

LOCAL EMBEDDABILITY OF REAL ANALYTIC PATH GEOMETRIES

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ABSTRACT. An almost complex structure \mathfrak{J} on a 4-manifold X may be described in terms of a rank 2 vector bundle $\Lambda_{\mathfrak{J}} \subset \Lambda^2 TX^*$. We call a pair of line subbundles L_1, L_2 of $\Lambda^2 TX^*$ a splitting of \mathfrak{J} if $\Lambda_{\mathfrak{J}} = L_1 \oplus L_2$. A hypersurface $M \subset X$ satisfying a nondegeneracy condition inherits a CR-structure from \mathfrak{J} and a path geometry from the splitting (L_1, L_2) . Using the Cartan-Kähler theorem we show that locally every real analytic path geometry is induced by an embedding into \mathbb{C}^2 equipped with the splitting generated by the real and imaginary part of $dz^1 \wedge dz^2$. As a corollary we obtain the well-known fact that every 3-dimensional nondegenerate real analytic CR-structure is locally induced by an embedding into \mathbb{C}^2 .

1. INTRODUCTION

Motivated by the well-known fact (see for instance [6]) that an almost complex structure \mathfrak{J} on a 4-manifold X admits a description in terms of a rank 2 vector bundle $\Lambda_{\mathfrak{J}} \subset \Lambda^2 TX^*$, we introduce the notion of a splitting of an almost complex structure: A pair of line subbundles L_1, L_2 of $\Lambda^2 TX^*$ is called a *splitting* of \mathfrak{J} if $\Lambda_{\mathfrak{J}} = L_1 \oplus L_2$. A hypersurface $M \subset X$ satisfying a nondegeneracy condition inherits a CR-structure from \mathfrak{J} and a path geometry from the splitting (L_1, L_2) . The purpose of this Note is to show that locally every real analytic path geometry is induced by an embedding into $\mathbb{R}^4 \simeq \mathbb{C}^2$ equipped with the splitting generated by the real and imaginary part of $dz^1 \wedge dz^2$. This will be done using the Cartan-Kähler theorem. As a corollary we obtain the well-known fact that every 3-dimensional nondegenerate real analytic CR-structure is locally induced by an embedding into \mathbb{C}^2 . It follows with Nirenberg's example of a smooth non-embeddable 3-dimensional CR-manifold that the real analyticity in our main statement is necessary.

The notation and terminology for the Cartan-Kähler theorem and exterior differential systems are chosen to be consistent with [2, 7]. Moreover we adhere to the convention of summing over repeated indices.

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2. PRELIMINARIES

2.1. Pairs of 2-forms. Throughout this section, let V denote an oriented 4-dimensional real vector space. Fix a volume form $\varepsilon \in \Lambda^4 V^*$ which induces the given



orientation. Given two 2-forms $\omega, \phi \in \Lambda^2 V^*$, we may write $\omega \wedge \phi = \langle \omega, \phi \rangle \varepsilon$ for some unique real number $\langle \omega, \phi \rangle$. Clearly the map $(\omega, \phi) \mapsto \langle \omega, \phi \rangle$ defines a symmetric bilinear form on the 6-dimensional real vector space $\Lambda^2 V^*$ which is easily seen to be nondegenerate and of signature $(3, 3)$. Replacing ε with another orientation compatible volume form gives a bilinear form which is a positive multiple of $\langle \cdot, \cdot \rangle$. Consequently, the wedge product may be thought of as a conformal structure of split signature on $\Lambda^2 V^*$.

Definition. A pair of 2-forms $\omega, \phi \in \Lambda^2 V^*$ is called *elliptic* if

$$\langle \omega, \omega \rangle \langle \phi, \phi \rangle > \langle \omega, \phi \rangle^2.$$

It is a natural problem to classify the pairs of elliptic 2-forms on V . This is a special case of a more general problem: Let $\omega \in \Lambda^2 V^*$ be a symplectic 2-form whose stabiliser subgroup will be denoted by $\text{Sp}(\omega) \subset \text{GL}(V)$. The natural representation of $\text{Sp}(\omega)$ on $\Lambda^2 V^*$ decomposes as $\Lambda^2 V^* = \{\omega\} \oplus \omega^\perp$ where both summands are irreducible $\text{Sp}(\omega)$ -modules.¹ Here ω^\perp is the 5-dimensional linear subspace of $\Lambda^2 V^*$ consisting of 2-forms orthogonal to ω . One can ask to classify the orbits of $\text{Sp}(\omega)$ on ω^\perp . This has been carried out in [8]. In the elliptic case one obtains:

