On Binary Search Tree Recursions with Monomials as Toll Functions

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Abstract

We consider distributional recursions which appear in the study of random binary search trees with monomials as toll functions. This extends classical parameters as the internal path length in binary search trees. As our main results we derive asymptotic expansions for the moments of the random variables under consideration as well as limit laws and properties of the densities of the limit distributions. The analysis is based on the contraction method.

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1 Introduction

We consider a sequence (X_n) of random variables with distributions given by $X_0 = 0$ and the recursion

$$X_n \stackrel{d}{=} X_{\lfloor nU \rfloor} + X^*_{\lfloor n(1-U) \rfloor} + t_n, \quad n \geq 1, \tag{1}$$

with (X_n), (X^*_n), U being independent, (X^*_n) being distributed as (X_n), and U a uniform $[0,1]$ distributed random variable. The symbol $\stackrel{d}{=}$ denotes equality of distributions. Throughout this work we assume monomials $t_n = n^\alpha$ as toll functions with $\alpha \in \mathbb{R}$ and $\alpha > 1$.

For the special choice $t_n = n - 1$ the X_n are distributed as the internal path length in random binary search trees. By a well-known equivalence this is also the number of key comparisons needed by Hoare's sorting algorithm `Quicksort` to sort a list of n randomly permuted items.

In the context of random search trees it is a common phenomenon that different parameters of the same tree satisfy distributional recursions of type (1) which only differ in the toll function t_n. Typically, the branching factor of the tree is reflected in the number of independent copies of the parameter on the right side of the equation (here in (1) these are the two sequences (X_n) and (X^*_n)), the splitting procedure settles the random indices of these sequences, and the special parameter under consideration determines the toll function; see, e.g., Devroye [2] for a list of random search trees fitting in this scheme.

The aim of this note is twofold. First we study the asymptotic behavior of the moments and distributions of X_n for our toll functions n^α. The investigation of (1) with non-standard toll functions was recently started by Panholzer and Prodinger [6] who considered the harmonic toll function $t_n = H_n := \sum_{i=1}^n 1/i$. Their study was motivated by the occurrence of a logarithmic toll function in Grabner and Prodinger [4]. It is our second intention to add a further example to the list of applications due to the contraction method which is applied in our analysis.

The contraction method was introduced by Rösler [8] for the distributional analysis of the `Quicksort` algorithm, i.e. our recursion (1) with $t_n = n - 1$. This method was further developed independently in Rösler [9] and Rachev and Rüschendorf [7], and later on in Rösler [10]. A survey of the method including the major applications is given in Rösler and Rüschendorf [11].

Characteristic for recursion (1) from the point of view of the contraction method is that mean and standard deviation of X_n are of the same order of magnitude. As long as we make use of the minimal L_2-metric ℓ_2 this implies that only knowledge of the leading term in the expansion of the mean is required in order to derive weak convergence for the scaled versions of X_n. This is in contrast to the `Quicksort` case $\alpha = 1$ where mean and standard deviation are of different orders of magnitude and the knowledge of the second term in the expansion of the mean is necessary; see [5] for a discussion of this problem in the context of the internal path length in random split trees. Note that the limit distributions for the problems considered in this work are determined by a type of fixed-point equation which has not so far appeared in other applications.

We proceed as follows: In the second section we derive the dominant term in the expansion of the mean of X_n. The third section gives the limit law for (X_n) by the approach of the contraction method. In the fourth section first order expansions for the variance and higher moments of X_n and information on the Laplace transform as well as tail estimates are derived. In the last section it is proved by arguments of Fill and Janson [3] that the limit distribution has a density which belongs to the class of rapidly decreasing C^∞ functions.

We denote by ℓ_2 the minimal L_2-metric acting on the space of probability distributions with finite second moment (see [1]). Convergence in the ℓ_2-metric is equivalent to weak convergence plus convergence of the second moments. We write also $\ell_2(X,Y) := \ell_2(L(X), L(Y))$ for random variables X, Y with laws $L(X)$, $L(Y)$.

