Mathematatik für Naturwissenschaftler I
Musterlösung Blatt 8

Aufgabe 8.1:

Allgemeine Formel für Substitution:

$$\boldsymbol{\int_a^b f(x)dx = \int_\alpha^\beta f\big(x(t)\big) \cdot x'(t)dt}$$

a)

$$\int_0^1 \frac{1}{\sqrt{1+2x}}dx$$

Substitution:

$$x(t) \ = \frac{t-1}{2}, \quad x' = \frac{1}{2}, \quad t(x) = 2x+1$$

Neue Grenzen:

$$0 = \frac{\alpha-1}{2} \to \alpha = 1 \qquad\qquad 1 = \frac{\beta-1}{2} \to \beta = 3$$

Einsetzen:

$$\int_0^1 \frac{1}{\sqrt{1+2x}}dx = \int_1^3 \frac{1}{\sqrt{1+2\frac{t-1}{2}}} \cdot \frac{1}{2}dt = \int_1^3 \frac{1}{2\sqrt{t}}dt = \left[\sqrt{t}\right]_1^3 = \sqrt{3}-\sqrt{1} = \boldsymbol{\sqrt{3}-1}$$

Stammfunktion berechnen mittels Resubstitution:

$$\sqrt{t} = \sqrt{2x+1}$$

Stammfunktion:

$$\boldsymbol{\int \frac{1}{\sqrt{1+2x}}dx = \sqrt{2x+1}+C}$$

b)

$$\int_0^1 \frac{x^3}{\sqrt{1+x^2}}\,dx$$

Substitution:

$$x(t) = \sqrt{t-1}, \quad x' = \frac{1}{2\sqrt{t-1}}, \qquad t(x) = x^2 + 1$$

Neue Grenzen:

$$0 = \sqrt{t-1} \rightarrow t = 1 \qquad\qquad 1 = \sqrt{t-1} \rightarrow \beta = 2$$

Einsetzen:

$$\int_0^1 \frac{x^3}{\sqrt{1+x^2}}\,dx = \int_1^2 \frac{\sqrt{t-1}^3}{\sqrt{1+\sqrt{t-1}^2}} \cdot \frac{1}{2\sqrt{t-1}}\,dt = \int_1^2 \frac{\sqrt{t-1}^{\cancel{3}2}}{\sqrt{\cancel{1}+t\cancel{-1}}} \cdot \frac{1}{2\cancel{\sqrt{t-1}}}\,dt = \int_1^2 \frac{t-1}{2\sqrt{t}}\,dt$$

$$= \frac{1}{2}\int_1^2 \frac{t}{\sqrt{t}} - \frac{1}{\sqrt{t}}\,dt = \frac{1}{2}\int_1^2 t^{1/2} - t^{-1/2}\,dt = \frac{1}{2}\left[\frac{2}{3}\cdot t^{\frac{3}{2}} - 2\cdot t^{1/2}\right]_1^2$$

$$= \frac{1}{2}\left[\left(\frac{2}{3}\cdot 2^{\frac{3}{2}} - 2\cdot 2^{\frac{1}{2}}\right) - \left(\frac{2}{3}\cdot 1^{\frac{3}{2}} - 2\cdot 1^{\frac{1}{2}}\right)\right] = \frac{1}{2}\left[\left(\frac{4}{3}\cdot\sqrt{2} - 2\cdot\sqrt{2}\right) - \left(\frac{2}{3} - 2\right)\right]$$

$$= -\frac{\sqrt{2}}{3} + \frac{2}{3} = \frac{2-\sqrt{2}}{3} = \mathbf{0.195}$$

Stammfunktion berechnen mittels Resubstitution:

$$\frac{1}{2}\left(\frac{2}{3}\cdot t^{\frac{3}{2}} - 2\cdot t^{\frac{1}{2}}\right) = \frac{2}{6}\cdot t^{\frac{3}{2}} - t^{\frac{1}{2}} = \frac{1}{3}\cdot(x^2+1)^{\frac{3}{2}} - (x^2+1)^{\frac{1}{2}}$$

Vereinfache:

$$\frac{1}{3}\cdot(x^2+1)^{\frac{3}{2}} - (x^2+1)^{\frac{1}{2}} = \frac{(x^2+1)^{\frac{3}{2}}}{3} - \frac{3(x^2+1)^{\frac{1}{2}}}{3} = \frac{(x^2+1)^{\frac{3}{2}} - 3(x^2+1)^{\frac{1}{2}}}{3}$$

$$= \frac{(x^2+1)^{\frac{1}{2}}\cdot(x^2+1-3)}{3} = \frac{\sqrt{x^2+1}\cdot(x^2-2)}{3}$$

