

# GL(2)-Structures in Dimension Four, H-Flatness and Integrability

WOJCIECH KRYŃSKI AND THOMAS METTLER

**ABSTRACT.** We show that torsion-free four-dimensional GL(2)-structures are flat up to a coframe transformation with a mapping taking values in a certain subgroup  $H \subset \mathrm{SL}(4, \mathbb{R})$ , which is isomorphic to a semidirect product of the three-dimensional continuous Heisenberg group  $H_3(\mathbb{R})$  and the Abelian group  $\mathbb{R}$ . In addition, we show that the relevant PDE system is integrable in the sense that it admits a dispersionless Lax-pair.

## 1. INTRODUCTION

A GL(2)-structure on a smooth 4-manifold  $M$  is given by a smoothly varying family of twisted cubic curves, one in each projectivised tangent space of  $M$ . Equivalently, a GL(2)-structure is the same as  $G$ -structure  $\pi: B \rightarrow M$  on  $M$ , where  $G$  is the image subgroup of the faithful irreducible 4-dimensional representation of  $\mathrm{GL}(2, \mathbb{R})$  on the space of homogeneous polynomials of degree three with real coefficients in two real variables. A GL(2)-structure is called *torsion-free* if its associated  $G$ -structure is torsion-free. Torsion-free GL(2)-structures are of particular interest, as they provide examples of torsion-free connections with exotic holonomy group  $\mathrm{GL}(2, \mathbb{R})$ . However, the local existence of torsion-free GL(2)-structures is highly non-trivial, even when applying the Cartan–Kähler machinery, which is particularly well-suited for the construction of torsion-free connections with special holonomy. Adapting methods of Hitchin [9], Bryant [2] gave an elegant twistorial construction of real-analytic torsion-free GL(2)-structures in dimension four, thus providing the first example of an irreducibly-acting holonomy group of a (non-metric) torsion-free connection missing from Berger’s list [1] of such connections.

A natural source for GL(2)-structures are differential operators. Recall that the principal symbol  $\sigma(D)$  of a  $k$ -th order linear differential operator  $D: C^\infty(M, \mathbb{R}^n) \rightarrow C^\infty(M, \mathbb{R}^m)$  assigns to each point  $p \in M$  a homogeneous polynomial of degree  $k$  on  $T_p^*M$ , with values in  $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ . Therefore, in each projectivised cotangent space  $\mathbb{P}(T_p^*M)$  of  $M$  we obtain the so-called *characteristic variety*  $\Xi_p$  of  $D$ , consisting of those  $[\xi] \in \mathbb{P}(T_p^*M)$ , for which the linear mapping  $\sigma_\xi(D): \mathbb{R}^n \rightarrow \mathbb{R}^m$  fails to be injective. Given a (possibly non-linear) differential operator  $D$  and a smooth  $\mathbb{R}^n$ -valued function  $u$  defined on some open subset  $U \subset M$  and which satisfies  $D(u) = 0$ , we may ask that the linearisation  $L_u(D)$  of  $D$  around  $u$  has



characteristic varieties all of which are the tangential variety of the twisted cubic curve. Consequently, one obtains a  $GL(2)$ -structure on the domain of definition of each solution  $u$  of the PDE  $D(u) = 0$  for an appropriate class of differential operators. Various examples of such operators have recently been given by Ferapontov–Kruglikov [6]. In particular, they show that locally all torsion-free  $GL(2)$ -structures arise in this fashion for some *second order* operator  $D$ , which furthermore has the property that the PDE  $D(u) = 0$  admits a dispersionless Lax representation. We also refer the reader to [7] for an application of similar ideas to the case of three-dimensional Einstein–Weyl structures.

Here we show that if a 4-manifold  $M$  carries a torsion-free  $GL(2)$ -structure  $\pi: B \rightarrow M$ , then for every point  $p \in M$  there exists a  $p$ -neighbourhood  $U_p$ , local coordinates  $x: U_p \rightarrow \mathbb{R}^4$  and a mapping  $h: U_p \rightarrow H$  into a certain 4-dimensional subgroup  $H \subset SL(4, \mathbb{R})$ , so that the coframing  $\eta = h dx$  is a local section of  $\pi: B \rightarrow M$ . The group  $H$  is isomorphic to a semidirect product of the three-dimensional continuous Heisenberg group  $H_3(\mathbb{R})$  and the Abelian group  $\mathbb{R}$ . Moreover, the mapping  $h$  satisfies a *first order* quasi-linear PDE system which admits a dispersionless Lax-pair. As in [6], linearising the PDE system around a solution  $h$  gives a linear first order differential operator whose characteristic variety is the tangential variety of the twisted cubic curve. Also, note that our result shows that 4-dimensional torsion-free  $GL(2)$ -structures are *H-flat*, that is, flat up to a coframe transformation with a mapping taking values in  $H$ .

Along the way (see **Theorem 2.4**), we derive a first order PDE describing general *H-flat* torsion-free  $G$ -structures which may be of independent interest.

**Acknowledgments.** The authors would like to thank Maciej Dunajski and Evgeny Ferapontov for helpful conversations and correspondence. The authors are also grateful to the anonymous referees for their careful reading and useful suggestions.

## 2. $G$ -STRUCTURES AND $H$ -FLATNESS

In this section we collect some elementary facts about  $G$ -structures, introduce the notion of *H-flatness* and derive the first order PDE system describing *H-flat* torsion-free  $G$ -structures. Throughout the article all manifolds and maps are assumed to be smooth, that is  $C^\infty$ .