2 Expectations

In our subsequent distributional analysis it turns out that the knowledge of the dominant term in the expansion of the mean is sufficient in order to obtain a limit law for (X_n). This leading term can be explored by well-known elementary methods. We denote $a_n := \mathbb{E}\, X_n$. The random indices in (1) are uniformly distributed on $\{0, \dots, n-1\}$. Thus, (1) implies

$$a_n = n^\alpha + \frac{2}{n} \sum_{i=0}^{n-1} a_i, \quad n \geq 1,$$

with initializing value $a_0 = 0$. This implies for $n \geq 1$

$$n a_n = n^{\alpha+1} + 2 \sum_{i=0}^{n-1} a_i \quad \text{and} \quad (n-1) a_{n-1} = (n-1)^{\alpha+1} + 2 \sum_{i=0}^{n-2} a_i.$$

Subtracting these two relations and using the expansion

$$(n-1)^{\alpha+1} = n^{\alpha+1} - (\alpha+1)n^{\alpha} + O(n^{\alpha-1}) \tag{2}$$

we deduce $na_n - (n+1)a_{n-1} = (\alpha+1)n^{\alpha} + O(n^{\alpha-1})$. This implies

$$\begin{aligned} a_n &= \frac{n+1}{n}a_{n-1} + (\alpha+1)n^{\alpha-1} + O(n^{\alpha-2}) \\ &= \sum_{i=0}^{n-1} \frac{n+1}{n+1-i}\left((\alpha+1)(n-i)^{\alpha-1} + O((n-i)^{\alpha-2})\right) \\ &= (n+1)\left((\alpha+1)\frac{1}{\alpha-1}n^{\alpha-1} + o(n^{\alpha-1}) + O(n^{\alpha-2})\right) \\ &= \frac{\alpha+1}{\alpha-1}n^{\alpha} + o(n^{\alpha}). \end{aligned} \tag{3}$$

For resolving the sum in (3) we used the estimate

$$\begin{aligned} \sum_{i=1}^{n} \frac{i^{\alpha-1}}{i+1} &= \sum_{i=1}^{n}\left(1 - \frac{1}{i+1}\right) i^{\alpha-2} \\ &= \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{\alpha-2}\right] n^{\alpha-1} + O(n^{\alpha-2}) \\ &= \left(\frac{1}{\alpha-1} + o(1)\right) n^{\alpha-1} + O(n^{\alpha-2}), \end{aligned}$$

where the Riemann integral $\int_0^1 x^{\alpha-2}\,dx$ is convergent due to our general assumption $\alpha > 1$. Using more terms in the expansion (2) may give a refined asymptotic expansion for a_n. For example, for $\alpha = 2, 3, 4$ we get the exact expressions

$$\begin{aligned} a_n &= 3n^2 - 6nH_n + 10n - 6H_n \quad \text{for} \quad \alpha = 2, \\ a_n &= 2n^3 - 6n^2 + 14nH_n - 23n + 14H_n, \quad \text{for} \quad \alpha = 3, \\ a_n &= \frac{5}{3}n^4 - \frac{10}{3}n^3 + \frac{40}{3}n^2 - 30nH_n + \frac{148}{3}n - 30H_n, \quad \text{for} \quad \alpha = 4. \end{aligned}$$

Using an expansion of H_n leads to asymptotic expressions for the a_n.

For our further probabilistic analysis we will only need the first order growth of (a_n):

Lemma 2.1 *The mean of the sequence* (X_n) *given in (1) with* $t_n = n^{\alpha}$, $\alpha > 1$, *satisfies*

$$\mathbb{E}\, X_n = a_n \sim \frac{\alpha+1}{\alpha-1}\, n^{\alpha} \quad \textit{as} \quad n \to \infty. \tag{4}$$

