Rates of Convergence for Quicksort

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Abstract

The normalized number of key comparisons needed to sort a list of randomly permuted items by the Quicksort algorithm is known to converge in distribution. We identify the rate of convergence to be of the order $\Theta(\ln(n)/n)$ in the Zolotarev metric. This implies several $\ln(n)/n$ estimates for other distances and local approximation results as for characteristic functions, for density approximation, and for the integrated distance of the distribution functions.

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1 Introduction and main result

The distribution of the number of key comparisons X_n of the Quicksort algorithm needed to sort an array of n randomly permuted items is known to converge after normalization in distribution as $n \to \infty$; see Régnier [9], Rösler [10]. Recently, some estimates for the rate were obtained by Fill and Janson [4], who roughly speaking get upper estimates $O(n^{-1/2})$ for the convergence in the minimal L_p-metrics ℓ_p, $p \geq 1$, and $O(n^{-1/2+\varepsilon})$ for the Kolmogorov metric for all $\varepsilon > 0$ as well as the lower estimates $\Omega(\ln(n)/n)$ for the ℓ_p metrics, $p \geq 2$, and $\Omega(1/n)$ for the Kolmogorov metric.

After presenting their results at "The Seventh Seminar on Analysis of Algorithms" on Tatihou in July, 2001, some indication was given at the meeting that $\Theta(\ln(n)/n)$ might be the right order of the rate of convergence for many metrics of interest. In this note we confirm this conjecture for the Zolotarev metric ζ_3. Since ζ_3 serves as an upper bound for several other distance measures this implies $\ln(n)/n$ bounds as well for some local metrics, for characteristic functions, and for weighted global metrics. For the proof we use a form of the contraction method as developed in Rachev and Rüschendorf [8] and Cramer and Rüschendorf [1]. We establish explicit estimates to identify the rate of convergence.

The paper is organized as follows: In this section we recall some known properties of the sequence (X_n), introduce the Zolotarev metric ζ_3, and state our main theorem, which is proved in section 2.

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In the last section implications of the ζ_3 convergence rate are drawn based on several inequalities between probability metrics.

The sequence of the number of key comparisons (X_n) needed by the Quicksort algorithm to sort an array of n randomly permuted items satisfies $X_0 = 0$ and the recursion

$$X_n \stackrel{\mathcal{D}}{=} X_{I_n} + X'_{n-1-I_n} + n - 1, \quad n \geq 1, \tag{1}$$

where $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution, $(X_k), (X'_k), I_n$ are independent, I_n is uniformly distributed on $\{0, \dots, n-1\}$, and $X_k \sim X'_k$, $k \geq 0$, where $\sim$ also denotes equality of distributions. The mean and variance of X_n are exactly known and satisfy

$$\mathbb{E}\, X_n = 2n \ln(n) + (2\gamma - 4)n + O(\ln(n)), \quad \mathrm{Var}(X_n) = \sigma^2 n^2 - 2n\ln(n) + O(n),$$

where γ denotes Euler's constant and $\sigma := \sqrt{7 - 2\pi^2/3} > 0$. We introduce the normalized quantities $Y_0 := 0$ and

$$Y_n := \frac{X_n - \mathbb{E}\, X_n}{n}, \quad n \geq 1,$$

which satisfy, see Régnier [9], Rösler [10], a limit law $Y_n \to Y$ in distribution as $n \to \infty$. Rösler [10] showed that Y satisfies the distributional fixed-point equation

$$Y \stackrel{\mathcal{D}}{=} UY + (1-U)Y' + g(U), \tag{2}$$

where Y, Y', U are independent, $Y \sim Y'$, U is uniform $[0,1]$ distributed, and $g(u) := 1 + 2u\ln(u) + 2(1-u)\ln(1-u)$, $u \in [0,1]$. Moreover this identity, subject to $\mathbb{E}\, Y = 0$, characterizes Y, and convergence and finiteness of the moment generating functions hold (see Rösler [10] and Fill and Janson [2]). We will use subsequently that $\mathrm{Var}(Y) = \sigma^2$ and $\|Y\|_3 < \infty$, where $\|Y\|_p := (\mathbb{E}\,|Y|^p)^{1/p}$, $1 \leq p < \infty$, denotes the L_p-norm.