Lemma 1 (see [8]). *Let $\omega, \phi \in \Lambda^2 V^*$ be a pair of elliptic orthogonal 2-forms, then there exists a positive real number κ and a basis e^i of V^* such that*

$$\omega = e^1 \wedge e^3 - e^2 \wedge e^4, \quad \phi = \kappa (e^1 \wedge e^4 + e^2 \wedge e^3).$$

The constant κ is an $\text{Sp}(\omega)$ -invariant and thus parametrises the set of elliptic $\text{Sp}(\omega)$ -orbits. Ellipticity will be useful because of the following:

Lemma 2. *Let W be 3-dimensional real vector space. Then the pullback of an elliptic pair of 2-forms $\omega, \phi \in \Lambda^2 V^*$ with any injective linear map $A : W \rightarrow V$ gives two linearly independent 2-forms on W .*

Proof. The ellipticity condition is equivalent to every nonzero linear combination of (ω, ϕ) being symplectic. Suppose (ω, ϕ) is an elliptic pair of 2-forms. Then for every choice of real numbers $(\lambda_1, \lambda_2) \neq 0$, the 2-form $\tau = \lambda_1 \omega + \lambda_2 \phi$ is symplectic. Since there are no isotropic subspaces of dimension greater than 2 in the symplectic vector space (V, τ) , it follows that $A^* \tau = \lambda_1 A^* \omega + \lambda_2 A^* \phi \neq 0$ for every linear injective map $A : W \rightarrow V$. \square

2.2. Splittings of complex structures. Let $\mathcal{C}^+(V)$ denote space of complex structures on V which are compatible with the orientation, i.e. its points $J \in \text{End}(V)$ satisfy $\varepsilon(v_1, Jv_1, v_2, Jv_2) \geq 0$ for all vectors $v_1, v_2 \in V$. Moreover let $G_2^+(\Lambda^2 V^*, \wedge_+)$ denote the submanifold of the Grassmannian of oriented 2-planes in $\Lambda^2 V^*$ to whose elements the wedge product restricts to be positive definite. Given a $(2,0)$ -form $\alpha \in \Lambda^{2,0} V^*$ with respect to some $J \in \mathcal{C}^+(V)$, let $\Lambda_J \in G_2^+(\Lambda^2 V^*, \wedge_+)$ denote the 2-dimensional linear subspace spanned by $\text{Re}(\alpha)$, $\text{Im}(\alpha)$ and orient Λ_J by declaring $\text{Re}(\alpha), \text{Im}(\alpha)$ to be positively oriented. Clearly Λ_J and its orientation are independent of the chosen $(2,0)$ -form α and one thus obtains a map

¹We denote by $\{\cdot\}$ the linear span of the elements within. In the case of smooth differential forms, the coefficients are smooth real-valued functions.

$\psi : \mathcal{C}^+(V) \rightarrow G_2^+(\Lambda^2 V^*, \wedge_+)$ given by $J \mapsto \Lambda_J$. Note that $G = \text{GL}^+(V)$ acts smoothly and transitively from the left on $\mathcal{C}^+(V)$ via $(A, J) \mapsto A^{-1}JA$. Every element of $G_2^+(\Lambda^2 V^*, \wedge_+)$ admits a positively oriented elliptic conformal basis. It follows with [Lemma 1](#) that via pushforward, $\text{GL}^+(V)$ acts smoothly and transitively from the left on $G_2^+(\Lambda^2 V^*, \wedge_+)$ as well.