3 Limit Laws

We will show later in Theorem 4.2 that the variance $\mathrm{Var}\ X_n$ admits an expansion

$$\mathrm{Var}\ X_n \sim v n^{2\alpha},$$

with some constant $v > 0$ depending on α. Therefore mean and standard deviation are of the same order of magnitude. Thus, in order to derive a limit law for X_n we could scale by

$$Y_n := \frac{X_n}{n^\alpha} \quad \text{or} \quad Z_n := \frac{X_n - \mathbb{E}\, X_n}{n^\alpha} \tag{5}$$

and expect that weak limits Y, Z of (Y_n) and (Z_n) respectively satisfy $\mathbb{E}\, Y = (\alpha+1)/(\alpha-1)$ and $\mathbb{E}\, Z = 0$. For technical reasons we will use both sequences (Z_n), (Y_n) in our analysis. Our original recursion (1) modifies for the scaled quantities to

$$\begin{aligned} Z_n &\stackrel{d}{=} \left(\frac{\lfloor nU \rfloor}{n}\right)^\alpha Z_{\lfloor nU \rfloor} + \left(\frac{\lfloor n(1-U) \rfloor}{n}\right)^\alpha Z^*_{\lfloor n(1-U) \rfloor} \\ &\quad + \frac{1}{n^\alpha}\left(a_{\lfloor nU \rfloor} + a_{\lfloor n(1-U) \rfloor} + n^\alpha - a_n\right) \qquad (6) \\ &= \left(\frac{\lfloor nU \rfloor}{n}\right)^\alpha Z_{\lfloor nU \rfloor} + \left(\frac{\lfloor n(1-U) \rfloor}{n}\right)^\alpha Z^*_{\lfloor n(1-U) \rfloor} \\ &\quad + \frac{\alpha+1}{\alpha-1}\left(U^\alpha + (1-U)^\alpha\right) - \frac{2}{\alpha-1} + o(1), \qquad (7) \end{aligned}$$

where the expansion (4) is used and again (Z_n), (Z^*_n), U are independent, (Z^*_n) is distributed as (Z_n), and U is uniform $[0,1]$ distributed. The $o(1)$ depends on randomness but the convergence is uniform. From this modified recursion one can guess a limiting form by looking for stabilization for $n \to \infty$. This suggests that a limit Z of (Z_n) should satisfy the fixed-point equation

$$Z \stackrel{d}{=} U^\alpha Z + (1-U)^\alpha Z^* + \frac{\alpha+1}{\alpha-1}\left(U^\alpha + (1-U)^\alpha\right) - \frac{2}{\alpha-1}, \tag{8}$$

with Z, Z^*, U being independent, Z, Z^* identically distributed and U uniformly on $[0,1]$ distributed. The translated version $Y = Z + (\alpha+1)/(\alpha-1)$ then solves the simpler fixed-point equation

$$Y \stackrel{d}{=} U^\alpha Y + (1-U)^\alpha Y^* + 1, \tag{9}$$

with relations analogous to (8). According to the idea of the contraction method the limits Z of (Z_n) and Y of (Y_n) should be characterized as the unique solutions of (8), (9) respectively subject to the constraints $\mathbb{E}\, Z = 0$ and $\mathrm{Var}\, Z < \infty$, and — for the translated case — $\mathbb{E}\, Y = (\alpha+1)/(\alpha-1)$ and $\mathrm{Var}\, Y < \infty$. For the proof of the uniqueness of such solutions and the weak convergence we can appeal to general theorems ([9, 10]), due to the standard form of our recursion.

Theorem 3.1 *Let (X_n) be given by (1) with $t_n = n^\alpha$, $\alpha > 1$. The fixed-point equation (8) has a unique distributional solution Z subject to $\mathbb{E}\, Z = 0$ and Var $Z < \infty$ and it holds the limit law*

$$\ell_2\left(\frac{X_n - \mathbb{E}\, X_n}{n^\alpha}, Z\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Proof: For the uniqueness of the fixed-point we apply Theorem 3 in [9]. The T_1, T_2, C occurring there are given here by

$$T_1 := U^\alpha, \quad T_2 := (1-U)^\alpha, \quad C := \frac{\alpha+1}{\alpha-1}\left(U^\alpha + (1-U)^\alpha\right) - \frac{2}{\alpha-1}.$$

It is

$$\mathbb{E}\sum_{i=1}^{2} T_i^2 = \frac{2}{2\alpha+1} < 1, \quad \mathbb{E}\,C^2 < \infty, \quad \text{and}$$
$$\mathbb{E}\,C = \frac{\alpha+1}{\alpha-1}\left(\frac{1}{\alpha+1} + \frac{1}{\alpha+1}\right) - \frac{2}{\alpha-1} = 0.$$

Thus the conditions of Rösler's theorem are satisfied and it follows that (8) has a unique distributional fixed-point in the space of centered probability distributions with finite second moment.