**2.1. The coframe bundle and  $G$ -structures.** Let  $M$  be an  $n$ -manifold and  $V$  a real  $n$ -dimensional vector space. A  $V$ -valued coframe at  $p \in M$  is a linear isomorphism  $f: T_p M \rightarrow V$ . The set  $F_p M$  of  $V$ -valued coframes at  $p \in M$  is the fibre of the principal right  $GL(V)$  coframe bundle  $v: FM \rightarrow M$ , where the right action  $R_a: FM \rightarrow FM$  is defined by the rule  $R_a(f) = a^{-1} \circ f$  for all  $a \in GL(V)$  and  $f \in FM$ . Of course, we may identify  $V \simeq \mathbb{R}^n$ , but it is often advantageous to allow  $V$  to be an abstract vector space, in which case we say  $FM$  is *modelled* on  $V$ . The coframe bundle carries a tautological  $V$ -valued 1-form defined by  $\omega_f = f \circ v_*$  so that we have the equivariance property  $R_a^* \omega = a^{-1} \omega$ . A local  $v$ -section  $\eta: U \rightarrow FM$  is called a *coframing* on  $U \subset M$  and a choice of a basis of  $V$  identifies  $\eta$  with  $n$  linearly independent 1-forms on  $U$ .

Let  $G \subset GL(V)$  be a closed subgroup. A  $G$ -structure on  $M$  is a reduction  $\pi: B \rightarrow M$  of the coframe bundle with structure group  $G$ , equivalently, a smooth section of the fibre bundle  $FM/G \rightarrow M$ . For local considerations we may take  $M = V$ . Note that in this case  $M$  is equipped with a coframing  $\eta_0$  defined by the exterior derivative of the identity map  $\eta_0 = d\text{Id}_V$ . Consequently, the coframe bundle of  $V$  may naturally be identified with  $V \times GL(V)$  and hence the set of  $G$ -structures on  $V$  is in one-to-one correspondence with the space of smooth maps  $V \rightarrow GL(V)/G$ . In particular, a smooth map  $h: V \rightarrow GL(V)$  defines a  $G$ -structure on  $V$  by composing  $h$  with the quotient projection  $GL(V) \rightarrow GL(V)/G$ .

**2.2.  $H$ -flatness.** A  $G$ -structure  $\pi: B \rightarrow M$  is called *flat* if in a neighbourhood  $U_p$  of every point  $p \in M$  there exist local coordinates  $x: U_p \rightarrow V$ , so that  $dx: U_p \rightarrow FM$  takes values in  $B$ . We remark that flat  $G$ -structures also are often called *integrable*. Suppose  $H \subset GL(V)$  is a closed subgroup. We say a  $G$ -structure is  *$H$ -flat* if in a neighbourhood  $U_p$  of every point  $p \in M$  there exist local coordinates  $x: U_p \rightarrow V$  and a mapping  $h: U_p \rightarrow H$ , so that  $hdx: U_p \rightarrow FM$  takes values in  $B$ . Clearly, every  $G$ -structure is  $GL(V)$ -flat and a  $G$ -structure is flat in the usual sense if and only if it is  $\{e\}$ -flat, where  $\{e\}$  denotes the trivial subgroup of  $GL(V)$ .

**Example 2.1.** Every  $O(2)$ -structure is  $\mathbb{R}^+$ -flat, where  $\mathbb{R}^+$  denotes the group of uniform scaling transformations of  $\mathbb{R}^2$  with positive scale factor. This is the existence of local isothermal coordinates for Riemannian metrics in two-dimensions. Likewise, conformally flat Riemannian metrics in dimensions  $n > 2$  yield examples of  $O(n)$ -structures that are  $\mathbb{R}^+$ -flat.

**Remark 2.2.** Note that if a  $G$ -structure is  $H$ -flat for some Lie group  $H \subset G$ , then it is  $\{e\}$ -flat.

**2.3. A PDE for  $H$ -flat torsion-free  $G$ -structures.** A  $G$ -structure  $\pi: B \rightarrow M$  is called *torsion-free* if there exists a principal  $G$ -connection  $\theta$  on  $B$ , so that Cartan's first structure equation

$$(2.1) \quad d\omega = -\theta \wedge \omega$$

holds. Recall that a principal  $G$ -connection on  $B$  is a 1-form  $\theta$  on  $B$  with values in the Lie algebra  $\mathfrak{g}$  of  $G$  that pulls back to each  $\pi$ -fibre to be the canonical left invariant 1-form on  $G$  and that is equivariant with respect to the adjoint action of  $G$ , that is,  $\theta$  satisfies  $R_g^*\theta = \text{Ad}(g^{-1})\theta$  for all  $g \in G$ .

**Remark 2.3.** We remark that a weaker notion of torsion-freeness is also in use, see for instance [3, 10]. Namely, a  $G$ -structure  $\pi: B \rightarrow M$  is called torsion-free if there exists a  $\mathfrak{g}$ -valued 1-form  $\theta$  on  $B$  so that (2.1) holds.

We may ask when a  $G$ -structure on  $V$  induced by a mapping  $h: V \rightarrow H \subset GL(V)$  is torsion-free. To this end let  $A \subset V^* \otimes V$  be a linear subspace. Denote by

$$\delta: V^* \otimes V^* \otimes V \rightarrow \Lambda^2(V^*) \otimes V$$

the natural skew-symmetrisation map. Recall that the Spencer cohomology group  $H^{0,2}(A)$  of  $A$  is the quotient

$$H^{0,2}(A) = (\Lambda^2(V^*) \otimes V) / \delta(V^* \otimes A).$$

Let

$$\Pi_A: \Lambda^2(V^*) \otimes V \rightarrow H^{0,2}(A)$$

denote the quotient projection and let  $\mu_H$  denote the Maurer–Cartan form of  $H$ . Note that  $\psi_h = h^* \mu_H$  is a 1-form on  $V$  with values in the Lie algebra  $\mathfrak{h}$  of  $H$ , that is, a smooth map

$$\psi_h: V \rightarrow V^* \otimes \mathfrak{h} \subset V^* \otimes \mathfrak{gl}(V) \simeq V^* \otimes V^* \otimes V.$$

We define  $\tau_h = \delta \psi_h$ , so that  $\tau_h$  is a 2-form on  $V$  with values in  $V$ . We now have:

