On Kähler Metrisability of Two-dimensional Complex Projective Structures

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ABSTRACT. We derive necessary conditions for a complex projective structure on a complex surface to arise via the Levi-Civita connection of a (pseudo-)Kähler metric. Furthermore we show that the (pseudo-)Kähler metrics defined on some domain in the projective plane which are compatible with the standard complex projective structure are in one-to-one correspondence with the hermitian forms on $\mathbb{C}^3$ whose rank is at least two. This is achieved by prolonging the relevant finite-type first order linear differential system to closed form. Along the way we derive the complex projective Weyl and Liouville curvature using the language of Cartan geometries.

1. Introduction

Recall that an equivalence class of affine torsion-free connections on the tangent bundle of a smooth manifold N is called a (real) projective structure **[11, 38, 39]**. Two connections ∇ and ∇' are *projectively equivalent* if they share the same unparametrised geodesics. This condition is equivalent to ∇ and ∇' inducing the same parallel transport on the projectivised tangent bundle $\mathbb{P}TN$.

It is a natural task to (locally) characterise the projective structures arising via the Levi-Civita connection of a (pseudo-)Riemannian metric. R. Liouville **[25]** made the crucial observation that the Riemannian metrics on a surface whose Levi-Civita connection belongs to a given projective class precisely correspond to nondegenerate solutions of a certain projectively invariant finite-type linear system of partial differential equations. In **[3]** Bryant, Eastwood and Dunajski used Liouville's observation to solve the two-dimensional version of the Riemannian metrisability problem. In another direction it was shown in **[29]** that on a surface locally every affine torsion-free connection is projectively equivalent to a conformal connection (see also **[28]**). Local existence of a connection with skew-symmetric Ricci tensor in a given projective class was investigated in **[36]** (see also **[23]** for a connection to Veronese webs). Liouville's result generalises to higher dimensions **[30]** and the corresponding finite-type differential system was prolonged to closed form in **[14, 30]**. Several necessary conditions for Riemann metrisability of a projective structure in dimensions larger than two were given in **[33]**. See also **[7, 16]** for the role of Einstein metrics in projective geometry.

Now let M be a complex manifold of complex dimension $d > 1$ with integrable almost complex structure map J. Two affine torsion-free connections ∇ and ∇' on TM which preserve J are called *complex projectively equivalent* if they share the same *generalised geodesics* (for the notion of a curved complex projective

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structure on Riemann surfaces see [5]). A *generalised geodesic* is a smoothly immersed curve $\gamma \subset M$ with the property that the 2-plane spanned by $\dot{\gamma}$ and $J\dot{\gamma}$ is parallel along γ. Complex projective geometry was introduced by Otsuki and Tashiro [35, 37]. Background on the history of complex projective geometry and its recently discovered connection to Hamiltonian 2-forms (see [1] and references therein) may be found in [26].

In the complex setting it is natural to study the *Kähler metrisability problem*, i.e. try to (locally) characterise the complex projective structures which arise via the Levi-Civita connection of a (pseudo-)Kähler metric. Similar to the real case, the Kähler metrics whose Levi-Civita connection belongs to a given complex projective class precisely correspond to nondegenerate solutions of a certain complex projectively invariant finite-type linear system of partial differential equations [12, 26, 31].

In this note we prolong the relevant differential system to closed form in the surface case. In doing so we obtain necessary conditions for Kähler metrisability of a complex projective structure $[\nabla]$ on a complex surface and show in particular that the generic complex projective structure is not Kähler metrisable. Furthermore we show that the space of Kähler metrics compatible with a given complex projective structure is algebraically constrained by the complex projective Weyl curvature of $[\nabla]$. We also show that the (pseudo-)Kähler metrics defined on some domain in $\mathbb{CP}^2$ which are compatible with the standard complex projective structure are in one-to-one correspondence with the hermitian forms on $\mathbb{C}^3$ whose rank is at least two. A result whose real counterpart is a well-known classical fact. This note concerns itself with the complex 2-dimensional case, but there are obvious higher dimensional generalisations that can be treated with the same techniques.

The reader should be aware that the results presented here can also be obtained by using the elegant and powerful theory of Bernstein–Gelfand–Gelfand (BGG) sequences developed by Čap, Slovák and Souček [10] (see also the article of Calderbank and Diemer [6]). In particular, the prolongation computed here is an example of a prolongation connection of a first BGG equation in parabolic geometry and may be derived using the techniques developed in [18].

This note aims at providing an intermediate analysis between the abstract BGG machinery and pure local coordinate computations. This is achieved by carrying out the computations on the parabolic Cartan geometry of a complex projective surface.

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2. Complex projective surfaces

2.1. Definitions. Let M be a complex 2-manifold with integrable almost complex structure map J and ∇ an affine torsion-free connection on TM. We call ∇ *complex-linear* if $\nabla J = 0$. A *generalised geodesic* for ∇ is a smoothly immersed curve $\gamma \subset M$ with the property that the 2-plane spanned by $\dot\gamma$ and $J\dot\gamma$ is parallel along γ, i.e. γ satisfies the reparametrisation invariant condition

$$\nabla_{\dot\gamma}\dot\gamma \wedge \dot\gamma \wedge J\dot\gamma = 0. \tag{2.1}$$

We call two complex linear torsion-free connections ∇ and ∇' on M *complex projectively equivalent*, if they have the same generalised geodesics. An equivalence class of complex projectively equivalent connections is called a *complex projective structure* and will be denoted by $[\nabla]$. A complex 2-manifold equipped with a complex projective structure will be called a *complex projective surface*.

Remark 2.1. What we here call a complex projective structure was originally called a *holomorphic projective structure* by Tashiro **[37]** and others. Once it was realised that in general complex projective structures are not holomorphic in any reasonable way, the name *h-projective structure* was used – and is still so – see for instance **[15, 21, 26]**. Furthermore, what we here call generalised geodesics are called *h-planar curves* in the literature using the name h-projective. One might argue that the notion of a complex projective structure can be confused with well-established notions in algebraic geometry. For this reason complex projective is sometimes also abbreviated to c-projective (see for instance **[2]**).

