

# Local Embeddability of Real Analytic Path Geometries

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**ABSTRACT.** An almost complex structure  $\mathfrak{J}$  on a 4-manifold  $X$  may be described in terms of a rank 2 vector bundle  $\Lambda_{\mathfrak{J}} \subset \Lambda^2 TX^*$ . We call a pair of line subbundles  $L_1, L_2$  of  $\Lambda^2 TX^*$  a splitting of  $\mathfrak{J}$  if  $\Lambda_{\mathfrak{J}} = L_1 \oplus L_2$ . A hypersurface  $M \subset X$  satisfying a nondegeneracy condition inherits a CR-structure from  $\mathfrak{J}$  and a path geometry from the splitting  $(L_1, L_2)$ . Using the Cartan-Kähler theorem we show that locally every real analytic path geometry is induced by an embedding into  $\mathbb{C}^2$  equipped with the splitting generated by the real and imaginary part of  $dz^1 \wedge dz^2$ . As a corollary we obtain the well-known fact that every 3-dimensional nondegenerate real analytic CR-structure is locally induced by an embedding into  $\mathbb{C}^2$ .

## 1. INTRODUCTION

Motivated by the well-known fact (see for instance [6]) that an almost complex structure  $\mathfrak{J}$  on a 4-manifold  $X$  admits a description in terms of a rank 2 vector bundle  $\Lambda_{\mathfrak{J}} \subset \Lambda^2 TX^*$ , we introduce the notion of a splitting of an almost complex structure: A pair of line subbundles  $L_1, L_2$  of  $\Lambda^2 TX^*$  is called a *splitting* of  $\mathfrak{J}$  if  $\Lambda_{\mathfrak{J}} = L_1 \oplus L_2$ . A hypersurface  $M \subset X$  satisfying a nondegeneracy condition inherits a CR-structure from  $\mathfrak{J}$  and a path geometry from the splitting  $(L_1, L_2)$ . The purpose of this Note is to show that locally every real analytic path geometry is induced by an embedding into  $\mathbb{R}^4 \simeq \mathbb{C}^2$  equipped with the splitting generated by the real and imaginary part of  $dz^1 \wedge dz^2$ . This will be done using the Cartan-Kähler theorem. As a corollary we obtain the well-known fact that every 3-dimensional nondegenerate real analytic CR-structure is locally induced by an embedding into  $\mathbb{C}^2$ . It follows with Nirenberg's example of a smooth non-embeddable 3-dimensional CR-manifold that the real analyticity in our main statement is necessary.

The notation and terminology for the Cartan-Kähler theorem and exterior differential systems are chosen to be consistent with [2, 7]. Moreover we adhere to the convention of summing over repeated indices.

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## 2. PRELIMINARIES

**2.1. Pairs of 2-forms.** Throughout this section, let  $V$  denote an oriented 4-dimensional real vector space. Fix a volume form  $\varepsilon \in \Lambda^4 V^*$  which induces the



given orientation. Given two 2-forms  $\omega, \phi \in \Lambda^2 V^*$ , we may write  $\omega \wedge \phi = \langle \omega, \phi \rangle \varepsilon$  for some unique real number  $\langle \omega, \phi \rangle$ . Clearly the map  $(\omega, \phi) \mapsto \langle \omega, \phi \rangle$  defines a symmetric bilinear form on the 6-dimensional real vector space  $\Lambda^2 V^*$  which is easily seen to be nondegenerate and of signature  $(3, 3)$ . Replacing  $\varepsilon$  with another orientation compatible volume form gives a bilinear form which is a positive multiple of  $\langle \cdot, \cdot \rangle$ . Consequently, the wedge product may be thought of as a conformal structure of split signature on  $\Lambda^2 V^*$ .