The purpose of the present note is to estimate the rate of the convergence $Y_n \to Y$. Our basic distance is the Zolotarev metric ζ_3 given for distributions $\mathcal{L}(V), \mathcal{L}(W)$ by

$$\zeta_3(\mathcal{L}(V), \mathcal{L}(W)) := \sup_{f \in \mathcal{F}_3} |\mathbb{E}\, f(V) - \mathbb{E}\, f(W)|,$$

where $\mathcal{F}_3 := \{f \in C^2(\mathbb{R}, \mathbb{R}) : |f''(x) - f''(y)| \leq |x - y|\}$ is the space of all twice differentiable functions with second derivative being Lipschitz continuous with Lipschitz constant 1. We will use the short notation $\zeta_3(V, W) := \zeta_3(\mathcal{L}(V), \mathcal{L}(W))$. It is well known that convergence in ζ_3 implies weak convergence and that $\zeta_3(V, W) < \infty$ if $\mathbb{E}\, V = \mathbb{E}\, W$, $\mathbb{E}\, V^2 = \mathbb{E}\, W^2$, and $\|V\|_3, \|W\|_3 < \infty$. The metric ζ_3 is ideal of order 3, i.e., we have for T independent of (V, W) and $c \neq 0$

$$\zeta_3(V + T, W + T) \leq \zeta_3(V, W), \quad \zeta_3(cV, cW) = |c|^3 \zeta_3(V, W).$$

For general reference and properties of ζ_3 we refer to Zolotarev [12] and Rachev [6].

Our main result states:

Theorem 1.1 *The number of key comparisons (X_n) needed by the Quicksort algorithm to sort an array of n randomly permuted items satisfies*

$$\zeta_3\left(\frac{X_n - \mathbb{E}\, X_n}{\sqrt{\mathrm{Var}(X_n)}}, X\right) = \Theta\left(\frac{\ln(n)}{n}\right), \quad (n \to \infty),$$

where $X := Y/\sigma$ is a scaled version of the limiting distribution given in (2).

For related results with respect to other distance measures see section 3.

2 The proof

In the following lemma we state two simple bounds for the Zolotarev metric ζ_3, for which we do not claim originality. The upper bound involves the minimal L_3-metric ℓ_3 given by

$$\ell_p(\mathcal{L}(V), \mathcal{L}(W)) := \ell_p(V, W) := \inf\{\|\bar{V} - \bar{W}\|_p : \bar{V} \sim V, \bar{W} \sim W\}, \quad p \geq 1. \tag{3}$$

Lemma 2.1 *For V, W with identical first and second moment and $\|V\|_3, \|W\|_3 < \infty$, we have*

$$\frac{1}{6}\left|\mathbb{E}\, V^3 - \mathbb{E}\, W^3\right| \leq \zeta_3(V, W) \leq \frac{1}{2}\left(\|V\|_3^2 + \|V\|_3\|W\|_3 + \|W\|_3^2\right)\ell_3(V, W).$$

Proof: The left inequality follows from the fact that we have $f \in \mathcal{F}_3$ for $f(x) := x^3/6, x \in \mathbb{R}$. For the right inequality we use the estimate $\zeta_3(V, W) \leq (1/2)\kappa_3(V, W)$, see Zolotarev [11, p. 729], where κ_3 denotes the third difference pseudomoment, which has the representation (see Rachev [6, p. 271])

$$\kappa_3(V, W) = \inf\left\{\mathbb{E}\,|\bar{V}^3 - \bar{W}^3| : \bar{V} \sim V, \bar{W} \sim W\right\}.$$

From $\left|\bar{V}^3 - \bar{W}^3\right| = \left|\bar{V}^2 + \bar{V}\bar{W} + \bar{W}^2\right|\left|\bar{V} - \bar{W}\right|$ and Hölder's inequality we obtain

$$\begin{aligned}\mathbb{E}\,\left|\bar{V}^3 - \bar{W}^3\right| &\leq \left\|\bar{V}^2 + \bar{V}\bar{W} + \bar{W}^2\right\|_{3/2}\left\|\bar{V} - \bar{W}\right\|_3 \\ &\leq \left(\left\|\bar{V}\right\|_3^2 + \left\|\bar{V}\right\|_3\left\|\bar{W}\right\|_3 + \left\|\bar{W}\right\|_3^2\right)\left\|\bar{V} - \bar{W}\right\|_3.\end{aligned}$$