Extending ∇ to the complexified tangent bundle $T^{\mathbb{C}}M \to M$, it follows from the complex linearity of ∇ that for every local holomorphic coordinate system $z = (z^i) : U \to \mathbb{C}^2$ on M there exist unique complex-valued functions Γ^i_{jk} on U, so that

$$\nabla_{\partial_{z^j}}\partial_{z^k} = \Gamma^i_{jk}\partial_{z^i}.$$

We call the functions Γ^i_{jk} the *complex Christoffel symbols* of ∇. Tashiro showed **[37]** that two torsion-free complex linear connections ∇ and ∇' on M are complex projectively equivalent if and only if there exists a (1,0)-form $\beta \in \Omega^{1,0}(M,\mathbb{R})$ so that

$$\nabla'_Z W - \nabla_Z W = \beta(Z)W + \beta(W)Z \tag{2.2}$$

for all (1,0) vector fields $Z, W \in \Gamma(T^{1,0}M)$. In analogy to the real case one can use **(2.2)** to show that ∇ and ∇' are complex projectively equivalent if and only if they induce the same parallel transport on the complex projectivised tangent bundle $\mathbb{P}T^{1,0}M$.

Writing Γ^i_{jk} and $\hat\Gamma^i_{jk}$ for the complex Christoffel symbols of ∇ and ∇' with respect to some holomorphic coordinates $z = (z^i)$ and $\beta = \beta_i \mathrm{d}z^i$, equation **(2.2)** translates to

$$\hat\Gamma^i_{jk} = \Gamma^i_{jk} + \delta^i_j\beta_k + \delta^i_k\beta_j. \tag{2.3}$$

Note that formally equation **(2.3)** is identical to the equation relating two real projectively equivalent connections on a real manifold. In particular, similarly to the real case (see **[11, 38]**), the functions

$$\Pi^i_{jk} = \Gamma^i_{jk} - \frac{1}{3}\left(\Gamma^l_{lj}\delta^i_k + \Gamma^l_{lk}\delta^i_j\right) \tag{2.4}$$

are complex projectively invariant in the sense that they only depend on the coordinates z. Moreover locally $[\nabla]$ can be recovered from the functions Π^i_{jk} and two torsion-free complex linear connections are complex projectively equivalent if and only if they give rise to the same functions Π^i_{jk} in every holomorphic coordinate system.

A complex projective structure $[\nabla]$ is called *holomorphic* if the functions Π^i_{jk} are holomorphic in every holomorphic coordinate system. Gunning **[17]** obtained relations on characteristic classes of complex manifolds carrying holomorphic projective structures. The condition on a manifold to carry a holomorphic projective structure is particularly restrictive in the case of compact complex surfaces. See also the beautiful twistorial interpretation of holomorphic projective surfaces by Hitchin **[19]** and **Remark 2.4**.

2.2. Cartan geometry.

A complex projective structure admits a description in terms of a *normal Cartan geometry* modelled on complex projective space $\mathbb{CP}^n$, following the work of Ochiai **[34]**: see **[20]** and **[40]**. The reader unfamiliar with Cartan geometries may consult **[9]** for a modern introduction. We will restrict to the construction in the complex two-dimensional case.

Let $\mathrm{PSL}(3,\mathbb{C})$ act on $\mathbb{CP}^2$ from the left in the obvious way and let P denote the stabiliser subgroup of the element $[1,0,0]^t \in \mathbb{CP}^2$. We have:

Theorem 2.1. *Suppose $(M, J, [\nabla])$ is a complex projective surface. Then there exists (up to isomorphism) a unique real Cartan geometry $(\pi : B \to M, \theta)$ of type $(\mathrm{PSL}(3,\mathbb{C}), P)$ such that for every local holomorphic coordinate system $z = (z^i) : U \to \mathbb{C}^2$, there exists a unique section $\sigma_z : U \to B$ satisfying*

$$(\sigma_z)^*\theta = \begin{pmatrix} 0 & \phi^0_1 & \phi^0_2 \\ \phi^1_0 & \phi^1_1 & \phi^1_2 \\ \phi^2_0 & \phi^2_1 & \phi^2_2 \end{pmatrix} \tag{2.5}$$

where

$$\phi^i_0 = \mathrm{d}z^i, \quad \text{and} \quad \phi^i_j = \Pi^i_{jk}\mathrm{d}z^k, \quad \text{and} \quad \phi^0_i = \Pi_{ik}\mathrm{d}z^k,$$

with

$$\Pi_{ij} = \Pi^k_{il}\Pi^l_{jk} - \frac{\partial \Pi^k_{ij}}{\partial z^k}$$

and Π^i_{jk} denote the complex projective invariants with respect to z^i defined in **(2.4)**.

Remark 2.2. Suppose $\varphi : (M, J, [\nabla]) \to (M', J', [\nabla]')$ is a biholomorphism between complex projective surfaces identifying the complex projective structures, then there exists a diffeomorphism $\hat{\varphi} : B \to B'$ which is a P-bundle map covering φ and which satisfies $\hat{\varphi}^*\theta' = \theta$. Conversely, every diffeomorphism $\Phi : B \to B'$ that is a P-bundle map and satisfies $\Phi^*\theta' = \theta$ is of the form $\Phi = \hat{\varphi}$ for a unique biholomorphism $\varphi : M \to M'$ identifying the complex projective structures.

Example 2.1. Let $B = \mathrm{PSL}(3,\mathbb{C})$ and let θ denote its Maurer-Cartan form. Setting $M = B/P \simeq \mathbb{CP}^2$ and $\pi : \mathrm{PSL}(3,\mathbb{C}) \to \mathbb{CP}^2$ the natural quotient projection, one obtains a complex projective structure on $\mathbb{CP}^2$ whose generalised geodesics are the smoothly immersed curves $\gamma \subset \mathbb{CP}^1$ where $\mathbb{CP}^1 \subset \mathbb{CP}^2$ is any linearly embedded projective line. This is precisely the complex projective structure associated to the Levi-Civita connection of the Fubini-Study metric on $\mathbb{CP}^2$. This example satisfies $\mathrm{d}\theta + \theta \wedge \theta = 0$ and is hence called *flat*.

Let $(\pi : B \to M, \theta)$ be the Cartan geometry of a complex projective structure $(J, [\nabla])$ on a simply-connected surface M whose Cartan connection satisfies $\mathrm{d}\theta + \theta \wedge \theta = 0$. Then there exists a local diffeomorphism $\Phi : B \to \mathrm{PSL}(3, \mathbb{C})$ pulling back the Maurer-Cartan form of $\mathrm{PSL}(3, \mathbb{C})$ to θ and consequently, a local biholomorphism $\varphi : M \to \mathbb{CP}^2$ identifying the projective structure on M with the standard flat structure on $\mathbb{CP}^2$.