Proposition 1. *The map $\psi : \mathcal{C}^+(V) \rightarrow G_2^+(\Lambda^2 V^*, \wedge_+)$, $J \mapsto \Lambda_J$ is a G -equivariant diffeomorphism.*

Proof. Clearly the map ψ is G -equivariant. To prove that ψ is a diffeomorphism it is sufficient to show that $G_J = G_{\psi(J)}$ for all $J \in \mathcal{C}^+(V)$ where G_J and $G_{\psi(J)}$ denote the stabiliser subgroups of G with respect to J and $\psi(J)$ respectively. Choose $J \in \mathcal{C}^+(V)$, then we have $G_J \subset G_{\psi(J)}$. Write

$$J(v) = -e^2(v)e_1 + e^1(v)e_2 - e^4(v)e_3 + e^3(v)e_4$$

for some basis (e_i) of V and dual basis (e^i) of V^* . Then

$$\omega = e^1 \wedge e^3 - e^2 \wedge e^4 = \frac{1}{2} w_{kl} e^k \wedge e^l, \quad \phi = e^1 \wedge e^4 + e^2 \wedge e^3 = \frac{1}{2} f_{kl} e^k \wedge e^l$$

is a positively oriented conformal basis of Λ_J . Consequently every $A \in G_{\psi(J)}$ satisfies $A^*\omega = x\omega + y\phi$ and $A^*\phi = -y\omega + x\phi$, for some real numbers $(x, y) \neq 0$. The matrix representation a of A with respect to the basis (e_i) thus satisfies

$$a^t w a = xw + yf, \quad a^t f a = -yw + xf.$$

From this one easily concludes $awf = wfa$ which is equivalent to A commuting with J . \square

[Proposition 1](#) motivates the following:

Definition. A *splitting* of a complex structure J on V is a pair of lines $L_1, L_2 \in \mathbb{P}(\Lambda^2 V^*)$ such that $\Lambda_J = L_1 \oplus L_2$.

Call two 4-dimensional real vector spaces V, V' equipped with complex structures J, J' and splittings $(L_1, L_2), (L'_1, L'_2)$ equivalent, if there exists a complex linear map $A : V \rightarrow V'$ such that $A^*(L'_i) = L_i$ for $i = 1, 2$.

On $V = \mathbb{R}^4$ let $\omega_0 = e^1 \wedge e^3 - e^2 \wedge e^4$ and $\phi_0 = e^1 \wedge e^4 + e^2 \wedge e^3$ where e^1, \dots, e^4 denotes the standard basis of $(\mathbb{R}^4)^*$. Define $L_1 = \{\omega_0\}$ and $L_2 = \{\alpha\omega_0 + \phi_0\}$ for some nonnegative real number α . Orient $L_1 \oplus L_2$ by declaring ω_0, ϕ_0 to be a positively oriented basis and let J_0 be the associated complex structure. Then $S_\alpha = (L_1, L_2)$ is a splitting of J_0 .

Proposition 2. *Every pair (V, J) equipped with a splitting (L_1, L_2) is equivalent to (\mathbb{R}^4, J_0) equipped with the splitting S_α for some unique $\alpha \in \mathbb{R}_0^+$.*

Proof. Let $L_1 = \{\omega\}$ and $L_2 = \{\omega'\}$ for some 2-forms $\omega, \omega' \in \Lambda^2 V^*$. Since the wedge product restricts to be positive definite on $L_1 \oplus L_2$ we have $\omega \wedge \omega > 0$ and there exists a real number α , such that $\omega' = \alpha\omega + \phi$ for some 2-form ϕ satisfying $\omega \wedge \phi = 0$ and $\phi \wedge \phi > 0$. After possibly rescaling ω' we can assume that $\phi \wedge \phi = \omega \wedge \omega$ and that α is nonnegative. It follows with [Lemma 1](#) that there exists a linear map $A : V \rightarrow \mathbb{R}^4$ which identifies ω with ω_0 and ϕ with ϕ_0 , in particular A is complex linear. To prove uniqueness of α suppose $A : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ satisfies

$A^*\omega_0 = x\omega_0$ and $A^*(\alpha\omega_0 + \phi_0) = y(\beta\omega_0 + \phi_0)$ for some real numbers $x, y \neq 0$ and some nonnegative real numbers α, β . Then $A^*(\omega_0 \wedge \omega_0) = x^2\omega_0 \wedge \omega_0$ and consequently

$$A^*(\omega_0 \wedge (\alpha\omega_0 + \phi_0)) = \alpha x^2\omega_0 \wedge \omega_0 = xy\beta\omega_0 \wedge \omega_0,$$

which is equivalent to $\alpha x = \beta y$. We also have

$$A^*((\alpha\omega_0 + \phi_0) \wedge (\alpha\omega_0 + \phi_0)) = x^2(\alpha^2 + 1)\omega_0 \wedge \omega_0 = y^2(\beta^2 + 1)\omega_0 \wedge \omega_0,$$

which implies $x^2 = y^2$ and thus $\alpha^2 = \beta^2$. Since $\alpha, \beta \geq 0$, the claim follows. \square