**Definition.** A pair of 2-forms  $\omega, \phi \in \Lambda^2 V^*$  is called *elliptic* if

$$\langle \omega, \omega \rangle \langle \phi, \phi \rangle > \langle \omega, \phi \rangle^2.$$

It is a natural problem to classify the pairs of elliptic 2-forms on  $V$ . This is a special case of a more general problem: Let  $\omega \in \Lambda^2 V^*$  be a symplectic 2-form whose stabiliser subgroup will be denoted by  $\text{Sp}(\omega) \subset \text{GL}(V)$ . The natural representation of  $\text{Sp}(\omega)$  on  $\Lambda^2 V^*$  decomposes as  $\Lambda^2 V^* = \{\omega\} \oplus \omega^\perp$  where both summands are irreducible  $\text{Sp}(\omega)$ -modules.<sup>1</sup> Here  $\omega^\perp$  is the 5-dimensional linear subspace of  $\Lambda^2 V^*$  consisting of 2-forms orthogonal to  $\omega$ . One can ask to classify the orbits of  $\text{Sp}(\omega)$  on  $\omega^\perp$ . This has been carried out in [8]. In the elliptic case one obtains:

**Lemma 1** (see [8]). *Let  $\omega, \phi \in \Lambda^2 V^*$  be a pair of elliptic orthogonal 2-forms, then there exists a positive real number  $\kappa$  and a basis  $e^i$  of  $V^*$  such that*

$$\omega = e^1 \wedge e^3 - e^2 \wedge e^4, \quad \phi = \kappa (e^1 \wedge e^4 + e^2 \wedge e^3).$$

The constant  $\kappa$  is an  $\text{Sp}(\omega)$ -invariant and thus parametrises the set of elliptic  $\text{Sp}(\omega)$ -orbits. Ellipticity will be useful because of the following:

**Lemma 2.** *Let  $W$  be 3-dimensional real vector space. Then the pullback of an elliptic pair of 2-forms  $\omega, \phi \in \Lambda^2 V^*$  with any injective linear map  $A : W \rightarrow V$  gives two linearly independent 2-forms on  $W$ .*

*Proof.* The ellipticity condition is equivalent to every nonzero linear combination of  $(\omega, \phi)$  being symplectic. Suppose  $(\omega, \phi)$  is an elliptic pair of 2-forms. Then for every choice of real numbers  $(\lambda_1, \lambda_2) \neq 0$ , the 2-form  $\tau = \lambda_1 \omega + \lambda_2 \phi$  is symplectic. Since there are no isotropic subspaces of dimension greater than 2 in the symplectic vector space  $(V, \tau)$ , it follows that  $A^* \tau = \lambda_1 A^* \omega + \lambda_2 A^* \phi \neq 0$  for every linear injective map  $A : W \rightarrow V$ .  $\square$

**2.2. Splittings of complex structures.** Let  $\mathcal{C}^+(V)$  denote space of complex structures on  $V$  which are compatible with the orientation, i.e. its points  $J \in \text{End}(V)$  satisfy  $\varepsilon(v_1, Jv_1, v_2, Jv_2) \geq 0$  for all vectors  $v_1, v_2 \in V$ . Moreover let  $G_2^+(\Lambda^2 V^*, \wedge_+)$  denote the submanifold of the Grassmannian of oriented 2-planes in  $\Lambda^2 V^*$  to

<sup>1</sup>We denote by  $\{\cdot\}$  the linear span of the elements within. In the case of smooth differential forms, the coefficients are smooth real-valued functions.

whose elements the wedge product restricts to be positive definite. Given a  $(2,0)$ -form  $\alpha \in \Lambda^{2,0}V^*$  with respect to some  $J \in \mathcal{C}^+(V)$ , let  $\Lambda_J \in G_2^+(\Lambda^2V^*, \wedge_+)$  denote the 2-dimensional linear subspace spanned by  $\operatorname{Re}(\alpha)$ ,  $\operatorname{Im}(\alpha)$  and orient  $\Lambda_J$  by declaring  $\operatorname{Re}(\alpha)$ ,  $\operatorname{Im}(\alpha)$  to be positively oriented. Clearly  $\Lambda_J$  and its orientation are independent of the chosen  $(2,0)$ -form  $\alpha$  and one thus obtains a map  $\psi : \mathcal{C}^+(V) \rightarrow G_2^+(\Lambda^2V^*, \wedge_+)$  given by  $J \mapsto \Lambda_J$ . Note that  $G = \operatorname{GL}^+(V)$  acts smoothly and transitively from the left on  $\mathcal{C}^+(V)$  via  $(A, J) \mapsto A^{-1}JA$ . Every element of  $G_2^+(\Lambda^2V^*, \wedge_+)$  admits a positively oriented elliptic conformal basis. It follows with **Lemma 1** that via pushforward,  $\operatorname{GL}^+(V)$  acts smoothly and transitively from the left on  $G_2^+(\Lambda^2V^*, \wedge_+)$  as well.