2.3. Bianchi-identities. **Theorem 2.1** implies that the curvature form $\Theta = \mathrm{d}\theta + \theta \wedge \theta$ satisfies

$$\Theta = \mathrm{d}\theta + \theta \wedge \theta = \begin{pmatrix} 0 & \Theta^0_1 & \Theta^0_2 \\ 0 & \Theta^1_1 & \Theta^1_2 \\ 0 & \Theta^2_1 & \Theta^2_2 \end{pmatrix} \tag{2.6}$$

with

$$\Theta^0_i = L_i \theta^1_0 \wedge \theta^2_0 + K_{il\bar{j}} \theta^l_0 \wedge \overline{\theta^j_0}, \quad \Theta^i_k = W^i_{kl\bar{j}} \theta^l_0 \wedge \overline{\theta^j_0}$$

for unique complex-valued functions L_i, $K_{il\bar{j}}$, and $W^i_{kl\bar{j}}$ on B satisfying $W^l_{li\bar{j}} = 0$. Note that by construction, with respect to local holomorphic coordinates $z = (z^i)$, we obtain

$$(\sigma_z)^* W^i_{kl\bar{j}} = -\frac{\partial \Pi^i_{kl}}{\partial \bar{z}^j}. \tag{2.7}$$

Differentiation of the structure equations **(2.6)** gives

$$0 = \mathrm{d}^2\theta^i_0 = W^i_{lk\bar{j}} \theta^l_0 \wedge \theta^k_0 \wedge \overline{\theta^j_0}, \quad \text{and} \quad 0 = \mathrm{d}^2\theta^0_0 = K_{ik\bar{j}} \theta^i_0 \wedge \theta^k_0 \wedge \overline{\theta^j_0}$$

which yields the algebraic Bianchi-identities

$$W^i_{lk\bar{j}} = W^i_{kl\bar{j}}, \quad \text{and} \quad K_{ik\bar{j}} = K_{ki\bar{j}}.$$

2.3.1. *Complex projective Weyl curvature.* The identities $\mathrm{d}^2\theta^i_k = 0$ yield

$$\kappa^i_{kl\bar{j}} \wedge \theta^l_0 \wedge \overline{\theta^j_0} = 0$$

with

$$\kappa^i_{kl\bar{j}} = \mathrm{d}W^i_{kl\bar{j}} + W^i_{kl\bar{j}} \left(\theta^0_0 + \overline{\theta^0_0}\right) + K_{kl\bar{j}} \theta^i_0 - W^i_{ls\bar{j}} \theta^s_k - W^i_{ks\bar{j}} \theta^s_l + W^s_{kl\bar{j}} \theta^i_s - W^i_{kl\bar{s}} \overline{\theta^s_l}$$

which implies that there exist complex-valued functions $W^i_{kl\bar{j}\bar{s}}$ and $W^i_{kl\bar{j}s}$ on B satisfying

$$W^i_{kl\bar{j}\bar{s}} = W^i_{lk\bar{j}\bar{s}} = W^i_{kl\bar{s}\bar{j}}, \quad W^k_{kl\bar{j}\bar{s}} = W^k_{kl\bar{j}s} = 0, \quad W^i_{kl\bar{j}s} = W^i_{lk\bar{j}s}$$

such that

$$\mathrm{d}W^i_{kl\bar{j}} = \left(W^i_{kl\bar{j}s} + \delta^i_k K_{sl\bar{j}} + \delta^i_l K_{sk\bar{j}} - 3\delta^i_s K_{kl\bar{j}}\right) \theta^s_0 + W^i_{kl\bar{j}\bar{s}} \overline{\theta^s_0} + \varphi^i_{kl\bar{j}} \tag{2.8}$$

where

$$\varphi^i_{kl\bar{j}} = -W^i_{kl\bar{j}} \left(\theta^0_0 + \overline{\theta^0_0}\right) + W^i_{ls\bar{j}} \theta^s_k + W^i_{ks\bar{j}} \theta^s_l - W^s_{kl\bar{j}} \theta^i_s + W^i_{kl\bar{s}} \overline{\theta^s_j}. \tag{2.9}$$

Let $\mathrm{End}_0(TM,J)$ denote the bundle whose fibre at $p\in M$ consists of the J-linear endomorphisms of T_pM which are complex-traceless. It follows with the structure equations (**Eq. (2.6)**,**Eq. (2.8)**,**Eq. (2.9)**) and straightforward computations, that there exists a unique (1,1)-form W on M with values in $\mathrm{End}_0(TM,J)$ for which

$$W\left(\frac{\partial}{\partial z^l},\frac{\partial}{\partial \overline{z^j}}\right)\frac{\partial}{\partial z^k}=(\sigma_z)^*W^i_{kl\bar{j}}\frac{\partial}{\partial z^i}=-\frac{\partial \Pi^i_{kl}}{\partial \bar{z}^j}\frac{\partial}{\partial z^i}$$

in every local holomorphic coordinate system $z=(z^i)$ on M. Here, as usual, we extend tensor fields on M complex multilinearly to the complexified tangent bundle of M. The bundle-valued 2-form W is called the *complex projective Weyl curvature* of $[\nabla]$. We obtain:

Proposition 2.1. *A complex projective structure $[\nabla]$ on a complex surface (M,J) is holomorphic if and only if the complex projective Weyl tensor of $[\nabla]$ vanishes.*

2.3.2. *Complex projective Liouville curvature.* From $\mathrm{d}^2\theta^0_i\wedge\overline{\theta^1_0}\wedge\overline{\theta^2_0}=0$ one sees after a short computation that

$$\mathrm{d}L_i=-4L_i\theta^0_0+L_j\theta^j_i+L_{ij}\theta^j_0+L_{i\bar{j}}\overline{\theta^j_0} \tag{2.10}$$

for unique complex-valued functions $L_{i\bar{j}}, L_{ij}$ on B. Using this last equation it is easy to check that the π-semibasic quantity

$$(L_1\theta^1_0+L_2\theta^2_0)\otimes(\theta^1_0\otimes\theta^2_0) \tag{2.11}$$

is invariant under the P right action and thus the π-pullback of a tensor field λ on M which is called the *complex projective Liouville curvature* (see the note of R. Liouville **[24]** for the construction of λ in the real case).

Remark 2.3. In the case of real projective structures on surfaces, the projective Weyl curvature vanishes identically. Furthermore, note that contrary to the complex projective Liouville curvature, the complex projective Weyl tensor exists as well in higher dimensions, but also contains (2,0) parts (see **[37]** for details).