Taking the infimum we obtain the assertion. ■

Proof of Theorem 1.1: First we prove the easier lower bound, where only information on the moments of (X_n) is needed. Throughout we use constants $\sigma(n) \geq 0$ defined by

$$\sigma^2(n) := \mathrm{Var}(Y_n) = \sigma^2 - 2\frac{\ln(n)}{n} + O\left(\frac{1}{n}\right). \tag{4}$$

Lower bound: By Lemma 2.1 we have the basic estimate

$$\zeta_3\left(\frac{X_n - \mathbb{E}\, X_n}{\sqrt{\mathrm{Var}(X_n)}}, X\right) \geq \frac{1}{6}\left|\mathbb{E}\left(\frac{1}{\sigma(n)}Y_n\right)^3 - \mathbb{E}\left(\frac{1}{\sigma}Y\right)^3\right|.$$

The third moment of Y_n satisfies

$$\mathbb{E}\, Y_n^3 = \frac{1}{n^3}\,\mathbb{E}\,(X_n - \mathbb{E}\, X_n)^3 = \frac{1}{n^3}\varkappa_3(X_n) = M + O\left(\frac{1}{n}\right),$$

with $M = \mathbb{E}\, Y^3 = 16\zeta(3) - 19 > 0$, where we use the expansion of the third cumulant $\varkappa_3(X_n)$ of X_n given by Hennequin [5, p. 136]. From (4) we obtain

$$\frac{1}{\sigma^3(n)} = \frac{1}{\sigma^3} + \frac{3}{\sigma^5}\frac{\ln(n)}{n} + O\left(\frac{1}{n}\right),$$

thus

$$\frac{1}{6}\left|\mathbb{E}\left(\frac{1}{\sigma(n)}Y_n\right)^3 - \mathbb{E}\left(\frac{1}{\sigma}Y\right)^3\right| = \frac{M}{2\sigma^5}\frac{\ln(n)}{n} + O\left(\frac{1}{n}\right),$$

which gives the lower estimate of the theorem.

Upper bound: The scaled variates Y_n satisfy the modified recursion

$$Y_n \stackrel{\mathcal{D}}{=} \frac{I_n}{n} Y_{I_n} + \frac{n-1-I_n}{n} Y'_{n-1-I_n} + g_n(I_n), \quad n \geq 1, \tag{5}$$

where, as in (1), $(Y_k), (Y'_k), I_n$ are independent, $Y_k \sim Y'_k$ for all $k \geq 0$, and

$$g_n(k) := \frac{1}{n} \left(\mu(k) + \mu(n-1-k) - \mu(n) + n - 1 \right),$$

with $\mu(n) := \mathbb{E}\, X_n$, $n \geq 0$. Furthermore, we define $Z_0 := Z'_0 := 0$ and

$$Z_n := \frac{\sigma(n)}{\sigma} Y, \quad Z'_n := \frac{\sigma(n)}{\sigma} Y', \quad n \geq 1,$$

where Y, Y' are independent copies of the limit distribution also independent of I_n. Finally, we define the accompanying sequence (Z^*_n) by $Z^*_0 := 0$,

$$Z^*_n \stackrel{\mathcal{D}}{:=} \frac{I_n}{n} Z_{I_n} + \frac{n-1-I_n}{n} Z'_{n-1-I_n} + g_n(I_n), \quad n \geq 1. \tag{6}$$

Note that Y_n, Z_n, Z^*_n have identical first and second moment and finite third absolute moment for all $n \geq 0$, thus ζ_3-distances between these quantities are finite. We will show

$$\zeta_3(Y_n, Z_n) = O\left(\frac{\ln(n)}{n} \right). \tag{7}$$

From this estimate the upper bound follows immediately since we have $(X_n - \mathbb{E}\, X_n)/\sqrt{\mathrm{Var}(X_n)} = Y_n/\sigma(n)$, $X \sim Z_n/\sigma(n)$, and therefore

$$\zeta_3\left(\frac{X_n - \mathbb{E}\, X_n}{\sqrt{\mathrm{Var}(X_n)}}, X \right) = \frac{1}{\sigma^3(n)} \zeta_3(Y_n, Z_n) = O\left(\frac{\ln(n)}{n} \right),$$

since $(\sigma(n))$ has a nonzero limit.