**Proposition 1.** *The map  $\psi : \mathcal{C}^+(V) \rightarrow G_2^+(\Lambda^2V^*, \wedge_+)$ ,  $J \mapsto \Lambda_J$  is a  $G$ -equivariant diffeomorphism.*

*Proof.* Clearly the map  $\psi$  is  $G$ -equivariant. To prove that  $\psi$  is a diffeomorphism it is sufficient to show that  $G_J = G_{\psi(J)}$  for all  $J \in \mathcal{C}^+(V)$  where  $G_J$  and  $G_{\psi(J)}$  denote the stabiliser subgroups of  $G$  with respect to  $J$  and  $\psi(J)$  respectively. Choose  $J \in \mathcal{C}^+(V)$ , then we have  $G_J \subset G_{\psi(J)}$ . Write

$$J(v) = -e^2(v)e_1 + e^1(v)e_2 - e^4(v)e_3 + e^3(v)e_4$$

for some basis  $(e_i)$  of  $V$  and dual basis  $(e^i)$  of  $V^*$ . Then

$$\omega = e^1 \wedge e^3 - e^2 \wedge e^4 = \frac{1}{2}w_{kl}e^k \wedge e^l, \quad \phi = e^1 \wedge e^4 + e^2 \wedge e^3 = \frac{1}{2}f_{kl}e^k \wedge e^l$$

is a positively oriented conformal basis of  $\Lambda_J$ . Consequently every  $A \in G_{\psi(J)}$  satisfies  $A^*\omega = x\omega + y\phi$  and  $A^*\phi = -y\omega + x\phi$ , for some real numbers  $(x, y) \neq 0$ . The matrix representation  $a$  of  $A$  with respect to the basis  $(e_i)$  thus satisfies

$$a^t wa = xw + yf, \quad a^t fa = -yw + xf.$$

From this one easily concludes  $awf = wfa$  which is equivalent to  $A$  commuting with  $J$ .  $\square$

**Proposition 1** motivates the following:

**Definition.** A *splitting* of a complex structure  $J$  on  $V$  is a pair of lines  $L_1, L_2 \in \mathbb{P}(\Lambda^2V^*)$  such that  $\Lambda_J = L_1 \oplus L_2$ .

Call two 4-dimensional real vector spaces  $V, V'$  equipped with complex structures  $J, J'$  and splittings  $(L_1, L_2), (L'_1, L'_2)$  equivalent, if there exists a complex linear map  $A : V \rightarrow V'$  such that  $A^*(L'_i) = L_i$  for  $i = 1, 2$ .

On  $V = \mathbb{R}^4$  let  $\omega_0 = e^1 \wedge e^3 - e^2 \wedge e^4$  and  $\phi_0 = e^1 \wedge e^4 + e^2 \wedge e^3$  where  $e^1, \dots, e^4$  denotes the standard basis of  $(\mathbb{R}^4)^*$ . Define  $L_1 = \{\omega_0\}$  and  $L_2 = \{\alpha\omega_0 + \phi_0\}$  for some nonnegative real number  $\alpha$ . Orient  $L_1 \oplus L_2$  by declaring  $\omega_0, \phi_0$  to be a positively oriented basis and let  $J_0$  be the associated complex structure. Then  $S_\alpha = (L_1, L_2)$  is a splitting of  $J_0$ .

**Proposition 2.** *Every pair  $(V, J)$  equipped with a splitting  $(L_1, L_2)$  is equivalent to  $(\mathbb{R}^4, J_0)$  equipped with the splitting  $S_\alpha$  for some unique  $\alpha \in \mathbb{R}_0^+$ .*