The differential Bianchi-identity **(2.8)** implies that if the functions $W^i_{kl\bar{j}}$ vanish identically, then the functions $K_{ik\bar{j}}$ must vanish identically as well. We have thus shown:

Proposition 2.2. *A complex projective structure $[\nabla]$ on a complex surface (M,J) is flat if and only the complex projective Liouville and Weyl curvature vanish.*

Remark 2.4. In **[22]** Kobayashi and Ochiai classified compact complex surfaces carrying flat complex projective structures. More recently Dumitrescu **[13]** showed among other things that a holomorphic projective structure on a compact complex surface must be flat (see also the results by McKay about holomorphic Cartan geometries **[27]**).

2.3.3. *Further identities.* We also obtain

$$0=\mathrm{d}^2\theta^0_i=\kappa_{ik\bar{j}}\wedge\overline{\theta^j_0}\wedge\theta^k_0$$

with[1]

$$\kappa_{ik\bar{j}} = -\,\mathrm{d}K_{ik\bar{j}} + \frac{1}{2}\varepsilon_{sk}L_{i\bar{j}}\theta^s_0 - K_{ik\bar{j}}\left(2\theta^0_0 + \overline{\theta^0_0}\right) + K_{sk\bar{j}}\theta^s_i + K_{si\bar{j}}\theta^s_k - \\ - W^s_{ik\bar{j}}\theta^0_s + K_{ik\bar{s}}\overline{\theta^s_j}.$$

It follows that there are complex-valued functions $K_{ik\bar{j}l}$ and $K_{kl\bar{i}\bar{j}}$ on B satisfying

$$K_{ik\bar{j}l} = K_{ki\bar{j}l}, \quad \text{and} \quad K_{kl\bar{i}\bar{j}} = K_{lk\bar{i}\bar{j}} = K_{kl\bar{j}\bar{i}}$$

such that

$$\mathrm{d}K_{ik\bar{j}} = \left(K_{ik\bar{j}s} + \frac{1}{4}\left(\varepsilon_{sk}L_{i\bar{j}} + \varepsilon_{si}L_{k\bar{j}}\right)\right)\theta^s_0 + K_{ik\bar{j}\bar{s}}\overline{\theta^s_0} + \varphi_{ik\bar{j}} \tag{2.12}$$

where

$$\varphi_{ik\bar{j}} = -K_{ik\bar{j}}\left(2\theta^0_0 + \overline{\theta^0_0}\right) + K_{sk\bar{j}}\theta^s_i + K_{si\bar{j}}\theta^s_k - W^s_{ik\bar{j}}\theta^0_s + K_{ik\bar{s}}\overline{\theta^s_j}.$$

2.4. Complex and generalised geodesics.

It is worth explaining how the generalised geodesics of $[\nabla]$ appear in the Cartan geometry $(\pi : B \to M, \theta)$. To this end let $G \subset P \subset \mathrm{PSL}(3,\mathbb{C})$ denote the quotient group of the group of upper triangular matrices of unit determinant modulo its center. The quotient B/G is the total space of a fibre bundle over M whose fibre P/G is diffeomorphic to $\mathbb{CP}^1$. In fact, B/G may be identified with the total space of the the complex projectivised tangent bundle $\tau : \mathbb{P}(T^{1,0}M) \to M$ of (M,J). Writing $\theta = (\theta^i_j)_{i,j=0..2}$, **Theorem 2.1** implies that the real codimension 4-subbundle of TB defined by $\theta^2_0 = \theta^2_1 = 0$ descends to a real rank 2 subbundle $E \subset T\mathbb{P}(T^{1,0}M)$. The integral manifolds of E can most conveniently be identified in local coordinates. Let $z = (z^1, z^2) : U \to \mathbb{C}^2$ be a local holomorphic coordinate system on M and write ϕ for the pullback of θ with the unique section σ_z associated to z in **Theorem 2.1**. We obtain a local trivialisation of Cartan's bundle

$$\varphi : U \times P \to \pi^{-1}(U)$$

so that for $(z,p) \in U \times P$ we have

$$(\varphi^*\theta)_{(z,p)} = (\omega_P)_p + \mathrm{Ad}(p^{-1}) \circ \phi_z \tag{2.13}$$

where ω_P denotes the Maurer-Cartan form of P and Ad the adjoint representation of $\mathrm{PSL}(3,\mathbb{C})$. Consider the Lie group $\tilde{P} \subset \mathrm{SL}(3,\mathbb{C})$ whose elements are of the form

$$\begin{pmatrix} \det a^{-1} & b \\ 0 & a \end{pmatrix} \tag{2.14}$$

for $a \in \mathrm{GL}(2,\mathbb{C})$ and $b^t \in \mathbb{C}^2$. By construction, the elements of P are equivalence classes of elements in $\tilde{P}$ where two elements are equivalent if they differ by scalar multiplication with a complex cube root of 1. The canonical projection $\tilde{P} \to P$ will be denoted by ν. Note that a piece N of an integral manifold of E that is contained in $\tau^{-1}(U)$ is covered by a map

$$(z^1, z^2, p) : N \to U \times \tilde{P}$$

[1] We write ε_{ij} for the antisymmetric 2-by-2 matrix satisfying $\varepsilon_{12} = 1$ and ε^{ij} for the inverse matrix.

where $p : N \to \tilde{P}$ may be taken to be of the form

$$p = \begin{pmatrix} \frac{1}{(a_1)^2+(a_2)^2} & 0 & 0 \\ 0 & a_1 & -a_2 \\ 0 & a_2 & a_1 \end{pmatrix}$$

for smooth complex-valued functions $a_i : N \to \mathbb{C}$ satisfying $(a_1)^2 + (a_2)^2 \neq 0$.