**Theorem 2.4.** *Let  $h: V \rightarrow H$  be a smooth map. Then the  $G$ -structure defined by  $h$  is torsion-free if and only if*

$$(2.2) \quad \Pi_{\text{Ad}(h^{-1})\mathfrak{g}} \tau_h = 0.$$

**Remark 2.5.** In the case where  $H = G$  the  $H$ -structure defined by  $h$  is the same as the torsion-free  $H$ -structure defined by the map  $h \equiv \text{Id}_V: V \rightarrow \text{GL}(V)$ , hence (2.2) must be trivially satisfied. This is indeed the case. Since the adjoint action of  $H$  preserves  $\mathfrak{h}$ , we obtain for any map  $h: V \rightarrow H$

$$\Pi_{\text{Ad}(h^{-1})\mathfrak{h}} \tau_h = \Pi_{\mathfrak{h}} \tau_h = \Pi_{\mathfrak{h}} \delta \psi_h = 0.$$

*Proof of Theorem 2.4.* For the proof we fix an identification  $V \simeq \mathbb{R}^n$ . Let  $x = (x^i)$  denote the standard linear coordinates on  $\mathbb{R}^n$ . Furthermore let  $h: \mathbb{R}^n \rightarrow H \subset \text{GL}(n, \mathbb{R})$  be given and let  $\pi: B_h \rightarrow \mathbb{R}^n$  denote the  $G$ -structure defined by  $h$ , that is,

$$B_h = \{ (x, a) \in \mathbb{R}^n \times \text{GL}(n, \mathbb{R}) : a = h^{-1}(x)g, g \in G \}.$$

We have a  $G$ -bundle isomorphism

$$\psi: \mathbb{R}^n \times G \rightarrow B_h, \quad (x, g) \mapsto (x, h^{-1}(x)g).$$

The tautological 1-form  $\omega_0$  on  $F\mathbb{R}^n \simeq \mathbb{R}^n \times \text{GL}(n, \mathbb{R})$  satisfies  $(\omega_0)_{(x,a)} = a^{-1}dx$  for all  $(x, a) \in \mathbb{R}^n \times \text{GL}(n, \mathbb{R})$ . Continuing to write  $\omega_0$  for the pullback to  $B_h$  of  $\omega_0$ , we obtain

$$\omega_{(x,g)} := (\psi^* \omega_0)_{(x,g)} = g^{-1}h(x)dx.$$

Let  $\alpha$  be any 1-form on  $\mathbb{R}^n$  with values in  $\mathfrak{g}$ , the Lie-algebra of  $G$ . We obtain a principal  $G$ -connection  $\theta = (\theta^i_j)$  on  $\mathbb{R}^n \times G$  by defining

$$\theta = g^{-1}\alpha g + g^{-1}dg,$$

where  $g: \mathbb{R}^n \times G \rightarrow G \subset \text{GL}(n, \mathbb{R})$  denotes the projection onto the latter factor. Conversely, every principal  $G$ -connection on the trivial  $G$ -bundle  $\mathbb{R}^n \times G$  arises

in this fashion. The  $G$ -structure  $B_h$  is torsion-free if and only if there exists a principal  $G$ -connection  $\theta$  such that

$$d\omega + \theta \wedge \omega = 0,$$

which is equivalent to

$$0 = d(g^{-1}hdx) + (g^{-1}\alpha g + g^{-1}dg) \wedge g^{-1}hdx$$

or

$$0 = (dg^{-1} + g^{-1}dgg^{-1}) \wedge hdx + g^{-1}(dh \wedge dx + \alpha \wedge hdx).$$

Using  $0 = d(g^{-1}g)$ , we see that the  $G$ -structure defined by  $h$  is torsion-free if and only if there exists a 1-form  $\alpha$  on  $V$  with values in  $\mathfrak{g}$  such that

$$0 = dh \wedge dx + \alpha \wedge hdx.$$

This is equivalent to

$$(h^{-1}dh + h^{-1}\alpha h) \wedge dx = 0$$

or

$$(2.3) \quad (\psi_h + \text{Ad}(h^{-1})\alpha) \wedge dx = 0,$$

where  $\psi_h = h^{-1}dh$  denotes the  $h$ -pullback of the Maurer–Cartan form of  $H$  and  $\text{Ad}(h)v = hvh^{-1}$  the adjoint action of  $h \in H$  on  $v \in \mathfrak{gl}(n, \mathbb{R})$ . Now (2.3) is equivalent to

$$\delta \psi_h + \delta \text{Ad}(h^{-1})\alpha = 0.$$

Since  $\alpha$  takes values in  $\mathfrak{g}$ , this implies that  $\tau_h = \delta \psi_h$  lies in the  $\delta$ -image of  $V^* \otimes \text{Ad}(h^{-1})\mathfrak{g}$ . Therefore, we obtain

$$\Pi_{\text{Ad}(h^{-1})\mathfrak{g}} \tau_h = 0.$$

Conversely, suppose  $\tau_h$  lies in the  $\delta$ -image of  $V^* \otimes \text{Ad}(h^{-1})\mathfrak{g}$ . Then there exists a 1-form  $\beta$  on  $V$  with values in  $h^{-1}\mathfrak{g}h$  so that

$$\tau_h = \delta \psi_h = \delta \beta.$$

Hence, the  $\mathfrak{g}$ -valued 1-form  $\alpha$  on  $V$  defined by  $\alpha = -h\beta h^{-1}$  satisfies

$$\tau_h + \delta h^{-1}\alpha h = \delta \psi_h + \delta \text{Ad}(h^{-1})\alpha = 0,$$