*Proof.* Let  $L_1 = \{\omega\}$  and  $L_2 = \{\omega'\}$  for some 2-forms  $\omega, \omega' \in \Lambda^2 V^*$ . Since the wedge product restricts to be positive definite on  $L_1 \oplus L_2$  we have  $\omega \wedge \omega' > 0$  and there exists a real number  $\alpha$ , such that  $\omega' = \alpha\omega + \phi$  for some 2-form  $\phi$  satisfying  $\omega \wedge \phi = 0$  and  $\phi \wedge \phi > 0$ . After possibly rescaling  $\omega'$  we can assume that  $\phi \wedge \phi = \omega \wedge \omega$  and that  $\alpha$  is nonnegative. It follows with **Lemma 1** that there exists a linear map  $A : V \rightarrow \mathbb{R}^4$  which identifies  $\omega$  with  $\omega_0$  and  $\phi$  with  $\phi_0$ , in particular  $A$  is complex linear. To prove uniqueness of  $\alpha$  suppose  $A : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  satisfies  $A^*\omega_0 = x\omega_0$  and  $A^*(\alpha\omega_0 + \phi_0) = y(\beta\omega_0 + \phi_0)$  for some real numbers  $x, y \neq 0$  and some nonnegative real numbers  $\alpha, \beta$ . Then  $A^*(\omega_0 \wedge \omega_0) = x^2\omega_0 \wedge \omega_0$  and consequently

$$A^*(\omega_0 \wedge (\alpha\omega_0 + \phi_0)) = \alpha x^2 \omega_0 \wedge \omega_0 = xy\beta\omega_0 \wedge \omega_0,$$

which is equivalent to  $\alpha x = \beta y$ . We also have

$$A^*((\alpha\omega_0 + \phi_0) \wedge (\alpha\omega_0 + \phi_0)) = x^2(\alpha^2 + 1)\omega_0 \wedge \omega_0 = y^2(\beta^2 + 1)\omega_0 \wedge \omega_0,$$

which implies  $x^2 = y^2$  and thus  $\alpha^2 = \beta^2$ . Since  $\alpha, \beta \geq 0$ , the claim follows.  $\square$

For a splitting  $(L_1, L_2)$ , the unique nonnegative real number  $\alpha$  provided by **Proposition 2** will be called the *degree* of the splitting. A splitting of degree 0 will be called *orthogonal*.

### 3. LOCAL EMBEDDABILITY OF REAL ANALYTIC PATH GEOMETRIES

**3.1. Splittings of almost complex structures.** Let  $X$  be a smooth 4-manifold and  $\mathfrak{J}$  be an almost complex structure with associated rank 2 vector bundle  $\Lambda_{\mathfrak{J}} \subset \Lambda^2 TX^*$  whose fibre at  $p \in X$  is the linear subspace  $\Lambda_{\mathfrak{J}_p} \subset \Lambda^2 T_p X^*$  associated to  $\mathfrak{J}_p : T_p X \rightarrow T_p X$ . A *splitting* of  $\mathfrak{J}$  consists of a pair of smooth line bundles  $L_1, L_2 \subset \Lambda^2 TX^*$  so that  $\Lambda_{\mathfrak{J}} = L_1 \oplus L_2$ .

**3.2. Induced structure on hypersurfaces.** A *CR-structure* on a 3-manifold  $M$  consists of a rank 2 subbundle  $D \subset TM$  and a vector bundle endomorphism  $I : D \rightarrow D$  which satisfies  $I^2 = -\text{Id}_D$ . A CR-structure  $(D, I)$  is called *nondegenerate* if  $D$  is nowhere integrable, i.e. a contact plane field. A closely related notion is that of a path geometry (see for instance [7] for a motivation of the following definition). A path geometry on a 3-manifold  $M$  consists of a pair of line subbundles  $(P_1, P_2)$  of  $TM$  which span a contact plane field. A CR-structure  $(D, I)$  and a path geometry  $(P_1, P_2)$  on  $M$  will be called *compatible* if  $D = P_1 \oplus P_2$  and  $I(P_1) = P_2$ .

Let  $(L_1, L_2)$  be a splitting of the almost complex structure  $\mathfrak{J}$  on  $X$  and  $(\omega, \phi)$  a pair of 2-forms defined on some open subset  $\tilde{U} \subset X$  which span  $(L_1, L_2)$ . Then the pair  $(\omega, \phi)$  is elliptic, i.e.  $(\omega_p, \phi_p)$  is elliptic for every point  $p \in \tilde{U}$ . Suppose  $M \subset X$  is a hypersurface. Then **Lemma 2** implies that the 2-forms  $(\omega, \phi)$  remain linearly independent when pulled back to  $M \cap \tilde{U}$ . This is useful because of the following:

**Lemma 3.** *Let  $\beta_1, \beta_2$  be smooth linearly independent 2-forms on a 3-manifold  $M$ . Then there exists a local coframing  $\eta = (\eta^1, \eta^2, \eta^3)^t$  of  $M$  such that  $\beta_1 = \eta_2 \wedge \eta_1$  and  $\beta_2 = \eta_2 \wedge \eta_3$ .*

Recall that a (local) *coframing on  $M$*  consists of three smooth linearly independent 1-forms defined on (some proper open subset of)  $M$ .