For a splitting (L_1, L_2) , the unique nonnegative real number α provided by [Proposition 2](#) will be called the *degree* of the splitting. A splitting of degree 0 will be called *orthogonal*.

3. LOCAL EMBEDDABILITY OF REAL ANALYTIC PATH GEOMETRIES

3.1. Splittings of almost complex structures. Let X be a smooth 4-manifold and \mathfrak{J} be an almost complex structure with associated rank 2 vector bundle $\Lambda_{\mathfrak{J}} \subset \Lambda^2 TX^*$ whose fibre at $p \in X$ is the linear subspace $\Lambda_{\mathfrak{J}_p} \subset \Lambda^2 T_p X^*$ associated to $\mathfrak{J}_p : T_p X \rightarrow T_p X$. A *splitting* of \mathfrak{J} consists of a pair of smooth line bundles $L_1, L_2 \subset \Lambda^2 TX^*$ so that $\Lambda_{\mathfrak{J}} = L_1 \oplus L_2$.

3.2. Induced structure on hypersurfaces. A *CR-structure* on a 3-manifold M consists of a rank 2 subbundle $D \subset TM$ and a vector bundle endomorphism $I : D \rightarrow D$ which satisfies $I^2 = -\text{Id}_D$. A CR-structure (D, I) is called *nondegenerate* if D is nowhere integrable, i.e. a contact plane field. A closely related notion is that of a path geometry (see for instance [\[7\]](#) for a motivation of the following definition). A path geometry on a 3-manifold M consists of a pair of line subbundles (P_1, P_2) of TM which span a contact plane field. A CR-structure (D, I) and a path geometry (P_1, P_2) on M will be called *compatible* if $D = P_1 \oplus P_2$ and $I(P_1) = P_2$.

Let (L_1, L_2) be a splitting of the almost complex structure \mathfrak{J} on X and (ω, ϕ) a pair of 2-forms defined on some open subset $\tilde{U} \subset X$ which span (L_1, L_2) . Then the pair (ω, ϕ) is elliptic, i.e. (ω_p, ϕ_p) is elliptic for every point $p \in \tilde{U}$. Suppose $M \subset X$ is a hypersurface. Then [Lemma 2](#) implies that the 2-forms (ω, ϕ) remain linearly independent when pulled back to $M \cap \tilde{U}$. This is useful because of the following:

Lemma 3. *Let β_1, β_2 be smooth linearly independent 2-forms on a 3-manifold M . Then there exists a local coframing $\eta = (\eta^1, \eta^2, \eta^3)^t$ of M such that $\beta_1 = \eta_2 \wedge \eta_1$ and $\beta_2 = \eta_2 \wedge \eta_3$.*

Recall that a (local) *coframing* on M consists of three smooth linearly independent 1-forms defined on (some proper open subset of) M .

Proof of Lemma 3. Let $x : U \rightarrow \mathbb{E}^3$ be local coordinates on M with respect to which $\beta_1|_U = b_1 \cdot \star dx$ and $\beta_2|_U = b_2 \cdot \star dx$ for some smooth $b_i : U \rightarrow \mathbb{R}^3$ where \star denotes the Hodge-star of Euclidean space \mathbb{E}^3 . Define $e = (b_1 \times b_2) / |b_1 \times b_2| : U \rightarrow \mathbb{R}^3$ and

$$\eta_1 = (b_1 \times e) \cdot dx, \quad \eta_2 = e \cdot dx, \quad \eta_3 = (b_2 \times e) \cdot dx,$$

then (η^1, η^2, η^3) have the desired properties. \square

A local coframing of M obtained via [Lemma 3](#) and some (local) choice of 2-forms (ω, ϕ) spanning (L_1, L_2) will be called *adapted* to the structure induced by the splitting (L_1, L_2) . Independent of the particular adapted local coframings are the line subbundles P_1 and P_2 of TM , locally defined by

$$P_1 = \{\eta_1, \eta_2\}^\perp, \quad P_2 = \{\eta_2, \eta_3\}^\perp.$$

Call a hypersurface $M \subset X$ *nondegenerate* if $D = P_1 \oplus P_2$ is a contact plane field. Summarising, we have shown:

Proposition 3. *A nondegenerate hypersurface $M \subset X$ inherits a path geometry from the splitting (L_1, L_2) .*

Remark. Fixing a $(2,0)$ -form on X allows to define a coframing on a hypersurface $M \subset X$. For the construction of the coframing and its properties see [\[4\]](#).

3.3. Local embeddability. We conclude by using the Cartan-Kähler theorem to show that locally every real analytic path geometry is induced by an embedding into \mathbb{C}^2 equipped with the splitting $(\{\omega_0\}, \{\phi_0\})$. Here $\omega_0 = \text{Re}(dz^1 \wedge dz^2)$ and $\phi_0 = \text{Im}(dz^1 \wedge dz^2)$ where $z = (z^1, z^2)$ are standard coordinates on \mathbb{C}^2 . Writing $z^1 = x^1 + ix^2$ and $z^2 = x^3 + ix^4$ for standard coordinates $x = (x^i)$ on \mathbb{R}^4 , we have

$$\omega_0 = dx^1 \wedge dx^3 - dx^2 \wedge dx^4, \quad \phi_0 = dx^1 \wedge dx^4 + dx^2 \wedge dx^3.$$

In [\[5\]](#), as an application of his method of equivalence, Cartan has shown how to associate a Cartan geometry to every path geometry.

Definition. Let G be a Lie group and $H \subset G$ a Lie subgroup with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. A *Cartan geometry of type (G, H)* on a manifold M consists of a right principal H -bundle $\pi : B \rightarrow M$ together with a 1-form $\theta \in \mathcal{Q}^1(B, \mathfrak{g})$ which satisfies the following conditions:

- (i) $\theta_b : T_b B \rightarrow \mathfrak{g}$ is an isomorphism for every $b \in B$,
- (ii) $\theta(X_v) = v$ for every fundamental vector field $X_v, v \in \mathfrak{h}$,
- (iii) $(R_h)^* \theta = \text{Ad}_{\mathfrak{g}}(h^{-1}) \circ \theta$.

Here $\text{Ad}_{\mathfrak{g}}$ denotes the adjoint representation of G . The 1-form θ is called the *Cartan connection* of the Cartan geometry $(\pi : B \rightarrow M, \theta)$.

Denote by $H \subset \text{SL}(3, \mathbb{R})$ the Lie subgroup of upper triangular matrices. In modern language Cartan's result is as follows (for a proof see [\[3, 7\]](#)):

Theorem 1 (Cartan). *Given a path geometry (M, P_1, P_2) , then there exists a Cartan geometry $(\pi : B \rightarrow M, \theta)$ of type $(\text{SL}(3, \mathbb{R}), H)$ which has the following properties: Writing*

$$\theta = \begin{pmatrix} \theta_0^0 & \theta_1^0 & \theta_2^0 \\ \theta_0^1 & \theta_1^1 & \theta_2^1 \\ \theta_0^2 & \theta_1^2 & \theta_2^2 \end{pmatrix},$$

- (i) *for any section $\sigma : M \rightarrow B$, the 1-form $\phi = \sigma^* \theta$ satisfies $P_1 = \{\phi_1^2, \phi_0^2\}^\perp$ and $P_2 = \{\phi_0^1, \phi_0^2\}^\perp$. Moreover $\phi_0^1 \wedge \phi_0^2 \wedge \phi_1^2$ is a volume form on M .*

(ii) *The curvature 2-form $\Theta = d\theta + \theta \wedge \theta$ satisfies*

$$(3.1) \quad \Theta = \begin{pmatrix} 0 & \mathcal{W}_1 \theta_0^1 \wedge \theta_0^2 & (\mathcal{W}_2 \theta_0^1 + \mathcal{F}_2 \theta_1^2) \wedge \theta_0^2 \\ 0 & 0 & \mathcal{F}_1 \theta_1^2 \wedge \theta_0^2 \\ 0 & 0 & 0 \end{pmatrix}$$

for some smooth functions $\mathcal{W}_1, \mathcal{W}_2, \mathcal{F}_1, \mathcal{F}_2 : B \rightarrow \mathbb{R}$.