Stammfunktion:

$$\boldsymbol{\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \frac{\sqrt{x^2+1}\cdot(x^2-2)}{3} + C}$$

c)

$$\int_0^1 \frac{1}{1+x^2}\,dx$$

Substitution:

$$x = \tan t, \quad x' \overset{Aufgabe\ 5.3\ d)}{=} \frac{1}{\cos^2 t} = \frac{\sin^2 t + \cos^2 t}{\cos^2 t} = \tan^2 t + 1, \quad t(x) = \arctan x$$

Neue Grenzen:

$$0 = \tan\alpha \rightarrow \alpha = 0 \qquad\qquad 1 = \tan\beta \rightarrow \beta = \frac{\pi}{4}$$

Einsetzen:

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{\cancel{1+\tan^2 t}} \cdot \cancel{(\tan^2 t + 1)}\,dt = \int_0^{\frac{\pi}{4}} 1\,dt = [t]_0^{\frac{\pi}{4}} = \boldsymbol{\frac{\pi}{4}}$$

Stammfunktion berechnen mittels Resubstitution:

t $= \arctan x$

Stammfunktion:

$$\boldsymbol{\int \frac{1}{1+x^2}\,dx = \arctan x + C}$$

d)

$$\int_0^1 (1+x)\sqrt{1-x}dx$$

Substitution:

$x(t) = 1 - t^2, \quad x'(t) = -2t, \quad t(x) = \sqrt{1-x}$

Neue Grenzen:

$0 = 1 - \alpha^2 \rightarrow \alpha = 1 \qquad\qquad 1 = 1 - \beta^2 \rightarrow \beta = 0$

Einsetzen:

$$\int_0^1 (1+x)\sqrt{1-x}dx = \int_1^0 \left(1 + (1-t^2)\right)\sqrt{1-(1-t^2)} \cdot (-2t)dt = \int_1^0 (2-t^2)t \cdot (-2t)dt$$

$$= \int_1^0 2t^4 - 4t^2 dt = \left[\frac{2}{5}t^5 - \frac{4}{3}t^3\right]_1^0 = \frac{2}{5}0^5 - \frac{4}{3}0^3 - \left(\frac{2}{5}1^5 - \frac{4}{3}1^3\right) = \frac{4}{3} - \frac{2}{5}$$

$$= \frac{20-6}{15} = \frac{14}{15} = 0.933$$

Stammfunktion berechnen mittels Resubstitution:

$$\frac{2}{5}t^5 - \frac{4}{3}t^3 = \frac{2}{5}\sqrt{1-x}^{\,5} - \frac{4}{3}\sqrt{1-x}^{\,3}$$

Vereinfache:

$$\frac{2}{5}\sqrt{1-x}^{\,5} - \frac{4}{3}\sqrt{1-x}^{\,3} = \frac{6}{15}\sqrt{1-x}^{\,5} - \frac{20}{15}\sqrt{1-x}^{\,3} = \frac{6\sqrt{1-x}^{\,5} - 20\sqrt{1-x}^{\,3}}{15}$$

$$= \frac{\sqrt{1-x}^{\,3} \cdot (6(1-x) - 20)}{15} = \frac{\sqrt{1-x}^{\,3} \cdot (-6x - 14)}{15}$$

Stammfunktion:

$$\int \boldsymbol{(1+x)\sqrt{1-x}dx} = \frac{\boldsymbol{\sqrt{1-x}^{\,3} \cdot (-6x-14)}}{\mathbf{15}} + \boldsymbol{C}$$

Aufgabe 8.2:

a)

$$\int_0^1 \frac{1}{x^2}\,dx$$

→ Untere Grenze für Integranden nicht definiert!

$$\lim_{a\to 0}\int_a^1 \frac{1}{x^2}\,dx = \lim_{a\to 0}\left[-\frac{1}{x}\right]_a^1 = -1 - \lim_{a\to 0}\left(-\frac{1}{a}\right) = -1 + \text{"}\infty\text{"}$$

→ Nicht uneigentlich integrierbar. Integral existiert nicht.