We first consider the case where N is one-dimensional. We fix a local coordinate t on N. It follows with **(2.13)** and straightforward calculations that

$$\left(\varphi \circ (z^1, z^2, \nu \circ p)\right)^* \theta_0^2 = \frac{a_1 \dot{z}^2 - a_2 \dot{z}^1}{((a_1)^2 + (a_2)^2)^2} \mathrm{d}t$$

where $\dot{z}^i$ denote the derivative of z^i with respect to t. Hence we may take

$$a_1 = \dot{z}^1 \quad \text{and} \quad a_2 = \dot{z}^2.$$

Writing $\beta = \left(\varphi \circ (z^1, z^2, \nu \circ p)\right)^* \theta_1^2$ and using **(2.13)** again, we compute

$$\beta = \left[\dot{z}^1 \ddot{z}^2 - \dot{z}^2 \ddot{z}^1 + \left(\dot{z}^1 \dot{z}^2 (\Pi^2_{21} - \Pi^1_{11}) + (\dot{z}^1)^2 \Pi^2_{11} - (\dot{z}^2)^2 \Pi^1_{12}\right) \dot{z}^1 + \right.$$
$$\left. + \left(\dot{z}^1 \dot{z}^2 (\Pi^2_{22} - \Pi^1_{12}) + (\dot{z}^1)^2 \Pi^2_{12} - (\dot{z}^2)^2 \Pi^1_{22}\right) \dot{z}^2\right] \frac{\mathrm{d}t}{(\dot{z}^1)^2 + (\dot{z}^2)^2}.$$

Note that since $\Pi^i_{ik} = 0$ for $k = 1, 2$, it follows that $\beta \equiv 0$ is equivalent to (z^1, z^2) satisfying the following ODE system

$$\dot{z}^i \left(\ddot{z}^j + \Pi^j_{kl} \dot{z}^k \dot{z}^l\right) = \dot{z}^j \left(\ddot{z}^i + \Pi^i_{kl} \dot{z}^k \dot{z}^l\right), \quad i, j = 1, 2.$$

This last system is easily seen to be equivalent to the system **(2.1)**. Consequently, the one-dimensional integral manifolds of E are the generalised geodesics of $[\nabla]$.

Note that in the case of two-dimensional integral manifolds the above computations carry over where t is now a complex parameter, i.e. the two-dimensional integral manifolds are immersed complex curves $Y \subset M$ for which $\nabla_{\dot{Y}} \dot{Y}$ is proportional to $\dot{Y}$ for some (and hence any) $\nabla \in [\nabla]$. This last condition is equivalent to Y being a totally geodesic immersed complex curve with respect to $([\nabla], J)$ (c.f. **[32, Lemma 4.1]**) . A totally geodesic immersed complex curve $Y \subset M$ which is maximally extended is called a *complex geodesic*. Since the complex geodesics are the (maximally extended) two-dimensional integral manifolds of E, they exist only provided that E is integrable. We will next determine the integrability conditions for E. Recall that $E \subset T\mathbb{P}(T^{1,0}M)$ is defined by the equations $\theta_1^2 = \theta_0^2 = 0$ on B. It follows with the structure equations **(2.6)** that

$$\mathrm{d}\theta_0^2 = 0 \mod \theta_0^2, \theta_1^2$$

and

$$\mathrm{d}\theta_1^2 = W^2_{11\bar{j}} \theta_0^1 \wedge \overline{\theta_0^j} \mod \theta_0^2, \theta_1^2.$$

Consequently, E is integrable if and only if $W^2_{11\bar{1}} = W^2_{11\bar{2}} = 0$. As a consequence of **(2.8)** and $W^2_{11\bar{1}} = 0$ we obtain

$$0 = \varphi^2_{11\bar{1}} = -W^2_{11\bar{1}} \left(\theta_0^0 + \overline{\theta_0^0}\right) + W^2_{1s\bar{1}} \theta_1^s + W^2_{1s\bar{1}} \theta_1^s - W^s_{11\bar{1}} \theta_s^2 + W^2_{11\bar{s}} \overline{\theta_j^s},$$

which is equivalent to $2W^2_{12\bar{1}} = W^1_{11\bar{1}}$. Using the symmetries of the complex projective Weyl tensor we compute

$$W^1_{11\bar{1}} = -W^2_{21\bar{1}} = 2W^2_{12\bar{1}} = 2W^2_{21\bar{1}},$$

thus showing that $W^1_{11\bar{1}} = W^2_{12\bar{1}} = 0$. From this we obtain

$$0 = \varphi^1_{11\bar{1}} = 2W^1_{12\bar{1}}\theta^2_{\bar{1}} - W^2_{11\bar{1}}\theta^1_2 + W^1_{11\bar{2}}\overline{\theta^2_1}.$$

thus implying $W^1_{12\bar{1}} = W^2_{11\bar{1}} = W^1_{11\bar{2}} = 0$. Continuing in this vein allows to conclude that all components of the complex projective Weyl tensor must vanish. We may summarise:

Proposition 2.3. *Let $(M, J, [\nabla])$ be a complex projective surface. Then the following statements are equivalent:*

(i) *$[\nabla]$ is holomorphic;*
(ii) *The complex projective Weyl tensor of $[\nabla]$ vanishes;*
(iii) *The rank 2 bundle $E \to \mathbb{P}(T^{1,0}M)$ is Frobenius integrable;*
(iv) *Every complex line $L \subset T^{1,0}M$ is tangent to a unique complex geodesic.*

Remark 2.5. The standard flat complex projective structure on $\mathbb{CP}^2$ is holomorphic and the complex geodesics are simply the linearly embedded projective lines $\mathbb{CP}^1 \subset \mathbb{CP}^2$.

Remark 2.6. Note that the integrability conditions for E are a special case of a more general result obtained by Čap in **[8]**. There it is shown that E is part of an elliptic CR structure of CR dimension and codimension 2, which the complex projective structure induces on $\mathbb{P}(T^{1,0}M)$. Furthermore, it is also shown that the integrability of E is equivalent to the holomorphicity of the complex projective surface.

3. Kähler metrisability

In this section we will derive necessary conditions for a complex projective structure $[\nabla]$ on a complex surface (M, J) to arise via the Levi-Civita connection of a (pseudo-)Kähler metric. There exists a complex projectively invariant linear first order differential operator acting on J-hermitian (2,0) tensor fields on M with weight 1/3, i.e sections of the bundle $S^2_J(TM) \otimes \left(\Lambda^4(T^*M)\right)^{1/3}$. This differential operator has the property that nondegenerate sections in its kernel are in one-to-one correspondence with (pseudo-)Kähler metrics on M whose Levi-Civita connection is compatible with $[\nabla]$ (see **[12, 26, 31]**).

3.1. The differential analysis.

We will show that in the surface case, the (pseudo-)Kähler metrics on $(M, J, [\nabla])$ whose Levi-Civita connection is compatible with $[\nabla]$ can equivalently be characterised in terms of a differential system on Cartan's bundle $(\pi : B \to M, \theta)$.

