-2)+B(x-1) = x+4$$

4.Koeffizientenvergleich:

$$(A+B)x = x \Rightarrow (A+B) = 1$$

$$-B-2A = 4$$

$$\Rightarrow A = -5 \ , \ B = 6$$

5.Ausrechnen des Integrals:

$$\int \frac{x^3-2x^2+6}{x^2-3x+2}dx = \int (x+1+\frac{x+4}{x^2-3x+2})dx = \int (x+1)dx + \int \frac{x+4}{x^2-3x+2}dx$$

$$= \int (x+1)dx - \int \frac{5}{x-1}dx + \int \frac{6}{x-2}dx$$

$$\Rightarrow F(x) = \frac{1}{2}x^2 + x - 5\cdot\ln(|x-1|) + 6\cdot\ln(|x-2|) + d \ , \ d \in \mathbb{R}$$

(b)
$$\int \frac{x^5-x^3+x}{x^3-x}dx$$

1.Polynomdivision:

$$(x^5 - x^3 + x) : (x^3 - x) = x^2 + \frac{x}{x^3 - x} = x^2 + \frac{1}{x^2 - 1}$$

$$\frac{x^5 - x^3}{}$$

$$x$$

$$\Rightarrow \int (x^2 + \frac{1}{x^2-1})dx = \int (x^2)dx + \int \frac{1}{x^2-1}dx$$

Betrachte den letzten Term:

$$\int \frac{1}{x^2-1}dx$$

2.Linearfaktorzerlegung des Nenners(Nullstellen erraten oder durch pq-Formel bestimmen):

$$x^2 - 1 = (x-1)(x+1)$$

3.Partialbruchzerlegung:

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1) + B(x-1)}{x^2-1}$$

$$\Rightarrow A(x+1) + B(x-1) = 1$$

4.Koeffizientenvergleich:

$$(A+B)x = 0 \Rightarrow (A+B) = 0$$

$$A - B = 1$$

$$\Rightarrow A = \frac{1}{2} \text{ , } B = -\frac{1}{2}$$

5.Ausrechnen des Integrals:

$$\int x^2 dx + \frac{1}{2} \cdot \int \frac{1}{x-1}dx - \frac{1}{2} \cdot \int \frac{1}{x+1}dx$$

$$\Rightarrow F(x) = \frac{1}{3}x^3 + \frac{1}{2} \cdot \ln(|x-1|) - \frac{1}{2} \cdot \ln(|x+1|) + d \text{ , } d \in \mathbb{R}$$