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b)

$$\int_0^\infty \frac{\sin x}{e^x}\,dx$$

→ obere Grenze ist kein endlicher Wert

$$\int_0^\infty \frac{\sin x}{e^x}\,dx = \lim_{b\to\infty}\int_0^b \frac{\sin x}{e^x}\,dx$$

Partielle Integration:

$$\int f'g = fg - \int fg'$$

Erste partielle Integration:

$f = -e^{-x}$, $f' = e^{-x}$

$g = \sin x$, $g' = \cos x$

$$\int \frac{\sin x}{e^x}\,dx = -e^{-x}\sin x - \int \cos x\,(-e^{x})dx$$

Zweite partielle Integration:

$f = e^{-x}$, $f' = -e^{-x}$

$g = \cos x$, $g' = -\sin x$

$$\int \frac{\sin x}{e^x}\,dx = -e^{-x}\sin x - \left[e^{-x}\cos x - \int e^{-x}(-\sin x)dx\right]$$

$$\int \frac{\sin x}{e^x}\,dx = -e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\sin x\,dx \qquad \Big| + \int_0^\infty \frac{\sin x}{e^x}\,dx$$

$$2\cdot\int \frac{\sin x}{e^x}\,dx = -e^{-x}(\sin x + \cos x)$$

$$\int \frac{\sin x}{e^x}\,dx = -\frac{(\sin x + \cos x)}{2\cdot e^x}$$

→ für $b = \infty$: obere Grenze ist kein endlicher Wert

$$\lim_{b\to\infty}\int_0^b \frac{\sin x}{e^x}\,dx = \lim_{b\to\infty}\left[-\frac{(\sin x + \cos x)}{2\cdot e^x}\right]_0^b = \lim_{b\to\infty} -\frac{(\sin b + \cos b)}{2\cdot e^b} - \left(-\frac{(\sin 0 + \cos 0)}{2\cdot e^0}\right)$$

$$= 0 - \left(-\frac{0+1}{2}\right) = \mathbf{\frac{1}{2}}$$

→ Uneigentlich integrierbar. Integral existiert mit dem Wert ½.

c)

$$\int_0^{\frac{\pi}{2}} \frac{1}{\tan x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x}\,dx$$

→ Untere Grenze für Integranden nicht definiert.

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x}\,dx = \lim_{a\to 0}\int_a^{\frac{\pi}{2}} \frac{\cos x}{\sin x}\,dx$$

Nutze Integrationsregel

$$\int_a^b \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|dx|_a^b$$

$$\lim_{a\to 0}\int_a^{\frac{\pi}{2}} \frac{\cos x}{\sin x}\,dx = \lim_{a\to 0}[ln\mid sin\,x\mid]_a^{\pi/2} = ln\mid sin\frac{\pi}{2}\mid - \lim_{a\to 0}(ln|sin\,a|) = 0 + \text{"}\infty\text{„}$$

→ Nicht uneigentlich integrierbar. Integral existiert nicht.

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d)

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 x}\,dx$$

→ Obere Grenze für Integranden nicht definiert.

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 x}\,dx = \int_0^{b} \frac{1}{\cos^2 x}\,dx$$

Berechne $(\tan x)'$:

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \boldsymbol{\frac{1}{\cos^2 x}}$$

$$\lim_{b\to\pi/2}\int_0^{b} \frac{1}{\cos^2 x}\,dx = \lim_{b\to\pi/2}[\tan x]_0^b = \lim_{b\to\pi/2}\tan b - \tan 0 = \text{"}\infty\text{"} - 0$$

→ Nicht uneigentlich integrierbar. Integral existiert nicht.

e)

$$\int_{-1}^{1} t \cdot e^{(t^2)} dt$$

Substitution:
$x \ = t^2, \quad x' = 2t$
Neue Grenzen:
$a = (-1)^2 \to a = 1$ $\qquad b = (1)^2 \to b = 1$

Umschreiben der Ausgangsfunktion:

$$\int_{-1}^{1} t \cdot e^{(t^2)} dt = \frac{1}{2} \int_{-1}^{1} e^{(t^2)} 2t \, dt$$

Einsetzen (Substitution von „rechts nach links“):

$$\frac{1}{2} \int_{-1}^{1} e^{(t^2)} 2t \, dt = \frac{1}{2} \int_{1}^{1} e^{(x)} \, dx = \frac{1}{2} [e^x]_1^1 = \frac{1}{2} [e^1 - e^1] = 0$$

Integral existiert mit Wert 0.

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f)

$$\int \frac{\ln x}{x} dx$$

Substitution (von rechts nach links!):

$$t = \ln x, \quad t' = \frac{1}{x}$$

Einsetzen:

$$\int \frac{\ln x}{x} dx = \int \ln x \cdot \frac{1}{x} dx = \int t dt = \frac{1}{2} t^2 + C$$

Resubstitution:

$$\boldsymbol{\frac{1}{2} \ln^2 x + C}$$

Integral existiert.