For the proof of (7) we use the triangle inequality:

$$\zeta_3(Y_n, Z_n) \leq \zeta_3(Y_n, Z^*_n) + \zeta_3(Z^*_n, Z_n). \tag{8}$$

To estimate the first summand note that for any random variables V, W, T we obtain $|\mathbb{E}\, f(V) - \mathbb{E}\, f(W)| \leq \mathbb{E}\, |\mathbb{E}\,(f(V) \mid T) - \mathbb{E}\,(f(W) \mid T)|$ and that for (V, W) independent of (S, T) we have $\zeta_3(V+S, W+T) \leq \zeta_3(V, W) + \zeta_3(S, T)$. This implies using (5),(6), that ζ_3 is ideal of order 3, and conditioning on I_n,

$$\begin{aligned}
&\zeta_3(Y_n, Z^*_n) \\
&\quad \leq \sum_{k=0}^{n-1} \frac{1}{n} \zeta_3\left(\frac{k}{n} Y_k + \frac{n-1-k}{n} Y'_{n-1-k} + g_n(k), \frac{k}{n} Z_k + \frac{n-1-k}{n} Z'_{n-1-k} + g_n(k) \right) \\
&\quad \leq \sum_{k=0}^{n-1} \frac{1}{n} \left(\zeta_3\left(\frac{k}{n} Y_k, \frac{k}{n} Z_k \right) + \zeta_3\left(\frac{n-1-k}{n} Y'_{n-1-k}, \frac{n-1-k}{n} Z'_{n-1-k} \right) \right) \\
&\quad = \sum_{k=0}^{n-1} \frac{1}{n} \left(\left(\frac{k}{n} \right)^3 \zeta_3(Y_k, Z_k) + \left(\frac{n-1-k}{n} \right)^3 \zeta_3(Y_{n-1-k}, Z_{n-1-k}) \right) \\
&\quad = \frac{2}{n} \sum_{k=1}^{n-1} \left(\frac{k}{n} \right)^3 \zeta_3(Y_k, Z_k).
\end{aligned} \tag{9}$$

We will show below that $\zeta_3(Z_n^*, Z_n) = O(\ln(n)/n)$. Thus (noting that $\zeta_3(Z_1^*, Z_1) = 0$) there exists a constant $c > 0$ with

$$\zeta_3(Z_n^*, Z_n) \leq c\frac{\ln(n)}{n}, \quad n \geq 1. \tag{10}$$

Then we prove (7) by induction using the constant c from (10):

$$\zeta_3(Y_n, Z_n) \leq 3c\frac{\ln(n)}{n}, \quad n \geq 1. \tag{11}$$

Assertion (11) holds for $n = 1$. With (8),(9),(10) and the induction hypothesis we obtain

$$\begin{aligned}
\zeta_3(Y_n, Z_n) &\leq \frac{2}{n}\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^3 3c\frac{\ln(k)}{k} + c\frac{\ln(n)}{n} \\
&\leq 6c\frac{\ln(n)}{n}\sum_{k=1}^{n-1}\frac{k^2}{n^3} + c\frac{\ln(n)}{n} \\
&\leq \frac{\ln(n)}{n}\left(6c\frac{1}{3} + c\right) \\
&= 3c\frac{\ln(n)}{n}.
\end{aligned}$$

The proof is completed by showing (10): Since Y has a finite third absolute moment and $(\sigma(n))$ is bounded, we obtain that the third absolute moments of $(Z_n), (Z_n^*)$ are uniformly bounded, thus by Lemma 2.1 there exists a constant $L > 0$ with

$$\zeta_3(Z_n^*, Z_n) \leq L\ell_3(Z_n^*, Z_n), \quad n \geq 1. \tag{12}$$

By definition of Z_n and the fixed-point property of Y we obtain the relation

$$Z_n \stackrel{\mathcal{D}}{=} UZ_n + (1-U)Z_n' + \frac{\sigma(n)}{\sigma}g(U), \tag{13}$$

with U independent of (Z_n, Z_n') and U uniform $[0,1]$ distributed. We may choose $I_n = \lfloor nU \rfloor$; hence it holds that $|I_n/n - U| \leq 1/n$ pointwise. Replacing Z_n, Z_n^* by their representations (13) and (6) respectively we have

$$\begin{aligned}
&\ell_3(Z_n, Z_n^*) \\
&\leq \left\| \frac{I_n}{n}Z_{I_n} + \frac{n-1-I_n}{n}Z'_{n-1-I_n} + g_n(I_n) - \left(UZ_n + (1-U)Z_n' + \frac{\sigma(n)}{\sigma}g(U)\right)\right\|_3 \\
&\leq \left\|\frac{I_n}{n}Z_{I_n} - UZ_n\right\|_3 + \left\|\frac{n-1-I_n}{n}Z'_{n-1-I_n} - (1-U)Z_n'\right\|_3 + \left\|g_n(I_n) - \frac{\sigma(n)}{\sigma}g(U)\right\|_3.
\end{aligned} \tag{14}$$