Proposition 3.1. *Suppose the (pseudo-)Kähler metric g is compatible with $[\nabla]$. Then, writing $\pi^*g = g_{i\bar{j}}\theta^i_0 \circ \overline{\theta^j_0}$ and setting $h_{i\bar{j}} = g_{i\bar{j}}\left(g_{1\bar{1}}g_{2\bar{2}} - |g_{1\bar{2}}|^2\right)^{-2/3}$, we* *have*

$$\mathrm{d}h_{i\bar{j}} = h_{i\bar{j}}\left(\theta^0_0+\overline{\theta^0_0}\right) + h_{i\bar{s}}\overline{\theta^s_j} + h_{s\bar{j}}\theta^s_i + h_i\overline{\varepsilon_{sj}\theta^s_0} + \overline{h_j}\varepsilon_{si}\theta^s_0 \tag{3.1}$$

for some complex-valued functions h_i on B. Conversely, suppose there exist complex-valued functions $h_{i\bar{j}}=\overline{h_{j\bar{i}}}$ and h_i on B solving **(3.1)** *and satisfying $\left(h_{1\bar{1}}h_{2\bar{2}}-|h_{1\bar{2}}|^2\right)$ 0, then the symmetric 2-form*

$$h_{i\bar{j}}\left(h_{1\bar{1}}h_{2\bar{2}}-|h_{1\bar{2}}|^2\right)^{-2}\theta^i_0\circ\overline{\theta^j_0}$$

is the π-pullback of a $[\nabla]$-compatible (pseudo-)Kähler metric on M.

Proof. Let g be a (pseudo-)Kähler metric on (M,J) and write $g=g_{i\bar{j}}\mathrm{d}z^i\circ\mathrm{d}\overline{z^j}$ for local holomorphic coordinates $z=(z^1,z^2):U\to\mathbb{C}^2$ on M. Denoting by ∇ the Levi-Civita connection of g, on U the identity $\nabla g=0$ is equivalent to

$$\frac{\partial g_{k\bar{j}}}{\partial z^i}=g_{s\bar{j}}\Gamma^s_{ik}\qquad\text{and}\qquad\frac{\partial g_{k\bar{j}}}{\partial\overline{z^i}}=g_{k\bar{s}}\overline{\Gamma^s_{ij}},$$

where Γ^i_{jk} denote the complex Christoffel symbols of ∇. Abbreviate $G=\det g_{i\bar{j}}$, then we obtain

$$\frac{\partial G}{\partial z^i}=G\,\Gamma^s_{si}.$$

Hence, the partial derivative of $h_{k\bar{j}}=g_{k\bar{j}}\,G^{-2/3}$ with respect to z^i becomes

$$\begin{aligned}\frac{\partial h_{k\bar{j}}}{\partial z^i}&=g_{l\bar{j}}\,\Gamma^l_{ik}\,G^{-2/3}-\frac{2}{3}g_{k\bar{j}}\,\Gamma^s_{si}\,G^{-2/3}=h_{l\bar{j}}\left(\Gamma^l_{ik}-\frac{2}{3}\Gamma^s_{si}\delta^l_k\right)\\&=h_{l\bar{j}}\left(\Gamma^l_{ik}-\frac{1}{3}\Gamma^s_{si}\delta^l_k-\frac{1}{3}\Gamma^s_{sk}\delta^l_i\right)-\frac{1}{3}h_{l\bar{j}}\left(\Gamma^s_{si}\delta^l_k-\Gamma^s_{sk}\delta^l_i\right).\end{aligned}$$

Note that the last two summands in the last equation are antisymmetric in i,k, so that we may write

$$-\frac{1}{3}h_{l\bar{j}}\left(\Gamma^s_{si}\delta^l_k-\Gamma^s_{sk}\delta^l_i\right)=\overline{h_j}\varepsilon_{ik}$$

for unique complex-valued functions h_i on U. We thus get

$$\frac{\partial h_{k\bar{j}}}{\partial z^i}=h_{s\bar{j}}\Pi^s_{ik}+\overline{h_j}\varepsilon_{ik}. \tag{3.2}$$

In entirely analogous fashion we obtain

$$\frac{\partial h_{k\bar{j}}}{\partial\overline{z^i}}=h_{k\bar{s}}\overline{\Pi^s_{ij}}+h_k\overline{\varepsilon_{ij}}. \tag{3.3}$$

Recall from **Theorem 2.1** that the coordinate system $z:U\to\mathbb{C}^2$ induces a unique section $\sigma_z:U\to B$ of Cartan's bundle such that

$$(\sigma_z)^*\,\theta^0_0=0,\qquad(\sigma_z)^*\,\theta^i_0=\mathrm{d}z^i,\qquad(\sigma_z)^*\,\theta^i_j=\Pi^i_{jk}\mathrm{d}z^k. \tag{3.4}$$

Consequently, using (**Eq. (3.2)**, **Eq. (3.3)**, **Eq. (3.4)**) we see that **(3.1)** is necessary.

Conversely, suppose there exist complex-valued functions $h_{i\bar{j}}=\overline{h_{j\bar{i}}}$ and h_i on B solving **(3.1)** for which

$$\left(h_{1\bar{1}}h_{2\bar{2}}-|h_{1\bar{2}}|^2\right)\neq0.$$

Setting $g_{i\bar{j}}=h_{i\bar{j}}\left(h_{1\bar{1}}h_{2\bar{2}}-|h_{1\bar{2}}|^2\right)^{-2}$ we get

$$\mathrm{d}g_{i\bar{j}}=-g_{i\bar{j}}\left(\theta^0_0+\bar{\theta}^0_0\right)+g_{i\bar{s}}\overline{\theta^s_j}+g_{s\bar{j}}\theta^s_i+g_{i\bar{j}\bar{s}}\overline{\theta^s_0}+g_{i\bar{j}s}\theta^s_0 \tag{3.5}$$

with

$$g_{\bar{i}\bar{j}\bar{s}}=\frac{(h_{\bar{i}\bar{j}}h_{l\bar{s}}+h_{i\bar{s}}h_{l\bar{j}})\varepsilon^{lk}h_k}{(h_{1\bar{1}}h_{2\bar{2}}-|h_{1\bar{2}}|^2)^3}, \quad \text{and} \quad g_{\bar{i}\bar{j}k}=\frac{(h_{\bar{i}\bar{j}}h_{k\bar{s}}+h_{k\bar{j}}h_{i\bar{s}})\overline{\varepsilon^{su}h_u}}{(h_{1\bar{1}}h_{2\bar{2}}-|h_{1\bar{2}}|^2)^3}.$$