**Proof of Lemma 3.** Let  $x : U \rightarrow \mathbb{E}^3$  be local coordinates on  $M$  with respect to which  $\beta_1|_U = b_1 \cdot \star dx$  and  $\beta_2|_U = b_2 \cdot \star dx$  for some smooth  $b_i : U \rightarrow \mathbb{R}^3$  where  $\star$  denotes the Hodge-star of Euclidean space  $\mathbb{E}^3$ . Define  $e = (b_1 \times b_2) / |b_1 \times b_2| : U \rightarrow \mathbb{R}^3$  and

$$\eta_1 = (b_1 \times e) \cdot dx, \quad \eta_2 = e \cdot dx, \quad \eta_3 = (b_2 \times e) \cdot dx,$$

then  $(\eta^1, \eta^2, \eta^3)$  have the desired properties.  $\square$

A local coframing of  $M$  obtained via **Lemma 3** and some (local) choice of 2-forms  $(\omega, \phi)$  spanning  $(L_1, L_2)$  will be called *adapted* to the structure induced by the splitting  $(L_1, L_2)$ . Independent of the particular adapted local coframings are the line subbundles  $P_1$  and  $P_2$  of  $TM$ , locally defined by

$$P_1 = \{\eta_1, \eta_2\}^\perp, \quad P_2 = \{\eta_2, \eta_3\}^\perp.$$

Call a hypersurface  $M \subset X$  *nondegenerate* if  $D = P_1 \oplus P_2$  is a contact plane field. Summarising, we have shown:

**Proposition 3.** *A nondegenerate hypersurface  $M \subset X$  inherits a path geometry from the splitting  $(L_1, L_2)$ .*

**Remark.** Fixing a  $(2,0)$ -form on  $X$  allows to define a coframing on a hypersurface  $M \subset X$ . For the construction of the coframing and its properties see [4].

**3.3. Local embeddability.** We conclude by using the Cartan-Kähler theorem to show that locally every real analytic path geometry is induced by an embedding into  $\mathbb{C}^2$  equipped with the splitting  $(\{\omega_0\}, \{\phi_0\})$ . Here  $\omega_0 = \text{Re}(dz^1 \wedge dz^2)$  and  $\phi_0 = \text{Im}(dz^1 \wedge dz^2)$  where  $z = (z^1, z^2)$  are standard coordinates on  $\mathbb{C}^2$ . Writing  $z^1 = x^1 + ix^2$  and  $z^2 = x^3 + ix^4$  for standard coordinates  $x = (x^i)$  on  $\mathbb{R}^4$ , we have

$$\omega_0 = dx^1 \wedge dx^3 - dx^2 \wedge dx^4, \quad \phi_0 = dx^1 \wedge dx^4 + dx^2 \wedge dx^3.$$

In [5], as an application of his method of equivalence, Cartan has shown how to associate a Cartan geometry to every path geometry.

**Definition.** Let  $G$  be a Lie group and  $H \subset G$  a Lie subgroup with Lie algebras  $\mathfrak{h} \subset \mathfrak{g}$ . A *Cartan geometry of type  $(G, H)$*  on a manifold  $M$  consists of a right principal  $H$ -bundle  $\pi : B \rightarrow M$  together with a 1-form  $\theta \in \mathcal{A}^1(B, \mathfrak{g})$  which satisfies the following conditions:

- (i)  $\theta_b : T_b B \rightarrow \mathfrak{g}$  is an isomorphism for every  $b \in B$ ,
- (ii)  $\theta(X_v) = v$  for every fundamental vector field  $X_v, v \in \mathfrak{h}$ ,
- (iii)  $(R_h)^* \theta = \text{Ad}_{\mathfrak{g}}(h^{-1}) \circ \theta$ .

Here  $\text{Ad}_{\mathfrak{g}}$  denotes the adjoint representation of  $G$ . The 1-form  $\theta$  is called the *Cartan connection* of the Cartan geometry  $(\pi : B \rightarrow M, \theta)$ .