thus proving the claim.  $\square$

### 3. GL(2)-STRUCTURES

Let  $x, y$  denote the standard linear coordinates on  $\mathbb{R}^2$  and let  $\mathbb{R}[x, y]$  denote the polynomial ring with real coefficients generated by  $x$  and  $y$ . We let  $\text{GL}(2, \mathbb{R})$  act from the left on  $\mathbb{R}[x, y]$  via the usual linear action on  $x, y$ . We denote by  $\mathcal{V}_d$  the subspace consisting of homogeneous polynomials in degree  $d \geq 0$  and by  $G_d \subset \text{GL}(\mathcal{V}_d)$  the image subgroup of the  $\text{GL}(2, \mathbb{R})$  action on  $\mathcal{V}_3$ . The vector space  $\mathcal{V}_3$  carries a two-dimensional cone  $\tilde{\mathcal{C}}$  of distinguished polynomials, consisting of the perfect cubes, i.e., those that are of the form  $(ax + by)^3$  for  $ax + by \in \mathcal{V}_1$ . The reader may easily check that  $G_3$  is characterised as the subgroup of  $\text{GL}(\mathcal{V}_3)$  that preserves  $\tilde{\mathcal{C}}$ . The projectivisation of  $\tilde{\mathcal{C}}$  gives an algebraic curve  $\mathcal{C}$  of degree 3

in  $\mathbb{P}(\mathcal{V}_3)$ , which is linearly equivalent to the *twisted cubic curve*, i.e., the curve in  $\mathbb{RP}^3$  defined by the zero locus of the three homogeneous polynomials

$$P_0 = XZ - Y^2, \quad P_1 = YW - Z^2, \quad P_2 = XW - YZ,$$

where  $[X : Y : Z : W]$  are the standard homogeneous coordinates on  $\mathbb{RP}^3$ . The vector space  $\mathcal{V}_3$  carries another algebraic variety in its projectivisation besides the twisted cubic curve. Indeed, the polynomials having vanishing discriminant define a  $G_3$ -invariant quartic cone  $\tilde{Q}$  whose projectivisation  $Q$  defines a quartic hypersurface in  $\mathbb{P}(\mathcal{V}_3)$ . Furthermore, the singular locus of  $Q$  is the twisted cubic curve  $\mathcal{C}$  and the tangential variety of  $\mathcal{C}$  is  $Q$ .

Let  $M$  be a 4-manifold and let  $\nu: FM \rightarrow M$  denote its coframe bundle modelled on  $\mathcal{V}_3$ . A  $GL(2)$ -structure on  $M$  is a reduction  $\pi: B \rightarrow M$  of  $FM$  with structure group  $G_3 \simeq GL(2, \mathbb{R})$ . By definition, a  $GL(2)$ -structure identifies each tangent space of  $M$  with  $\mathcal{V}_3$  up to the action by  $GL(2, \mathbb{R})$ . Consequently, each projectivised tangent space  $\mathbb{P}(T_p M)$  of  $M$  carries an algebraic curve  $\mathcal{C}_p$ , which is linearly equivalent to the twisted cubic curve. Conversely, if  $\mathcal{C} \subset \mathbb{P}(TM)$  is a smooth subbundle having the property that each fibre  $\mathcal{C}_p$  is linearly equivalent to the twisted cubic curve, then one obtains a unique reduction of the coframe bundle of  $M$  whose structure group is  $G_3$ .

For what follows it will be convenient to identify  $\mathcal{V}_3 \simeq \mathbb{R}^4$  by the isomorphism  $\mathcal{V}_3 \rightarrow \mathbb{R}^4$  defined on the basis of monomials as

$$x^{(3-i)}y^i \mapsto e_{i+1},$$

where  $i = 0, 1, 2, 3$  and  $e_i$  denotes the standard basis of  $\mathbb{R}^4$ . Note that, under the identification  $T_p M = \mathcal{V}_3$ , the cone  $\tilde{C}$  of a  $GL(2)$ -structure at  $p$  can be written as

$$\tilde{C}_p = \{s^3 e_1 + 3s^2 t e_2 + 3s t^2 e_3 + t^3 e_4 \mid s, t \in \mathbb{R}\}.$$

We now have:

**Theorem 3.1.** *All torsion-free  $GL(2)$ -structures in dimension four are  $H$ -flat, where  $H \subset SL(4, \mathbb{R})$  is the subgroup consisting of matrices of the form*

$$(3.1) \quad \begin{pmatrix} 1 & A & B & D \\ 0 & 1 & A & C \\ 0 & 0 & 1 & A \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and where  $A, B, C, D$  are arbitrary real numbers.

**Remark 3.2.** We note that the group  $H$  is isomorphic to a semidirect product of the continuous three-dimensional Heisenberg group  $H_3(\mathbb{R})$  and the Abelian group  $\mathbb{R}$ , that is,  $H \simeq H_3(\mathbb{R}) \rtimes \mathbb{R}$ . Indeed,  $H_3(\mathbb{R})$  has a faithful (necessarily reducible) four-dimensional representation defined by the Lie group homomorphism  $\varphi: H_3(\mathbb{R}) \rightarrow SL(4, \mathbb{R})$

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a & \frac{1}{2}a^2 + b & \frac{1}{6}a^3 + ab - c \\ 0 & 1 & a & \frac{1}{2}a^2 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The homomorphism  $\varphi$  embeds  $H_3(\mathbb{R})$  as a normal subgroup of the group  $H$  and we think of  $\mathbb{R}$  as the Abelian subgroup of  $H$  defined by setting  $A = B = D = 0$  in (3.1).

**Remark 3.3.** In fact, the notion of a GL(2)-structure makes sense in all dimensions  $d \geq 3$ . However, torsion-free GL(2)-structures in dimensions exceeding four are  $\{e\}$ -flat [2], that is, flat in the usual sense. We refer the reader to [8, 16] for a comprehensive study of five-dimensional GL(2)-structures (with torsion).

**Remark 3.4.** Phrased differently, **Theorem 3.1** states that locally every torsion-free GL(2)-structure in dimension four is obtained from a solution to the first order PDE system (2.2), where  $h$  takes values in the aforementioned group  $H$ .