For the ℓ_2-convergence we apply Theorem 3 in [10]. The Z_1^n, Z_2^n, T_1^n, T_2^n, C^n occurring there are given here by

$$Z_1^n = \lfloor nU \rfloor, \quad Z_2^n = \lfloor n(1-U) \rfloor, \quad T_1^n = \left(\frac{\lfloor nU \rfloor}{n}\right)^{\alpha}, \quad \text{and}$$
$$T_2^n = \left(\frac{\lfloor n(1-U) \rfloor}{n}\right)^{\alpha}, \quad C^n = \frac{1}{n^\alpha}\left(a_{\lfloor nU \rfloor} + a_{\lfloor n(1-U) \rfloor} + n^\alpha - a_n\right). \tag{10}$$

We check the conditions of the theorem: That $\mathbb{E}\,C^n = 0$ holds follows by taking expectations in (6) and noting that the Z_i, Z_i^* there are centered. For any $n_1 \in \mathbb{N}$ we have

$$\begin{aligned}
&\sum_{i=1}^{2} \mathbb{E}\left[\mathbf{1}_{\{Z_i^n \le n_1\}} (T_i^n)^2\right] \\
&\quad = \sum_{j=0}^{n_1} \Big(\mathbb{P}(\lfloor nU \rfloor = j) + \mathbb{P}(\lfloor n(1-U) \rfloor = j)\Big) \left(\frac{j}{n}\right)^{2\alpha} \\
&\quad \le 2\mathbb{P}\left(U < \frac{n_1+1}{n}\right) \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

which is condition (21) in the cited theorem. Furthermore, it holds

$$\begin{aligned}
\ell_2^2(L(C^n, T^n), L(C, T)) &\le \mathbb{E}\,(C^n - C)^2 + \mathbb{E}\,(T_1^n - T_1)^2 + \mathbb{E}\,(T_2^n - T_2)^2 \\
&\le \mathbb{E}\,[o(1)^2] + 2\left(\frac{\alpha}{n}\right)^2 \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

where $o(1)$ is the uniformly converging $o(1)$ in (7). Now, Rösler's theorem implies convergence in the ℓ_2-metric. ∎

4 Higher moments and Laplace transforms

Similarly to Theorem 3.1, ℓ_2-convergence of (Y_n) to Y holds, where Y is the unique distributional fixed-point in (9) subject to $\mathbb{E}\,Y = (\alpha+1)/(\alpha-1)$ and $\mathrm{Var}\,Y < \infty$. Convergence in the ℓ_2-metric induces convergence of the second moments. This implies

$$\mathrm{Var}\,Y_n \to \mathrm{Var}\,Y \quad \text{and}$$
$$\mathrm{Var}\,X_n = \mathrm{Var}(n^\alpha Y_n) \sim \mathrm{Var}(Y)\,n^{2\alpha}.$$

The leading constant Var Y can be obtained form the fixed-point equation (9). We can also pump higher order moments of Y from the fixed-point equation. This implies asymptotic expansions for the moments of X_n as soon as we know that convergence of the moments of higher order of (Y_n) holds. This can be shown by analyzing the Laplace transforms of Z_n and Z. For this we apply the tools developed in Lemma 4.1 and Theorem 4.2 in [8].