The first and second summand are identical. We have

$$\left\|\frac{I_n}{n}Z_{I_n} - UZ_n\right\|_3 = \left\|\frac{I_n}{n}\frac{\sigma(I_n)}{\sigma}Y - \frac{\sigma(n)}{\sigma}UY\right\|_3 = \frac{\|Y\|_3}{\sigma}\left\|\sigma(I_n)\frac{I_n}{n} - \sigma(n)U\right\|_3$$

and

$$\left\|\sigma(I_n)\frac{I_n}{n} - \sigma(n)U\right\|_3 \leq \left\|(\sigma(I_n) - \sigma(n))\frac{I_n}{n}\right\|_3 + \sigma(n)\left\|\frac{I_n}{n} - U\right\|_3. \tag{15}$$

The second summand in (15) is $O(1/n)$ since $(\sigma(n))$ is bounded and $|I_n/n - U| \leq 1/n$. For the estimate of the first summand we use

$$\sigma^2(n) = \sigma^2 + R(n), \quad R(n) = O\left(\frac{\ln(n)}{n}\right),$$

and obtain for n sufficiently large such that $\sigma(n) \geq \sigma/2 > 0$

$$\begin{aligned} \left\| (\sigma(I_n) - \sigma(n)) \frac{I_n}{n} \right\|_3 &= \left\| \left(\sigma^2(I_n) - \sigma^2(n)\right) \frac{I_n}{n} \frac{1}{\sigma(n) + \sigma(I_n)} \right\|_3 \\ &\leq \frac{2}{\sigma} \left\| \left(\sigma^2(I_n) - \sigma^2(n)\right) \frac{I_n}{n} \right\|_3 \\ &= \frac{2}{n\sigma} \left\| I_n \left(\sigma^2 + R(I_n) - \sigma^2 - R(n)\right) \right\|_3 \\ &= O\left(\frac{\ln(n)}{n}\right). \end{aligned}$$

For the proof of the latter equality we use the triangle inequality for the L_3-norm as well as the finiteness of $\|\ln U\|_3$. This gives the $O(\ln(n)/n)$ bounds for the first and second summand in (14). The third summand in (14) is estimated by

$$\left\| g_n(I_n) - \frac{\sigma(n)}{\sigma} g(U) \right\|_3 \leq \|g_n(I_n) - g(U)\|_3 + \left| 1 - \frac{\sigma(n)}{\sigma} \right| \|g(U)\|_3.$$

We have $\|g_n(I_n) - g(U)\|_3 = O(\ln(n)/n)$ since the maximum norm satisfies $\|g_n(I_n) - g(U)\|_\infty = O(\ln(n)/n)$, see, e.g., Rösler [10, Prop. 3.2]. Finally, $\|g(U)\|_3 < \infty$ since $g(U)$ is bounded and

$$\left| 1 - \frac{\sigma(n)}{\sigma} \right| \leq \left| 1 - \frac{\sigma^2(n)}{\sigma^2} \right| = \frac{2}{\sigma^2} \frac{\ln(n)}{n} + O\left(\frac{1}{n}\right).$$

Thus we have $\ell_3(Z_n^*, Z_n) = O(\ln(n)/n)$ which by (12) implies $\zeta_3(Z_n^*, Z_n) = O(\ln(n)/n)$. ■

3 Related distances

In the following we compare several further distances to ζ_3 and obtain similar convergence rates for these distances. We denote the normalized version of X_n by

$$\widetilde{X}_n := \frac{X_n - \mathbb{E}\, X_n}{\sqrt{\mathrm{Var}(X_n)}}, \quad n \geq 3,$$

and X as in Theorem 1.1. Furthermore let $C > 0$ be a constant such that, by Theorem 1.1, $\zeta_3(\widetilde{X}_n, X) \leq C \ln(n)/n$ for $n \geq 3$.