Using this result and the Cartan-Kähler theorem we obtain local embeddability in the real analytic category:

Theorem 2. *Let (M, P_1, P_2) be a real analytic path geometry. Then for every point $p \in M$ there exists a p -neighbourhood $U_p \subset M$ and a real analytic embedding $\varphi : U_p \rightarrow \mathbb{C}^2$ such that the path geometry induced by the splitting $(\{\omega_0\}, \{\phi_0\})$ is (P_1, P_2) on U_p .*

Proof. Let $(\pi : B \rightarrow M, \theta)$ denote the Cartan geometry of the path geometry (M, P_1, P_2) . On $N = B \times \mathbb{R}^4$ consider the exterior differential system with independence condition (\mathfrak{L}, ζ) where $\zeta = \zeta^1 \wedge \zeta^2 \wedge \zeta^3$ with $\zeta^1 = \theta_0^1, \zeta^2 = \theta_0^2, \zeta^3 = \theta_1^2$ and the differential ideal \mathfrak{L} is generated by the two 2-forms

$$\chi_1 = \theta_0^2 \wedge \theta_0^1 - \omega_0, \quad \chi_2 = \theta_0^2 \wedge \theta_1^2 - \phi_0.$$

The dual vector fields to the coframing (θ_k^i, dx^l) of N will be denoted by (T_k^i, ∂_{x^l}) . Let $G_k(TN) \rightarrow N$ be the Grassmann bundle of k -planes on N and $G_3(TN, \zeta) = \{E \in G_3(TN) \mid \zeta_E \neq 0\}$ where ζ_E denotes the restriction of ζ to the 3-plane E . Let $V^k(\mathfrak{L})$ denote the set of k -dimensional *integral elements* of \mathfrak{L} , i.e. those $E \in G_k(TN)$ for which $\beta_E = 0$ for every form $\beta \in \mathfrak{L}^k = \mathfrak{L} \cap \mathcal{Q}^k(N)$. The flag of integral elements $F = (E^0, E^1, E^2, E^3)$ of \mathfrak{L} given by $E^0 = \{0\}$, $E^1 = \{v_1\}$, $E^2 = \{v_1, v_2\}$, $E^3 = \{v_1, v_2, v_3\}$ where

$$\begin{aligned} v_1 &= T_0^1 + T_0^2 + T_1^2 + \partial_{x^4}, \\ v_2 &= T_0^0 + T_0^1 - T_1^2 + \partial_{x^1} + \partial_{x^2}, \\ v_3 &= T_1^1 - T_1^2 + \partial_{x^1}, \end{aligned}$$

has Cartan characters $(s_0, s_1, s_2, s_3) = (0, 2, 4, 3)$. Therefore, by Cartan's test, $V^3(\mathfrak{L})$ has codimension at least 8 at E^3 . However the forms of \mathfrak{L}^3 which impose independent conditions on the elements of $G_3(TN, \zeta)$ are the eight 3-forms $d\chi_i, \chi_i \wedge \zeta^k, i = 1, 2, k = 1, 2, 3$. It follows that $V^3(\mathfrak{L}) \cap G_3(TN, \zeta)$ has codimension 8 in $G_3(TN)$. Moreover computations show that $V^3(\mathfrak{L}) \cap G_3(TN, \zeta)$ is a smooth submanifold near E^3 , thus the flag F is Kähler regular and therefore the ideal \mathfrak{L} is involutive. Pick points $p \in M$ and $q = (b, 0) \in N$ with $\pi(b) = p$. By the Cartan-Kähler theorem there exists a 3-dimensional integral manifold $\bar{\psi} = (\bar{s}, \bar{\varphi}) : \Sigma \rightarrow B \times \mathbb{R}^4$ of (\mathfrak{L}, ζ) passing through q and having tangent space E^3 at q . Every volume form on M pulls back under π to a nowhere vanishing multiple of ζ . Since $\bar{\varphi}^* \zeta = \bar{s}^* \zeta \neq 0$, $\pi \circ \bar{s} : \Sigma \rightarrow M$ is a local diffeomorphism. Therefore $p \in M$ has a neighbourhood U_p on which there exists a real analytic immersion $\psi = (s, \varphi) : U_p \rightarrow B \times \mathbb{R}^4$ such that the pair (ψ, U_p) is an integral manifold of the EDS (N, \mathfrak{L}, ζ) and s a local section of $\pi : B \rightarrow M$. After possibly shrinking U_p we can assume that φ is an embedding. Since by construction $\varphi^*(\omega_0 + i\phi_0) = s^*(\theta_0^2 \wedge (\theta_0^1 + i\theta_1^2))$, it follows that the path geometry induced by φ is (P_1, P_2) on U_p . \square