Theorem 4.1 *The scaled sequence (Z_n) given in (5) and the fixed-point Z of Theorem 3.1 satisfy for all $\lambda \in \mathbb{R}$*

$$\mathbb{E} \exp(\lambda Z_n) \to \mathbb{E} \exp(\lambda Z) < \infty \quad \textit{as} \quad n \to \infty.$$

Proof: In place of the random variable U_n in the proof of Lemma 4.1 in [8] we use

$$V_n := \left(\frac{\lfloor nU \rfloor}{n}\right)^{2\alpha} + \left(\frac{\lfloor n(1-U) \rfloor}{n}\right)^{2\alpha} - 1.$$

Then with C^n given by (10) it holds

$$\forall\, n \in \mathbb{N} : -1 \leq V_n < 0, \tag{11}$$
$$\sup_{n\in\mathbb{N}} \mathbb{E}\, V_n < 0, \tag{12}$$
$$\sup_{n\in\mathbb{N}} \|C^n\|_\infty < \infty. \tag{13}$$

The proof of (12) follows from $\mathbb{E}\, V_n < 0$ for all $n \in \mathbb{N}$ and from the convergence of the means, $\mathbb{E}\, V_n \to \mathbb{E}\,[U^{2\alpha} + (1-U)^{2\alpha} - 1] = 2/(2\alpha+1) - 1 < 0$. Relation (13) follows from the representation of C^n given in (7). Now, using (11)-(13) we can conclude as in Lemma 4.1 and Theorem 4.2 in [8] which leads to our assertion. ■

The convergence of the Laplace transform implies convergence of moments of arbitrary order. We can also deduce tail estimates from this convergence. Obviously, we do only have a right tail. Using Markov's inequality and $\mathbb{E}\, X_n = (\alpha+1)/(\alpha-1)n^\alpha + d_n$ with $d_n = o(n^\alpha)$ we derive

$$\begin{aligned}
&\mathbb{P}(X_n \geq b_n) \\
&\quad = \mathbb{P}\left(\exp\left(\lambda \frac{X_n - \mathbb{E}\, X_n}{n^\alpha}\right) \geq \exp\left(\lambda\left(\frac{b_n}{n^\alpha} - \frac{\alpha+1}{\alpha-1} + \frac{d_n}{n^\alpha}\right)\right)\right) \\
&\quad \leq \mathbb{E} \exp(\lambda Z_n) \exp\left(-\lambda\left(\frac{b_n}{n^\alpha} - \frac{\alpha+1}{\alpha-1} + \frac{d_n}{n^\alpha}\right)\right) \\
&\quad \leq c_{\alpha,\lambda} \exp\left(-\lambda \frac{b_n}{n^\alpha}\right)
\end{aligned}$$

for all positive sequences (b_n) with a constant $c_{\alpha,\lambda} > 0$. Now, we give the first order asymptotic expansion for the higher moments of X_n:

Theorem 4.2 *Let (X_n) be given by the recursion (1) with $t_n = n^\alpha$, $\alpha > 1$. Then for all $k \geq 0$ it holds*

$$\mathbb{E}\, X_n^k \sim \mu_k n^{k\alpha},$$

with $\mu_0 = 1$, $\mu_1 = (\alpha+1)/(\alpha-1)$, *and*

$$\mu_k = \frac{k\alpha+1}{k\alpha-1} \sum_{\substack{r+s+t=k\\ r,s<k}} \binom{k}{r,s,t} B(\alpha r+1, \alpha s+1)\mu_r\mu_s, \quad k \geq 2,$$

where $B(\,\cdot\,,\,\cdot\,)$ *denotes the Eulerian beta-integral. In particular the variance of* X_n *satisfies*

$$\mathit{Var}\ X_n \sim \frac{\alpha(\alpha+1)^2 B(\alpha,\alpha) + 2(\alpha^2-2\alpha-1)}{(2\alpha-1)(\alpha-1)^2}\, n^{2\alpha}.$$