3.1 Density approximation

Let ϑ be a random variable with support on $[0, 1]$ or $[-1/2, 1/2]$ and with a density f_ϑ being three times differentiable on the real line and suppose

$$C_{\vartheta,3} := \sup_{x \in \mathbb{R}} |f_\vartheta^{(3)}(x)| < \infty.$$

For random variables V, W with densities f_V, f_W let the sup-metric ℓ of the densities be denoted by

$$\ell(V,W) := \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f_V(x) - f_W(x)|.$$

For any distributions of V and W, the random variables $V + h\vartheta$ and $W + h\vartheta$ have densities with bounded third derivative. The smoothed sup-metric

$$\mu_{\vartheta,4}(V,W) := \sup_{h\in\mathbb{R}} |h|^4 \ell(V + h\vartheta, W + h\vartheta),$$

with ϑ independent of V, W, is ideal of order 3 and

$$\mu_{\vartheta,4}(V,W) \le C_{\vartheta,3}\zeta_3(V,W),$$

see Rachev [6, p. 269]. Therefore, from Theorem 1.1 we obtain the estimate

$$\mu_{\vartheta,4}(\widetilde{X}_n, X) \le CC_{\vartheta,3}\frac{\ln(n)}{n}, \quad n \ge 3.$$

This implies the following local approximation results for the densities of the smoothed random variates:

Corollary 3.1 *For any sequence (h_n) of positive numbers and any $n \ge 3$ we have*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}} \left| f_{\widetilde{X}_n + h_n\vartheta}(x) - f_{X+h_n\vartheta}(x)\right| \le CC_{\vartheta,3}\frac{\ln(n)}{nh_n^4}.$$

In particular for $h_n \equiv 1$ we obtain an $\ln(n)/n$ approximation bound.

For a related approximation result for the density f_X see Theorem 6.1 in Fill and Janson [4].

A global density approximation result holds in the following form. Assume

$$\bar{C}_{\vartheta,2} := \left\| f_\vartheta^{(2)} \right\|_1 := \int_{-\infty}^{\infty} \left| f_\vartheta^{(2)}(x)\right| dx < \infty \tag{16}$$

for some random variable ϑ with density f_ϑ twice differentiable on the line and with support of length bounded by one, which is independent of $\widetilde{X}_n, X$. Then the following holds:

Corollary 3.2 *For any sequence (h_n) of positive numbers and any $n \ge 3$ we have*

$$\left\| f_{\widetilde{X}_n + h_n\vartheta} - f_{X+h_n\vartheta}\right\|_1 \le C\bar{C}_{\vartheta,2}\frac{\ln(n)}{nh_n^3}. \tag{17}$$

Proof: Consider the smoothed total variation metric

$$\nu_{\vartheta,3}(V,W) := \sup_{h\in\mathbb{R}} |h|^3 \left\| f_{V+h\vartheta} - f_{W+h\vartheta}\right\|_1,$$

with ϑ independent of V, W, which is a probability metric, ideal of order 3, satisfying $\nu_{\vartheta,3}(V,W) \le \bar{C}_{\vartheta,2}\zeta_3(V,W)$, see Rachev [6, p. 269]. Therefore, Theorem 1.1 implies the estimate (17). ■

In particular, we obtain an $\ln(n)/n$ convergence rate for $h_n \equiv 1$. Note that the left-hand side of (17) is the total variation distance between the smoothed variables $\widetilde{X}_n + h_n\vartheta, X + h_n\vartheta$.

3.2 Characteristic function distances

For a random variable V denote by $\phi_V(t) := \mathbb{E}\exp(itV), t \in \mathbb{R}$, its characteristic function and by

$$\chi(V,W) := \sup_{t\in\mathbb{R}} |\phi_V(t) - \phi_W(t)|$$

the uniform distance between characteristic functions. We obtain the following approximation result.