Denote by  $H \subset \text{SL}(3, \mathbb{R})$  the Lie subgroup of upper triangular matrices. In modern language Cartan's result is as follows (for a proof see [3, 7]):

**Theorem 1** (Cartan). *Given a path geometry  $(M, P_1, P_2)$ , then there exists a Cartan geometry  $(\pi : B \rightarrow M, \theta)$  of type  $(\mathrm{SL}(3, \mathbb{R}), H)$  which has the following properties: Writing*

$$\theta = \begin{pmatrix} \theta_0^0 & \theta_1^0 & \theta_2^0 \\ \theta_0^1 & \theta_1^1 & \theta_2^1 \\ \theta_0^2 & \theta_1^2 & \theta_2^2 \end{pmatrix},$$

- (i) *for any section  $\sigma : M \rightarrow B$ , the 1-form  $\phi = \sigma^*\theta$  satisfies  $P_1 = \{\phi_1^2, \phi_0^2\}^\perp$  and  $P_2 = \{\phi_0^1, \phi_0^2\}^\perp$ . Moreover  $\phi_0^1 \wedge \phi_0^2 \wedge \phi_1^2$  is a volume form on  $M$ .*  
(ii) *The curvature 2-form  $\Theta = d\theta + \theta \wedge \theta$  satisfies*

$$(3.1) \quad \Theta = \begin{pmatrix} 0 & \mathcal{W}_1 \theta_0^1 \wedge \theta_0^2 & (\mathcal{W}_2 \theta_0^1 + \mathcal{F}_2 \theta_1^2) \wedge \theta_0^2 \\ 0 & 0 & \mathcal{F}_1 \theta_1^2 \wedge \theta_0^2 \\ 0 & 0 & 0 \end{pmatrix}$$

*for some smooth functions  $\mathcal{W}_1, \mathcal{W}_2, \mathcal{F}_1, \mathcal{F}_2 : B \rightarrow \mathbb{R}$ .*

Using this result and the Cartan-Kähler theorem we obtain local embeddability in the real analytic category:

**Theorem 2.** *Let  $(M, P_1, P_2)$  be a real analytic path geometry. Then for every point  $p \in M$  there exists a  $p$ -neighbourhood  $U_p \subset M$  and a real analytic embedding  $\varphi : U_p \rightarrow \mathbb{C}^2$  such that the path geometry induced by the splitting  $(\{\omega_0\}, \{\phi_0\})$  is  $(P_1, P_2)$  on  $U_p$ .*

*Proof.* Let  $(\pi : B \rightarrow M, \theta)$  denote the Cartan geometry of the path geometry  $(M, P_1, P_2)$ . On  $N = B \times \mathbb{R}^4$  consider the exterior differential system with independence condition  $(\mathcal{I}, \zeta)$  where  $\zeta = \zeta^1 \wedge \zeta^2 \wedge \zeta^3$  with  $\zeta^1 = \theta_0^1, \zeta^2 = \theta_0^2, \zeta^3 = \theta_1^2$  and the differential ideal  $\mathcal{I}$  is generated by the two 2-forms

$$\chi_1 = \theta_0^2 \wedge \theta_1^1 - \omega_0, \quad \chi_2 = \theta_0^2 \wedge \theta_1^2 - \phi_0.$$

The dual vector fields to the coframing  $(\theta_k^j, dx^l)$  of  $N$  will be denoted by  $(T_k^j, \partial_{x^l})$ . Let  $G_k(TN) \rightarrow N$  be the Grassmann bundle of  $k$ -planes on  $N$  and  $G_3(TN, \zeta) = \{E \in G_3(TN) \mid \zeta_E \neq 0\}$  where  $\zeta_E$  denotes the restriction of  $\zeta$  to the 3-plane  $E$ . Let  $V^k(\mathcal{I})$  denote the set of  $k$ -dimensional *integral elements* of  $\mathcal{I}$ , i.e. those  $E \in G_k(TN)$  for which  $\beta_E = 0$  for every form  $\beta \in \mathcal{I}^k = \mathcal{I} \cap \mathcal{A}^k(N)$ . The flag of integral elements  $F = (E^0, E^1, E^2, E^3)$  of  $\mathcal{I}$  given by  $E^0 = \{0\}$ ,  $E^1 = \{v_1\}$ ,  $E^2 = \{v_1, v_2\}$ ,  $E^3 = \{v_1, v_2, v_3\}$  where