*Proof of Theorem 3.1.* We shall prove that for a given torsion-free GL(2)-structure one can always choose local coordinates such that the cone  $\tilde{C}$  has the following form

$$\tilde{C} = \{s^3 V_0 + 3s^2 t V_1 + 3st^2 V_2 + t^3 V_3 \mid s, t \in \mathbb{R}\},$$

where the framing  $(V_0, V_1, V_2, V_3)$  is

$$(3.2) \quad \begin{aligned} V_0 &= \partial_0, & V_1 &= \partial_1 + \alpha \partial_0, & V_2 &= \partial_2 + \alpha \partial_1 + \beta \partial_0, \\ V_3 &= \partial_3 + \alpha \partial_2 + \gamma \partial_1 + \delta \partial_0, \end{aligned}$$

for some functions  $\alpha, \beta, \gamma$  and  $\delta$ . Then, the dual coframing is of the form  $h dx$ , where  $h$  takes values in  $H$  with

$$A = -\alpha, \quad B = -\beta + \alpha^2, \quad C = -\gamma + \alpha^2, \quad D = -\delta + \alpha(\gamma + \beta) - \alpha^3.$$

In order to derive the desired form of  $\tilde{C}$  we explore a correspondence between the torsion-free GL(2)-structures and classes of contact equivalent fourth order ODEs (compare the proof of [4, Theorem 1] and a similar correspondence in dimension 3). Indeed, it is proved in [2] that any torsion-free GL(2)-structure is defined by a fourth order ODE of the form

$$(3.3) \quad x^{(4)} = F(y, x, x', x'', x'''),$$

where the function  $F = F(y, x_0, x_1, x_2, x_3)$  satisfies a system of non-linear equations that we will refer to as the Bryant–Wünschmann condition. (Similar conditions in higher dimensions are known as the generalized Wünschmann conditions, because they generalize the classical 3-dimensional case, c.f. [5, 15].)

Above,  $(y, x_0, x_1, x_2, x_3)$  denote the standard coordinates on the space  $J^3(\mathbb{R}, \mathbb{R})$  of 3-jets of functions  $\mathbb{R} \rightarrow \mathbb{R}$  and the Bryant–Wünschmann condition is invariant with respect to the group of contact transformations of the coordinates. The GL(2)-structure corresponding to equation (3.3) is defined on the solution space of (3.3), i.e., on the quotient space  $J^3(\mathbb{R}, \mathbb{R})/X_F$ , where  $X_F = \partial_y + x_1 \partial_0 + x_2 \partial_1 + x_3 \partial_2 + F \partial_3$  is the total derivative. In order to define the structure, we first consider the following field of cones on  $J^3(\mathbb{R}, \mathbb{R})$  as in [11]

$$\hat{C} = \{s^3 \hat{V}_0 + 3s^2 t \hat{V}_1 + 3st^2 \hat{V}_2 + t^3 \hat{V}_3 \mid s, t \in \mathbb{R}\} \quad \text{mod } X_F$$

where

$$\begin{aligned}\hat{V}_0 &= \frac{3}{4}\partial_3, \\ \hat{V}_1 &= \frac{1}{2}\partial_2 + \frac{3}{8}\partial_3 F\partial_3, \\ \hat{V}_2 &= \frac{1}{2}\partial_1 + \frac{1}{4}\partial_3 F\partial_2 + \left(\frac{7}{20}\partial_2 F - \frac{3}{20}X_F(\partial_3 F) + \frac{9}{40}(\partial_3 F)^2\right)\partial_3, \\ \hat{V}_3 &= \partial_0 + \frac{1}{4}\partial_3 F\partial_1 + \left(\partial_2 F - \frac{5}{4}X_F(\partial_3 F) + \frac{7}{16}(\partial_3 F)^2 + \frac{7}{10}K\right)\partial_2 \\ &\quad + \left(\partial_1 F - \frac{3}{10}X_F(K) - X_F(\partial_2 F) + \frac{21}{40}K\partial_3 F\right. \\ &\quad \left.- \frac{27}{16}X_F(\partial_3 F)\partial_3 F - \frac{3}{4}\partial_2 F\partial_3 F + \frac{3}{4}X_F^2(\partial_3 F) + \frac{27}{64}(\partial_3 F)^3\right)\partial_3,\end{aligned}$$

with  $K = -\partial_2 F + \frac{3}{2}X(\partial_3 F) - \frac{3}{8}(\partial_3 F)^2$ . To define the cone one looks for  $(f, g)$  such that

$$(3.4) \quad \text{ad}_{fX_F}^4(g\partial_3) = 0 \pmod{X_F, \partial_3, \partial_2},$$

where  $\text{ad}_{X_F}^i$  stands for the iterated Lie bracket with the vector field  $X_F$ . Then  $\hat{C}_\rho$  is defined as the set of all  $(\text{ad}_{fX_F}^3(g\partial_3))(p)$ , where  $(f, g)$  solve (3.4). The explicit formula for  $\hat{C}$  can be found using [11, Proposition 4.1] and [11, Corollary 5.3]. The cone  $\hat{C}$  is invariant with respect to the flow of  $X_F$  if and only if (3.3) satisfies the Bryant–Wünschmann condition. In this case (3.4) takes the form  $\text{ad}_{fX_F}^4(g\partial_3) = 0 \pmod{X_F}$  (c.f. [12]). Then  $\hat{C}$  can be projected to the quotient space  $J^3(\mathbb{R}, \mathbb{R})/X_F$  and defines a  $\text{GL}(2)$ -structure there via the field of cones  $\tilde{C} = q_*\hat{C}$ , where  $q: J^3(\mathbb{R}, \mathbb{R}) \rightarrow J^3(\mathbb{R}, \mathbb{R})/X_F$  is the quotient map. Note that  $J^3(\mathbb{R}, \mathbb{R})/X_F$  can be identified with the hypersurface  $\{y = 0\} \subset J^3(\mathbb{R}, \mathbb{R})$ . Denoting

$$\begin{aligned}\alpha &= \partial_3 F|_{y=0}, \\ \beta &= \left(\frac{7}{20}\partial_2 F - \frac{3}{20}X(\partial_3 F) + \frac{9}{40}(\partial_3 F)^2\right)\Big|_{y=0}, \\ \gamma &= \left(\partial_2 F - \frac{5}{4}X_F(\partial_3 F) + \frac{7}{16}(\partial_3 F)^2 + \frac{7}{10}K\right)\Big|_{y=0}, \\ \delta &= \left(\partial_1 F - \frac{3}{10}X(K) - X(\partial_2 F) + \frac{21}{40}K\partial_3 F - \frac{27}{16}X(\partial_3 F)\partial_3 F\right. \\ &\quad \left.- \frac{3}{4}\partial_2 F\partial_3 F + \frac{3}{4}X^2(\partial_3 F) + \frac{27}{64}(\partial_3 F)^3\right)\Big|_{y=0}\end{aligned}$$