Proof: The convergence of arbitrary moments of Y_n implies

$$\mathbb{E}\, X_n^k \;=\; \mathbb{E}\left[(n^\alpha Y_n)^k\right] = \mathbb{E}\, Y_n^k n^{k\alpha} \sim \mathbb{E}\, Y^k n^{k\alpha},$$

thus our expansion holds for $\mu_k = \mathbb{E}\, Y^k$. This yields the values $\mu_0 = 1$, $\mu_1 = (\alpha+1)/(\alpha-1)$. Higher moments of Y can be derived straightforwardly from the fixed-point equation (9). By the binomial formula it is (the summation indices r, s, t being nonnegative integers)

$$\begin{aligned}
\mu_k = \mathbb{E}\, Y^k &= \mathbb{E} \sum_{r+s+t=k} \binom{k}{r,s,t} U^{r\alpha}(1-U)^{s\alpha} Y^r (Y^*)^s \\
&= \sum_{r+s+t=k} \binom{k}{r,s,t} B(r\alpha+1, s\alpha+1)\mu_r\mu_s \\
&= \frac{2}{k\alpha+1}\mu_k + \sum_{\substack{r+s+t=k\\ r,s<k}} \binom{k}{r,s,t} B(r\alpha+1, s\alpha+1)\mu_r\mu_s.
\end{aligned}$$

Resolving for μ_k leads to the recursion given in the theorem. The formula for the variance follows from $\mathrm{Var}\ Y = \mu_2 - ((\alpha+1)/(\alpha-1))^2$. ∎

5 Densities

In this section we provide information on the densities of the limit distributions following an approach of Fill and Janson [3] for the analysis of the `Quicksort` limit distribution. Fill and Janson analyze decay properties of the Fourier transform of a distributional fixed-point in order to prove the existence, differentiability properties, and bounds of a density and its derivatives. This analysis can be carried over to the family of distributions Y given by the fixed-point equation (9). The pure existence of a density could also be derived by the approach of Tan and Hadjicostas [12].

Let $\phi(t) := \mathbb{E}\exp(itY)$ be the characteristic function of the fixed-point Y of (9). It is $\phi(t) = \exp(it(\alpha+1)/(\alpha-1))\,\mathbb{E}\exp(itZ)$ with Z the limit distribution of Theorem 3.1, thus $|\phi(t)| = |\,\mathbb{E}\exp(itZ)|$. The fixed-point equation (9) translates into

$$\phi(t) = e^{it}\int_0^1 \phi(u^\alpha t)\phi((1-u)^\alpha t)\,du.$$

This implies in particular

$$|\phi(t)| \leq \int_0^1 |\phi(u^\alpha t)||\phi((1-u)^\alpha t)|\,du. \tag{14}$$

We define $h_{y,y^*}(u) := u^\alpha y + (1-u)^\alpha y^* + 1$ for $u \in [0,1]$ and $y, y^* \in \mathbb{R}$. The fixed-point equation (9) takes then the form $Y = h_{Y,Y^*}(U)$ in distribution. The approach of Fill and Janson consists of deriving first a decay rate for the characteristic function of $h_{y,y^*}(U)$ for all $y, y^* \in \mathbb{R}$ using a method of van der Corput. This bound carries over to the characteristic function of Y by mixing over the distribution of Y. Then the bound can be improved by successive substitution into (14). This leads to integrability properties of the characteristic function which imply the existence and further properties of a density of the fixed-point.

In contrast to the `Quicksort` limit distribution the fixed-point Y given by (9) does not have the whole real line as support. Since Y is the limit of non-negative random variables we obtain $Y \geq 0$ almost surely. Plugging this information into (9) we obtain $Y \geq 1$ almost surely. By induction and $U^\alpha + (1-U)^\alpha \geq 2^{1-\alpha}$ we increase this bound to $Y \geq \sum_{j=0}^n (2^{1-\alpha})^j$ for all $n \in \mathbb{N}$, thus

$$Y \geq L_\alpha := \frac{2^{\alpha-1}}{2^{\alpha-1}-1}$$

almost surely.