It follows with **(3.5)** that there exists a unique J-Hermitian metric g on M such that $\pi^*g=g_{i\bar{j}}\,\theta^i_0\circ\overline{\theta^j_0}$. Choose local holomorphic coordinates $z=(z^1,z^2):U\to\mathbb{C}^2$ on M. By abuse of notation we will write $g_{i\bar{j}},g_{i\bar{j}\bar{s}},g_{i\bar{j}k}$ for the pullback of the respective functions on B by the section $\sigma_z:U\to B$ associated to z. From **(3.5)** we obtain

$$\frac{\partial g_{i\bar{j}}}{\partial z^s}=g_{u\bar{j}}\Pi^u_{is}+g_{i\bar{j}s}=g_{u\bar{j}}\left(\Pi^u_{is}+g^{\bar{v}u}g_{i\bar{v}s}\right)=g_{u\bar{j}}\left(\Pi^u_{is}+\delta^u_i b_s+\delta^u_s b_i\right)=g_{u\bar{j}}\Gamma^u_{is}$$

where we write

$$b_i=\frac{h_{i\bar{s}}\overline{\varepsilon^{su}h_u}}{(h_{1\bar{1}}h_{2\bar{2}}-|h_{1\bar{2}}|^2)^{11/3}} \qquad \text{and} \qquad \Gamma^i_{jk}=\Pi^i_{jk}+\delta^i_j b_k+\delta^i_k b_j.$$

Likewise we obtain

$$\frac{\partial g_{i\bar{j}}}{\partial \overline{z^s}}=g_{i\bar{u}}\overline{\Gamma^u_{js}}.$$

It follows that there exists a complex-linear connection ∇ on U defining $[\nabla]$ and whose complex Christoffel symbols are given by Γ^i_{jk}. By construction, the connection ∇ preserves g and hence must be the Levi-Civita connection of g. Furthermore, ∇ being complex-linear implies that g is Kähler. This completes the proof. □

3.1.1. *First prolongation.* Differentiating **(3.1)** yields

$$0=\mathrm{d}^2h_{i\bar{j}}=\varepsilon_{si}\overline{\eta_j}\wedge\theta^s_0+\overline{\varepsilon_{sj}}\eta_i\wedge\overline{\theta^s_0}-(h_{s\bar{j}}W^s_{iv\bar{u}}+h_{i\bar{s}}\overline{W^s_{juv}})\overline{\theta^u_0}\wedge\theta^v_0 \tag{3.6}$$

with

$$\eta_k=\mathrm{d}h_k+h_k\left(\overline{\theta^0_0}-\theta^0_0\right)-h_j\theta^j_k+\overline{\varepsilon^{ij}}h_{k\bar{j}}\overline{\theta^0_i}.$$

This implies that we can write

$$\eta_i=a_{ij}\theta^j_0 \tag{3.7}$$

for unique complex-valued functions a_{ij} on B. Equations **(3.6)** and **(3.7)** imply

$$\varepsilon_{ki}\overline{a_{jl}}-\overline{\varepsilon_{lj}}a_{ik}=\overline{h_{j\bar{s}}}W^s_{ik\bar{l}}-h_{i\bar{s}}\overline{W^s_{jl\bar{k}}} \tag{3.8}$$

Contracting this last equation with $\overline{\varepsilon^{jl}}\varepsilon^{ik}$ implies that the function

$$h=-\frac{1}{2}\overline{\varepsilon^{ij}a_{ij}}$$

is real-valued. We get

$$a_{jl}=\varepsilon_{jl}h-\frac{1}{2}\overline{\varepsilon^{iu}}h_{s\bar{i}}W^s_{jl\bar{u}},$$

and thus

$$\mathrm{d}h_i=h_i\left(\theta^0_0-\overline{\theta^0_0}\right)+h_j\theta^j_i+h_{i\bar{s}}\overline{\varepsilon^{sl}\theta^0_l}+\left(\varepsilon_{ij}h-\frac{1}{2}\overline{\varepsilon^{uv}}h_{s\bar{u}}W^s_{ij\bar{v}}\right)\theta^j_0.$$

Plugging the formula for a_{ij} back into **(3.8)** yields the integrability conditions

$$h_{s\bar{j}}W^s_{ik\bar{l}}-h_{i\bar{s}}\overline{W^s_{jl\bar{k}}}=\frac{1}{2}\overline{\varepsilon_{lj}\varepsilon^{uv}}h_{s\bar{u}}W^s_{ik\bar{v}}-\frac{1}{2}\varepsilon_{ki}\varepsilon^{uv}h_{u\bar{s}}\overline{W^s_{jl\bar{v}}}.$$

This last equation can be simplified so that we obtain:

Proposition 3.2. *A necessary condition for a complex projective surface $(M, J, [\nabla])$ to be Kähler metrisable is that*

$$\overline{h_{j\bar{s}}}W^s_{ik\bar{l}} + \overline{h_{l\bar{s}}}W^s_{ik\bar{j}} = h_{k\bar{s}}\overline{W^s_{jl\bar{i}}} + h_{i\bar{s}}\overline{W^s_{jl\bar{k}}} \tag{3.9}$$

admits a nondegenerate solution $\overline{h_{i\bar{j}}} = h_{j\bar{i}}$.

Remark 3.1. Note that under suitable constant rank assumptions the system (3.9) defines a subbundle of the bundle over M whose sections are hermitian forms on (M, J). For a generic complex projective structure $[\nabla]$ this subbundle does have rank 0.

3.1.2. *Second prolongation.* We start by computing

$$0 = \mathrm{d}^2 h_i \wedge \theta^1_0 \wedge \theta^2_0 = -\left(h_{i\bar{j}}\overline{\varepsilon^{jk} L_k}\right)\theta^1_0 \wedge \overline{\theta^1_0} \wedge \theta^2_0 \wedge \overline{\theta^2_0}$$

which is equivalent to

$$\begin{pmatrix} h_{1\bar{1}} & h_{1\bar{2}} \\ h_{2\bar{1}} & h_{2\bar{2}} \end{pmatrix} \cdot \begin{pmatrix} \overline{L_2} \\ -\overline{L_1} \end{pmatrix} = 0$$

which cannot have any solution with $(h_{11}h_{22} - |h_{12}|^2) \neq 0$ unless $L_1 = L_2 = 0$. This shows:

Theorem 3.1. *A necessary condition for a complex projective surface to be Kähler metrisable is that it is Liouville-flat, i.e. its complex projective Liouville curvature vanishes.*

Remark 3.2. Note that the vanishing of the Liouville curvature is equivalent to requesting that the curvature of θ is of type (1,1) only, which agrees with general results in [9].