we get that

$$\tilde{C} = \{s^3 V_0 + 3s^2 t V_1 + 3st^2 V_2 + t^3 V_3 \mid s, t \in \mathbb{R}\}$$



where

$$\begin{aligned} V_0 &= \frac{3}{4}\partial_3, & V_1 &= \frac{1}{2}\partial_2 + \frac{3}{8}\alpha\partial_3, & V_2 &= \frac{1}{2}\partial_1 + \frac{1}{4}\alpha\partial_2 + \beta\partial_3, \\ V_3 &= \partial_0 + \frac{1}{4}\alpha\partial_1 + \gamma\partial_2 + \delta\partial_3. \end{aligned}$$

The following linear change of coordinates

$$(x_0, x_1, x_2, x_3) \mapsto \left( x_3, 2x_2, 2x_1, \frac{4}{3}x_0 \right)$$

transforms  $(V_0, V_1, V_2, V_3)$  to

$$\begin{aligned} V_0 &= \partial_0, & V_1 &= \partial_1 + \frac{1}{2}\alpha\partial_0, & V_2 &= \partial_2 + \frac{1}{2}\alpha\partial_1 + \frac{4}{3}\beta\partial_0, \\ V_3 &= \partial_3 + \frac{1}{2}\alpha\partial_2 + 2\gamma\partial_1 + \frac{4}{3}\delta\partial_0, \end{aligned}$$

which is equivalent to **(3.2)** up to constants.  $\square$

**Remark 3.5.** **Theorem 3.1** should be compared with **[6, Proposition 1]**, which can be rephrased that locally any torsion-free GL(2)-structure admits a coframing of the form  $h dx$  with

$$h = \begin{pmatrix} a_1 a_2 a_3 & a_0 a_2 a_3 & a_0 a_1 a_3 & a_0 a_1 a_2 \\ \frac{1}{3}(a_1 a_2 b_3 + a_1 b_2 a_3 + b_1 a_2 a_3) & \frac{1}{3}(a_0 a_2 b_3 + a_0 b_2 a_3 + b_0 a_2 a_3) & \frac{1}{3}(a_0 a_1 b_3 + a_0 b_1 a_2 + b_0 a_1 a_3) & \frac{1}{3}(a_0 a_1 b_2 + a_0 b_1 a_3 + b_0 a_1 a_2) \\ \frac{1}{3}(a_1 b_2 b_3 + b_1 a_2 b_3 + b_1 b_2 a_3) & \frac{1}{3}(a_0 b_2 b_3 + b_0 a_2 b_3 + b_0 b_2 a_3) & \frac{1}{3}(a_0 b_1 b_3 + b_0 a_1 b_3 + b_0 b_1 a_3) & \frac{1}{3}(a_0 b_1 b_2 + b_0 a_1 b_2 + b_0 a_1 b_2) \\ b_1 b_2 b_3 & b_0 b_2 b_3 & b_0 b_1 b_3 & b_0 b_1 b_2 \end{pmatrix},$$

where  $a_i = \left(\frac{\partial u}{\partial x_i}\right)^{-1}$  and  $b_i = \left(\frac{\partial v}{\partial x_i}\right)^{-1}$  for some real-valued functions  $u$  and  $v$  on  $\mathcal{V}_3 \simeq \mathbb{R}^4$ . One checks that  $h$  is not contained in any proper subgroup of  $GL(4, \mathbb{R})$ . It is an interesting problem to find the smallest possible dimension of the group  $H$ , such that all torsion-free GL(2)-structures are  $H$ -flat (we believe that dimension 4 from **Theorem 3.1** is optimal).

#### 4. INTEGRABILITY

In this section we derive the system **(2.2)** explicitly in terms of the functions  $A, B, C$  and  $D$  of **Theorem 3.1**. Moreover, we prove that it possesses a dispersionless Lax pair understood as a pair of commuting vector fields depending on a spectral parameter. Systems of this type, e.g., the dispersionless Kadomtsev-Petviashvili equation, often appear as dispersionless limits of integrable PDEs. Other examples include the Plebański heavenly equation or the Manakov-Santini system describing 3-dimensional Einstein-Weyl geometry. We refer to **[13, 14]** for general methods of integration of such systems. Let  $H \subset SL(4, \mathbb{R})$  be the subgroup of matrices **(3.1)**. Furthermore, let  $A_i, B_i, C_i$  and  $D_i$  denote  $\partial_i A, \partial_i B, \partial_i C$  and  $\partial_i D$ , respectively,

**Theorem 4.1.** *An  $H$ -flat  $GL(2)$ -structure defined by a coframing  $h dx$ , where  $h$  takes values in  $H$ , is torsion-free if and only if*

$$\begin{aligned}
 & V_2(D) - V_3(B) - AV_2(B) - CV_2(A) + AV_3(A) + A^2V_2(A) = 0 \\
 & 2V_1(D) - V_2(C) - 2AV_1(B) - V_3(A) + \\
 & \quad + AV_2(A) + 2A^2V_1(A) - 2CV_1(A) = 0 \\
 (4.1) \quad & V_0(D) - 2V_1(C) + 3V_1(B) - AV_0(B) - 2V_2(A) \\
 & \quad - AV_1(A) - CV_0(A) + A^2V_0(A) = 0 \\
 & V_0(C) - 2V_0(B) + V_1(A) + AV_0(A) = 0,
 \end{aligned}$$

and where the framing  $(V_0, V_1, V_2, V_3)$  dual to  $h dx$  is explicitly given by

$$\begin{aligned}
 V_0 &= \partial_0, & V_1 &= \partial_1 - A\partial_0, & V_2 &= \partial_2 - A\partial_1 - (B - A^2)\partial_0, \\
 V_3 &= \partial_3 - A\partial_2 - (C - A^2)\partial_1 - (D - (C + B)A + A^3)\partial_0.
 \end{aligned}$$