Remark. Every nondegenerate hypersurface $M \subset \mathbb{C}^2$ also inherits a CR-structure (D, I) from the complex structure J on \mathbb{C}^2 : For every $p \in M$ define D_p to be the largest J_p -invariant subspace of $T_p M$ and I_p to be the restriction of J_p to D_p . Then (D, I) is easily seen to be compatible with the path geometry induced on M by $(\{\omega_0\}, \{\phi_0\})$.

Using this remark and [Theorem 2](#) we get the well-known:

Corollary 1. *Let (D, I) be a nondegenerate real analytic CR-structure on a 3-manifold M . Then for every point $p \in M$ there exists a p -neighbourhood U_p and a real analytic embedding $\varphi : U_p \rightarrow \mathbb{C}^2$, such that (D, I) is the CR-structure on U_p induced by the embedding φ .*

Proof. Pick a line bundle $P_2 \subset D$, define $P_1 = I(P_2)$ and apply [Theorem 2](#). \square

Remark. [Corollary 1](#) also holds without the nondegeneracy assumption and in higher dimensions [\[1\]](#). In [\[9\]](#), Nirenberg has constructed a smooth nondegenerate 3-dimensional CR-structure which is not induced by an embedding into \mathbb{C}^2 . It follows that the real analyticity assumption in [Theorem 2](#) is necessary.

REFERENCES

- [1] Aldo Andreotti and C. Denson Hill, *Complex characteristic coordinates and tangential Cauchy-Riemann equations*, Ann. Scuola Norm. Sup. Pisa (3) **26** (1972), 299–324. MR 0460724 [7](#)
- [2] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, and P. A. Griffiths, *Exterior differential systems*, Mathematical Sciences Research Institute Publications, vol. 18, Springer-Verlag, New York, 1991. MR 1083148 [1](#)
- [3] Robert Bryant, Phillip Griffiths, and Lucas Hsu, *Toward a geometry of differential equations*, Geometry, topology, & physics, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995, pp. 1–76. MR 1358612 [5](#)
- [4] Robert L. Bryant, *Real hypersurfaces in unimodular complex surfaces*, [arXiv:math/0407472v1](#), 2004. [5](#)
- [5] Elie Cartan, *Sur les variétés à connexion projective*, Bull. Soc. Math. France **52** (1924), 205–241. MR 1504846 [5](#)
- [6] S. K. Donaldson, *Two-forms on four-manifolds and elliptic equations*, Inspired by S. S. Chern, Nankai Tracts Math., vol. 11, World Sci. Publ., Hackensack, NJ, 2006, pp. 153–172. MR 2313334 [1](#)
- [7] Thomas A. Ivey and J. M. Landsberg, *Cartan for beginners: differential geometry via moving frames and exterior differential systems*, Graduate Studies in Mathematics, vol. 61, American Mathematical Society, Providence, RI, 2003. MR 2003610 [1](#), [4](#), [5](#)
- [8] V. V. Lychagin and V. N. Rubtsov, *The theorems of Sophus Lie for the Monge-Ampère equations*, Dokl. Akad. Nauk BSSR **27** (1983), no. 5, 396–398. MR 703049 [2](#)
- [9] Louis Nirenberg, *On a question of Hans Lewy.*, Russ. Math. Surv. **29** (1974), no. 2, 251–262 (English. Russian original). [7](#)

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