Assuming henceforth $L_1 = L_2 = 0$ we also get

$$0 = \mathrm{d}^2 h_i = (\varepsilon_{ij}\eta + \varphi_{ij}) \wedge \theta^j_0 \tag{3.10}$$

with

$$\eta = \mathrm{d}h + 2h\mathrm{Re}(\theta^0_0) + 2\varepsilon^{ij}\mathrm{Re}(h_i\theta^0_j) - \frac{1}{2}\varepsilon^{kl}h_{k\bar{l}}\overline{\varepsilon^{ij}K_{js\bar{i}}\theta^s_0}$$

and

$$\varphi_{ij} = \mathrm{d}r_{ij} + r_{ij}\overline{\theta^0_0} - r_{si}\theta^s_j - r_{sj}\theta^s_i - h_l W^l_{ij\bar{s}}\overline{\theta^s_0} + \frac{1}{2}\overline{\varepsilon^{uv}}\left(h_{i\bar{u}}\overline{K_{vs\bar{j}}} + h_{j\bar{u}}\overline{K_{vs\bar{i}}}\right)\overline{\theta^s_0}$$

where

$$r_{ij} = -\frac{1}{2}\overline{\varepsilon^{uv}}h_{s\bar{u}}W^s_{ij\bar{v}}.$$

It follows with Cartan's lemma that there are functions $a_{ijk} = a_{ikj}$ such that

$$\varepsilon_{ij}\eta + \varphi_{ij} = a_{ijk}\theta^k_0.$$

Since φ_{ij} is symmetric in i, j, this implies

$$\eta = \frac{1}{2}\varepsilon^{ji}a_{ijs}\theta^s_0.$$

Since h is real-valued, we must have

$$\varepsilon^{ji}a_{ijs} = \overline{\varepsilon^{uv}}\varepsilon^{kl}h_{k\bar{u}}K_{ls\bar{v}}.$$

Concluding, we get

$$\mathrm{d}h = -2h\mathrm{Re}(\theta^0_0) + 2\varepsilon^{kl}\mathrm{Re}(h_l\theta^0_k) + \frac{1}{2}\overline{\varepsilon^{ij}}\varepsilon^{kl}\mathrm{Re}(h_{k\bar{i}}K_{ls\bar{j}}\theta^s_0).$$

This completes the prolongation procedure.

Remark 3.3. Note that further integrability conditions can be derived from **(3.10)**, we won't write these out though.

Using **Proposition 3.1** we obtain:

Theorem 3.2. *Let $(M, J, [\nabla])$ be a complex projective surface with Cartan geometry $(\pi : B \to M, \theta)$. If $U \subset B$ is a connected open set on which there exist functions $h_{i\bar{j}} = \overline{h_{j\bar{i}}}$, h_i and h that satisfy the rank 9 linear system*

$$\begin{aligned}
\mathrm{d}h_{i\bar{j}} &= 2h_{i\bar{j}}\mathrm{Re}(\theta^0_0) + h_{i\bar{s}}\overline{\theta^s_j} + h_{s\bar{j}}\theta^s_i + h_i\overline{\varepsilon_{sj}\theta^s_0} + \overline{h_j}\varepsilon_{si}\theta^s_0, \\
\mathrm{d}h_k &= 2\mathrm{i}h_k\mathrm{Im}(\theta^0_0) + h_l\theta^l_k + h_{k\bar{i}}\overline{\varepsilon^{ij}\theta^0_j} + \left(\varepsilon_{kl}h - \frac{1}{2}\overline{\varepsilon^{ij}}h_{s\bar{i}}W^s_{kl\bar{j}}\right)\theta^l_0, \\
\mathrm{d}h &= -2h\mathrm{Re}(\theta^0_0) - 2\varepsilon^{lk}\mathrm{Re}(h_l\theta^0_k) + \frac{1}{2}\overline{\varepsilon^{ij}}\varepsilon^{kl}\mathrm{Re}(h_{k\bar{i}}K_{ls\bar{j}}\theta^s_0),
\end{aligned} \tag{3.11}$$

and $(h_{1\bar{1}}h_{2\bar{2}} - |h_{1\bar{2}}|^2) \neq 0$, then the quadratic form

$$g = \frac{h_{i\bar{j}}\theta^i_0 \circ \overline{\theta^j_0}}{(h_{1\bar{1}}h_{2\bar{2}} - |h_{1\bar{2}}|^2)^2}$$

is the π-pullback to U of a (pseudo-)Kähler metric on $\pi(U) \subset M$ that is compatible with $[\nabla]$.

From this we get:

Corollary 3.1. *The Kähler metrics defined on some domain $U \subset \mathbb{CP}^2$ which are compatible with the standard complex projective structure on $\mathbb{CP}^2$ are in one-to-one correspondence with the hermitian forms on $\mathbb{C}^3$ whose rank is at least two.*

Proof. Suppose the complex projective structure $[\nabla]$ has vanishing complex projective Weyl and Liouville curvature. Then the differential system **(3.11)** may be written as

$$\mathrm{d}H + \theta H + H\theta^* = 0 \tag{3.12}$$

with

$$H = H^* = \begin{pmatrix} h & -\overline{h_2} & \overline{h_1} \\ -h_2 & -h_{22} & h_{21} \\ h_1 & h_{12} & -h_{11} \end{pmatrix}$$

where * denotes the conjugate transpose matrix. Recall that in the flat case $\theta = g^{-1}\mathrm{d}g$ for some smooth $g : B \to \mathrm{PSL}(3, \mathbb{C})$, hence the solutions to **(3.12)** are

$$H = g^{-1}C\left(g^{-1}\right)^*$$

where $C = C^*$ is a constant hermitian matrix of rank at least two. The statement now follows immediately with **Theorem 3.2**. □

Remark 3.4. On can deduce from **Corollary 3.1** that a Kähler metric g giving rise to flat complex projective structures must have constant holomorphic sectional curvature. A result first proved in [37] (in all dimensions).

Remark 3.5. One can also ask for existence of complex projective structures $[\nabla]$ whose *degree of mobility* is greater than one, i.e. they admit several (non-proportional) compatible Kähler metrics. In [15] (see also [21]) it was shown that the only closed complex projective manifold with degree of mobility greater than two is $\mathbb{CP}^n$ with the projective structure arising via the Fubini-Study metric.

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