The system (4.1) can be put in the Lax form  $[L_0, L_1] = 0$  with

$$\begin{aligned}
 L_0 &= \partial_3 + (-C + 2A\lambda - 3\lambda^2)\partial_1 \\
 & \quad + (-D + AC - 2A^2\lambda + 4A\lambda^2 - 2\lambda^3)\partial_0 + \nu(\lambda)\partial_\lambda, \\
 L_1 &= \partial_2 + (-A + 2\lambda)\partial_1 + (-B + A^2 - 2A\lambda + \lambda^2)\partial_0 + \mu(\lambda)\partial_\lambda
 \end{aligned}$$

and

$$\begin{aligned}
 \nu(\lambda) &= \left( \frac{1}{2}A^2A_1 - ABA_0 + AA_2 - AB_1 - \frac{1}{2}DA_0 - \frac{1}{2}C_2 \right. \\
 & \quad \left. + \frac{1}{2}AC_1 + \frac{1}{2}BC_0 - \frac{1}{2}CA_1 + \frac{1}{2}ACA_0 + \frac{1}{2}A_3 \right) \\
 & \quad + (3B_1 - C_1 - AA_1 - AC_0 + 2BA_0 - 2A_2)\lambda \\
 & \quad + (C_0 - A_1)\lambda^2 \\
 \mu(\lambda) &= \left( \frac{1}{2}AA_1 + \frac{1}{2}AC_0 - BA_0 + A_2 - B_1 \right) \\
 & \quad + \left( \frac{1}{2}A_1 - \frac{1}{2}C_0 \right) \lambda,
 \end{aligned}$$

for some auxiliary spectral coordinate  $\lambda$ .

**Remark 4.2.** The spectral parameter  $\lambda$  can be treated as an affine parameter on the fibres of  $\mathcal{C}$ . The theorem states that  $\mathcal{D} = \text{span}\{L_0, L_1\}$  is an integrable rank-2 distribution on  $\mathcal{C}$ . There is a 3-parameter family of integral manifolds of  $\mathcal{D}$ . Projections of these submanifolds to  $M$  give a 3-parameter family of 2-dimensional submanifolds of  $M$  tangent to the field of cones  $\tilde{\mathcal{C}}$ .

**Remark 4.3.** The space of integral manifolds of the aforementioned distribution  $\mathcal{D} = \text{span}\{L_0, L_1\}$  is the twistor space  $T$  of a torsion-free  $GL(2)$ -structure. In this context  $\mathcal{C}$  is the correspondence space and we have a double fibration picture  $M \longleftarrow \mathcal{C} \longrightarrow T$ , where the fibres of the second projection are tangent to  $\mathcal{D}$ . If the coefficients  $\mu$  and  $\nu$  in the Lax pair  $(L_0, L_1)$  vanish, then there is additional

natural projection, defined by the parameter  $\lambda$ , from  $T$  to one-dimensional projective space. In other words, for any fixed  $\lambda$ , the integral leaves of  $\mathcal{D}_\lambda = \text{span}\{L_0(\lambda), L_1(\lambda), L_2(\lambda)\}$  define a 2-dimensional foliation of  $M$ . Among these structures there is a subclass for which the distribution  $\text{span}\{L_0(\lambda), L_1(\lambda), \frac{d}{d\lambda}L_1(\lambda)\}$  is integrable and thus defines a 3-dimensional foliation. Such foliations are known as Veronese webs, c.f. [12]. From this point of view, the Veronese webs can be thought of as higher-dimensional counterparts of 3-dimensional hyper-CR Einstein-Weyl structures [?].

Veronese webs are described by a hierarchy of integrable systems introduced in [?], which generalize the dispersionless Hirota equation. It is worth seeing how the system (4.1) looks like in this case. For this we note that the  $H$ -flat form of 4-dimensional Veronese webs has been given in [?, Section 6] and in this case we get (after permutation of indices) the following coefficients

$$A = \frac{\partial_1 f}{\partial_0 f}, \quad B = C = \frac{\partial_2 f}{\partial_0 f}, \quad D = \frac{\partial_3 f}{\partial_0 f},$$

where  $f = f(x_0, x_1, x_2, x_3)$  is a function. Then, in terms of  $f$ , the system (4.1) takes the following simple form

$$\begin{aligned} f_2 f_{00} - f_0 f_{02} - f_1 f_{01} + f_0 f_{11} &= 0, \\ f_3 f_{00} - f_0 f_{03} - f_1 f_{02} + f_0 f_{12} &= 0, \\ f_3 f_{01} - f_0 f_{13} - f_2 f_{02} + f_0 f_{22} &= 0, \end{aligned}$$

which coincides with the system derived in [?, Theorem 6.1]. One can also set  $H_i = -\frac{f_{i+1}}{f_0}$  and pass to a system derived in [?, Theorem 6.2]. An example of such a structure is given by the equation  $x^{(4)} = (x^{(3)})^{4/3}$  from [?]. In this case, using the formulae given in the proof of Theorem 3.1, one finds  $\alpha = x_0^{1/3}$ ,  $\beta = \gamma = x_0^{2/3}$  and  $\delta = x_0$ . Thus  $A = -x_0^{1/3}$ ,  $B = C = D = 0$  and  $f(x_0, x_1, x_2, x_3) = x_1 - \frac{3}{2}x_0^{2/3}$ .

**Remark 4.4.** A Cartan–Kähler analysis reveals that the first order system (4.1) – or equivalently (2.2) – is involutive and has solutions depending on four functions of three variables, confirming the count of Bryant [2]. Moreover, straightforward computations show that the characteristic variety of the system (4.1) linearised along any solution  $(A, B, C, D)$  is the discriminant locus  $\mathcal{Q}$ , i.e., the tangential variety of  $\mathcal{C}$ .