$$\begin{aligned} v_1 &= T_0^1 + T_0^2 + T_1^2 + \partial_{x^4}, \\ v_2 &= T_0^0 + T_0^1 - T_1^2 + \partial_{x^1} + \partial_{x^2}, \\ v_3 &= T_1^1 - T_1^2 + \partial_{x^1}, \end{aligned}$$

has Cartan characters  $(s_0, s_1, s_2, s_3) = (0, 2, 4, 3)$ . Therefore, by Cartan's test,  $V^3(\mathcal{I})$  has codimension at least 8 at  $E^3$ . However the forms of  $\mathcal{I}^3$  which impose independent conditions on the elements of  $G_3(TN, \zeta)$  are the eight 3-forms  $d\chi_i, \chi_i \wedge \zeta^k, i = 1, 2, k = 1, 2, 3$ . It follows that  $V^3(\mathcal{I}) \cap G_3(TN, \zeta)$  has codimension 8 in  $G_3(TN)$ . Moreover computations show that  $V^3(\mathcal{I}) \cap G_3(TN, \zeta)$  is a smooth submanifold near  $E^3$ , thus the flag  $F$  is Kähler regular and therefore the ideal  $\mathcal{I}$  is involutive. Pick points  $p \in M$  and  $q = (b, 0) \in N$  with

$\pi(b) = p$ . By the Cartan-Kähler theorem there exists a 3-dimensional integral manifold  $\bar{\psi} = (\bar{s}, \bar{\varphi}) : \Sigma \rightarrow B \times \mathbb{R}^4$  of  $(\mathcal{I}, \zeta)$  passing through  $q$  and having tangent space  $E^3$  at  $q$ . Every volume form on  $M$  pulls back under  $\pi$  to a nowhere vanishing multiple of  $\zeta$ . Since  $\bar{\phi}^* \zeta = \bar{s}^* \zeta \neq 0$ ,  $\pi \circ \bar{s} : \Sigma \rightarrow M$  is a local diffeomorphism. Therefore  $p \in M$  has a neighbourhood  $U_p$  on which there exists a real analytic immersion  $\psi = (s, \varphi) : U_p \rightarrow B \times \mathbb{R}^4$  such that the pair  $(\psi, U_p)$  is an integral manifold of the EDS  $(N, \mathcal{I}, \zeta)$  and  $s$  a local section of  $\pi : B \rightarrow M$ . After possibly shrinking  $U_p$  we can assume that  $\varphi$  is an embedding. Since by construction  $\varphi^*(\omega_0 + i\phi_0) = s^*(\theta_0^2 \wedge (\theta_0^1 + i\theta_1^2))$ , it follows that the path geometry induced by  $\varphi$  is  $(P_1, P_2)$  on  $U_p$ .  $\square$

**Remark.** Every nondegenerate hypersurface  $M \subset \mathbb{C}^2$  also inherits a CR-structure  $(D, I)$  from the complex structure  $J$  on  $\mathbb{C}^2$ : For every  $p \in M$  define  $D_p$  to be the largest  $J_p$ -invariant subspace of  $T_p M$  and  $I_p$  to be the restriction of  $J_p$  to  $D_p$ . Then  $(D, I)$  is easily seen to be compatible with the path geometry induced on  $M$  by  $(\{\omega_0\}, \{\phi_0\})$ .

Using this remark and **Theorem 2** we get the well-known:

**Corollary 1.** *Let  $(D, I)$  be a nondegenerate real analytic CR-structure on a 3-manifold  $M$ . Then for every point  $p \in M$  there exists a  $p$ -neighbourhood  $U_p$  and a real analytic embedding  $\varphi : U_p \rightarrow \mathbb{C}^2$ , such that  $(D, I)$  is the CR-structure on  $U_p$  induced by the embedding  $\varphi$ .*

*Proof.* Pick a line bundle  $P_2 \subset D$ , define  $P_1 = I(P_2)$  and apply **Theorem 2**.  $\square$

**Remark.** **Corollary 1** also holds without the nondegeneracy assumption and in higher dimensions [1]. In [9], Nirenberg has constructed a smooth nondegenerate 3-dimensional CR-structure which is not induced by an embedding into  $\mathbb{C}^2$ . It follows that the real analyticity assumption in **Theorem 2** is necessary.

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