Corollary 3.3 *For all $t \in \mathbb{R}$ and for any $n \geq 3$ we have*

$$\left|\phi_{\widetilde{X}_n}(t) - \phi_X(t)\right| \leq Ct^3 \frac{\ln(n)}{n}. \qquad (18)$$

Proof: We define the weighted χ-metric χ_3 by

$$\chi_3(V,W) := \sup_{t\in\mathbb{R}} |t|^{-3} |\phi_V(t) - \phi_W(t)|.$$

Then χ_3 is a probability metric, ideal of order 3, satisfying $\chi_3 \leq \zeta_3$, see Rachev [6, p. 279]. Therefore, (18) follows from Theorem 1.1. ■

3.3 Approximation of distribution functions

In this section we consider the local and global approximation of the (smoothed) distribution functions. We denote by F_V the distribution function of a random variable V. Note that for integrable V, W we have the well-known representation of the ℓ_1-metric as defined in (3) due to Dall'Aglio (see Rachev [6, p. 153])

$$\ell_1(V,W) = \|F_V - F_W\|_1.$$

The Kolmogorov metric is denoted by

$$\varrho(V,W) := \sup_{x\in\mathbb{R}} |F_V(x) - F_W(x)|.$$

Let ϑ be a random variate, independent of $\widetilde{X}_n, X$, with density f_ϑ twice continuously differentiable and support of length bounded by one, and $\bar{C}_{\vartheta,2}$ as in (16). It is known that X has a bounded density, see Fill and Janson [3]. We obtain:

Corollary 3.4 *For any sequence (h_n) of positive numbers we have for any $n \geq 3$*

$$\ell_1(\widetilde{X}_n + h_n\vartheta, X + h_n\vartheta) \leq C\bar{C}_{\vartheta,2}\frac{\ln(n)}{nh_n^2}, \qquad (19)$$

$$\varrho(\widetilde{X}_n + h_n\vartheta, X + h_n\vartheta) \leq C\bar{C}_{\vartheta,2}\left(1 + \|f_X\|_\infty\right)\frac{\ln(n)}{nh_n^2}. \qquad (20)$$

Proof: Note that $\zeta_1 = \ell_1$ by the classical Kantorovich-Rubinstein duality theorem (see Rachev [6, p. 109]). Furthermore, between $\zeta_1 = \ell_1$ and ζ_3 we have the relation

$$\zeta_1(V + \vartheta, W + \vartheta) \leq \bar{C}_{\vartheta,2}\zeta_3(V,W),$$

see Zolotarev [12, Theorem 5], if V, W have identical first and second moments. This implies that for all $h \neq 0$

$$\ell_1(V + h\vartheta, W + h\vartheta) \leq \bar{C}_{h\vartheta,2}\zeta_3(V, W) = \frac{\bar{C}_{\vartheta,2}}{h^2}\zeta_3(V, W). \tag{21}$$

The inequality in (21) implies that the smoothed ℓ_1 metric

$$\bar{\ell_1}^{(2)}(V, W) := \sup_{h \in \mathbb{R}} |h|^2 \ell_1(V + h\vartheta, W + h\vartheta)$$

is bounded from above by $\bar{\ell_1}^{(2)}(V, W) \leq \bar{C}_{\vartheta,2}\zeta_3(V, W)$. With Theorem 1.1 this implies (19).

For the proof of (20) first note that $\|f_{X+h\vartheta}\|_\infty \leq \|f_X\|_\infty < \infty$ for all $h \neq 0$. With the stop loss metric

$$\mathrm{d}_1(V, W) := \sup_{t \in \mathbb{R}} \left| \mathbb{E}\,(V - t)^+ - \mathbb{E}\,(W - t)^+ \right|$$

we obtain from Rachev and Rüschendorf [7, (2.30),(2.26)] and Rachev [6, p. 325]

$$\begin{aligned} \varrho(X_n + h\vartheta, X + h\vartheta) &\leq (1 + \|f_X\|_\infty)\,\mathrm{d}_1(X_n + h\vartheta, X + h\vartheta) \\ &\leq \bar{C}_{h\vartheta,2}\,(1 + \|f_X\|_\infty)\,\zeta_3(X_n, X) \\ &= \frac{\bar{C}_{\vartheta,2}}{h^2}\,(1 + \|f_X\|_\infty)\,\zeta_3(X_n, X), \end{aligned}$$

which implies the assertion. ■

Concluding remark

Our results indicate that $\ln(n)/n$ is the relevant rate for the convergence $Y_n \to Y$ for several natural distances. We do however have no argument to decide the order of the rate of convergence in the Kolmogorov metric $\varrho(Y_n, Y)$ (without smoothing) nor in the ℓ_p-metrics as considered in Fill and Janson [4].

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