Lemma 5.1 *It holds $|\phi(t)| \leq (32/B_\alpha)^{1/2}|t|^{-1/2}$ for all $t \in \mathbb{R}$ with*

$$B_\alpha := \begin{cases} 2^{3-\alpha}\alpha(\alpha-1)L_\alpha & \text{for } 1 < \alpha \leq 2 \text{ or } \alpha \geq 3, \\ \alpha(\alpha-1)L_\alpha & \text{for } 2 < \alpha < 3. \end{cases}$$

Proof: It is for $u \in [0,1]$

$$h''_{y,y^*}(u) = \alpha(\alpha-1)\left[u^{\alpha-2}y + (1-u)^{\alpha-2}y^*\right],$$

thus for all $y, y^* \geq L_\alpha$ we obtain

$$h''_{y,y^*}(u) \geq \alpha(\alpha-1)L_\alpha \min_{u\in[0,1]}\left\{u^{\alpha-2} + (1-u)^{\alpha-2}\right\} = B_\alpha$$

for all $u \in [0,1]$. Now, the argument of Lemma 2.3 in [3] implies for all $y, y^* \geq L_\alpha$

$$|\,\mathbb{E}\exp(ith_{y,y^*}(U))| \leq \left(\frac{32}{B_\alpha}\right)^{1/2}|t|^{-1/2}, \quad t \in \mathbb{R}.$$

Note that the optimal choice of γ in the cited proof is here $(2/B_\alpha)^{1/2}$. Since $L(Y)$ has no mass on $(-\infty, L_\alpha)$ we obtain by conditioning

$$|\phi(t)| = \left|\int_{L_\alpha}^\infty \int_{L_\alpha}^\infty \mathbb{E}\exp(ith_{y,y^*}(U))\, d\sigma(y)d\sigma(y^*)\right| \leq \left(\frac{32}{B_\alpha}\right)^{1/2}|t|^{-1/2}$$

for all $t \in \mathbb{R}$, where σ denotes the distribution of Y. ■

This bound can be improved to superpolynomial decay of ϕ by successive substitution into (14):

Theorem 5.2 *For every real $p \geq 0$ there is a smallest constant $0 < c_p < \infty$ such that the characteristic function ϕ of Y satisfies*

$$|\phi(t)| \leq c_p|t|^{-p} \quad \text{for all} \quad t \in \mathbb{R}. \tag{15}$$

The constants c_p satisfy $c_{1/2} \leq (32/B_\alpha)^{1/2}$,

$$c_{2p} \leq \frac{\Gamma^2(1-\alpha p)}{\Gamma(2-2\alpha p)} c_p^2 \quad \textit{for} \quad 0 < p < \frac{1}{\alpha}, \tag{16}$$

$$c_{p+1/\alpha} \leq 2^{\alpha p+1} \frac{\alpha p}{\alpha p - 1} c_p^{1+1/(\alpha p)} \quad \textit{for} \quad p > \frac{1}{\alpha}. \tag{17}$$

Proof: First we show that if (15) holds for a $0 < p < 1/\alpha$ with $c_p < \infty$ then (15) holds also with p replaced by $2p$, where the estimate (16) is valid: By (14) we obtain

$$\begin{aligned} |\phi(t)| &\leq \int_0^1 c_p^2 |u^\alpha t|^{-p} |(1-u)^\alpha t|^{-p}\, du \\ &= c_p^2 |t|^{-2p} B(1-\alpha p, 1-\alpha p) \\ &= \frac{\Gamma^2(1-\alpha p)}{\Gamma(2-2\alpha p)} c_p^2 |t|^{-2p}. \end{aligned}$$

Next, if (15) holds for a $p > 1/\alpha$ with $c_p < \infty$ then (15) holds as well with p replaced by $p + 1/\alpha$ with (17) being valid: It is

$$|\phi(t)| \leq \int_0^1 \min\left\{\frac{c_p}{(u^\alpha t)^p}, 1\right\} \min\left\{\frac{c_p}{((1-u)^\alpha t)^p}, 1\right\} du.$$

Adapting the estimates of Fill and Janson we consider first $t \geq 2^\alpha c_p^{1/p}$ and split the domain of integration into the region $[c_p^{1/(\alpha p)} t^{1/\alpha}, 1 - c_p^{1/(\alpha p)} t^{1/\alpha}]$ and its complement. This implies (cf. Lemma 2.6 in [3])

$$|\phi(t)| \leq 2^{\alpha p+1} \frac{\alpha p}{\alpha p - 1} c_p^{1+1/(\alpha p)} t^{-(p+1/\alpha)}$$

for $t \geq 2^\alpha c_p^{1/p}$. For $0 < t < 2^\alpha c_p^{1/p}$ the right hand side is at least one and negative t are covered by $|\phi(-t)| = |\phi(t)|$.

Now, the proof is completed as follows: The assertion (15) trivially holds for $p = 0$ with $c_0 = 1$ and, by Lemma 5.1, for $p = 1/2$ with $c_{1/2}$ estimated in the Theorem. If $\alpha > 2$ then we iterate (17) starting with $p = 1/2$ and obtain (15) for all $p = 1/2 + j/\alpha$, $j \in \mathbb{N}$. Since $c_q^{1/q} \leq c_p^{1/p}$ for all $0 < q \leq p$ this gives the assertion for all $p \geq 0$. If $1 < \alpha < 2$ we apply (16) with $p = 1/2$ and obtain the assertion with $p = 1$. Then we iterate (17) as in the case $\alpha > 2$. Finally, for $\alpha = 2$ the assertion is true for $p = 1/2$ thus as well for $p = 1/3$. We apply (16) with $p = 1/3$ and obtain the assertion for $p = 2/3$. Then we can iterate (17) starting with $p = 2/3$. ■

As discussed in [3] our Theorems 4.1 and 5.2 together imply that ϕ belongs to the class of rapidly decreasing C^∞ functions, which is preserved under Fourier transform. Therefore, we obtain analogous decay properties for the density of the fixed-point Y and its translated version Z:

Theorem 5.3 *The limit random variable Z of Theorem 3.1 has an infinitely differentiable density function f. For all $p \geq 0$ and integer $k \geq 0$ there is a constant $C_{p,k}$ such that its k-th derivative $f^{(k)}$ satisfies*

$$|f^{(k)}(x)| \leq C_{p,k} |x|^{-p} \quad \textit{for all} \quad x \in \mathbb{R}.$$

Explicit bounds on the supremum norm of $f^{(k)}$ can as well be established using Theorem 5.2 and a Fourier inversion formula.

References

[1] Bickel, P. J. and P. A. Freedman (1981). Some asymptotic theory for the bootstrap. *Ann. Statist. 9*, 1196–1217.

[2] Devroye, L. (1998). Universal limit laws for the depths in random trees. *SIAM J. Comput. 28*, 409–432.

[3] Fill, J. A. and S. Janson (2000). Smoothness and decay properties of the limiting Quicksort density function. *Mathematics and computer science (Versailles, 2000)*, 53–64. Birkhäuser, Basel.

[4] Grabner, P. and H. Prodinger (2001). Sorting algorithms for broadcast communications: Mathematical analysis. *Theoret. Comput. Sci.*, to appear.

[5] Neininger, R. and L. Rüschendorf (1999). On the internal path length of d-dimensional quad trees. *Random Structures Algorithms 15*, 25–41.

[6] Panholzer, A. and H. Prodinger (2001). Binary search tree recursions with harmonic toll functions. *J. Comput. Appl. Math.*, to appear.

[7] Rachev, S. T. and L. Rüschendorf (1995). Probability metrics and recursive algorithms. *Adv. in Appl. Probab. 27*, 770–799.

[8] Rösler, U. (1991). A limit theorem for "quicksort". *RAIRO Inform. Théor. Appl. 25*, 85–100.

[9] Rösler, U. (1992). A fixed point theorem for distributions. *Stochastic Process. Appl. 42*, 195–214.

[10] Rösler, U. (2001). The analysis of stochastic divide and conquer algorithms. *Algorithmica 29*, 238–261.

[11] Rösler, U. and L. Rüschendorf (2001). The contraction method for recursive algorithms. *Algorithmica 29*, 3–33.

[12] Tan, K. H. and P. Hadjicostas (1995). Some properties of a limiting distribution in quicksort. *Statist. Probab. Lett. 25*, 87–94.