*Proof of Theorem 4.1.* The system (4.1) can be directly obtained by expanding (2.2) explicitly in terms of the functions  $A, B, C, D$ . Here we use a different method and apply [11, Corollary 7.4] to the framing  $(V_0, 3V_1, 3V_2, V_3)$ . Namely, denoting  $\lambda = \frac{s}{t}$ , we get that the curve  $\mathcal{C}$  in  $\mathbb{P}(TM)$  is the image of  $\lambda \mapsto \mathbb{R}V(\lambda) \in \mathbb{P}(TM)$ , where  $V(\lambda) = \lambda^3 V_0 + 3\lambda^2 V_1 + 3\lambda V_2 + V_3$  and the vector fields  $V_0, V_1, V_2$  and  $V_3$  are given by (3.2) with

$$\alpha = -A, \quad \beta = -B + A^2, \quad \gamma = -C + A^2, \quad \delta = -D + (C + B)A - A^3.$$

According to [11, Corollary 7.2], a GL(2)-structure is torsion-free if and only if

$$(4.2) \quad \left[ V(\lambda), \frac{d}{d\lambda} V(\lambda) \right] \in \text{span} \left\{ V(\lambda), \frac{d}{d\lambda} V(\lambda), \frac{d^2}{d\lambda^2} V(\lambda) \right\},$$

for any  $\lambda \in \mathbb{R}$ . This, due to [11, Corollary 7.4] applied to the framing

$$(V_0, 3V_1, 3V_2, V_3),$$

is expressed as eight linear equations for structural functions  $c_{ij}^k$  defined by  $[V_i, V_j] = \sum_k c_{ij}^k V_k$ . However, in the present case, the vector fields  $V_i$  are special and four equations are void. Indeed, the nontrivial equations are as follows:

$$\begin{aligned} c_{23}^0 &= 0, & c_{23}^1 - 2c_{13}^0 &= 0, \\ c_{23}^2 - 2c_{13}^1 + c_{03}^0 + 3c_{12}^0 &= 0, & c_{23}^3 - 2c_{13}^2 + c_{03}^1 + 3c_{12}^1 - 2c_{02}^0 &= 0 \end{aligned}$$

(the equations differ from equations in [11] because of the factor 3 next to  $V_1$  and  $V_2$  in the present paper). Substituting the structural functions, which can be easily computed, we get the system (4.1).

Now, we consider

$$L_0 = V(\lambda) - \left( \lambda - \frac{1}{3}A \right) \frac{d}{d\lambda} V(\lambda) \quad \text{mod } \partial_\lambda$$

and

$$L_1 = \frac{1}{3} \frac{d}{d\lambda} V(\lambda) \quad \text{mod } \partial_\lambda.$$

Due to (4.2), the commutator  $[L_0, L_1]$  lies in the span of  $\{L_0, L_1, \frac{d^2}{d\lambda^2} V(\lambda)\} \quad \text{mod } \partial_\lambda$ . Moreover, since

$$L_0 = \partial_3 \quad \text{mod } \partial_1, \partial_0, \partial_\lambda \quad \text{and} \quad L_1 = \partial_2 \quad \text{mod } \partial_1, \partial_0, \partial_\lambda,$$

we get  $[L_0, L_1] = \varphi \frac{d^2}{d\lambda^2} V(\lambda) \quad \text{mod } \partial_\lambda$  for some  $\varphi$ . One checks by direct computations that  $\mu(\lambda)$  and  $\nu(\lambda)$  are chosen such that  $\varphi = 0$  and the coefficient of  $[L_0, L_1]$  next to  $\partial_\lambda$  vanishes as well.  $\square$

## REFERENCES

- [1] M. BERGER, Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes, *Bull. Soc. Math. France* **83** (1955), 279–330. **MR 1**
- [2] R. L. BRYANT, Two exotic holonomies in dimension four, path geometries, and twistor theory, in *Complex geometry and Lie theory, Proc. Sympos. Pure Math.* **53**, Amer. Math. Soc., Providence, RI, 1991, pp. 33–88. **DOI MR 1, 7, 11**
- [3] R. L. BRYANT and P. A. GRIFFITHS, Characteristic cohomology of differential systems. II. Conservation laws for a class of parabolic equations, *Duke Math. J.* **78** (1995), 531–676. **DOI MR zbM 3**
- [4] M. DUNAJSKI, E. V. FERAPONTOV, and B. KRUGLIKOV, On the Einstein-Weyl and conformal self-duality equations, *J. Math. Phys.* **56** (2015), 083501, 10. **DOI MR 7**
- [5] M. DUNAJSKI and P. TOD, Paraconformal geometry of  $n$ th-order ODEs, and exotic holonomy in dimension four, *J. Geom. Phys.* **56** (2006), 1790–1809. **DOI MR 7**
- [6] E. V. FERAPONTOV and B. KRUGLIKOV, Dispersionless integrable hierarchies and  $GL(2, \mathbb{R})$  geometry, 2016. **arXiv 2, 9**
- [7] E. V. FERAPONTOV and B. S. KRUGLIKOV, Dispersionless integrable systems in 3D and Einstein-Weyl geometry, *J. Differential Geom.* **97** (2014), 215–254. **MR 2**
- [8] M. GODLINSKI and P. NUROWSKI,  $GL(2, \mathbb{R})$  geometry of ODE's, *J. Geom. Phys.* **60** (2010), 991–1027. **DOI MR 7**
- [9] N. HITCHIN, Complex manifolds and Einstein's equations, in *Twistor geometry and nonlinear systems, Lecture Notes in Math.* **970**, Springer, Berlin, 1982, pp. 73–99. **MR zbM 1**
- [10] T. A. IVEY and J. M. LANDSBERG, *Cartan for beginners: differential geometry via moving frames and exterior differential systems, Graduate Studies in Mathematics* **61**, American Mathematical Society, Providence, RI, 2003. **MR 3**

- [11] W. KRYŃSKI, Paraconformal structures and differential equations, *Differential Geom. Appl.* **28** (2010), 523–531. DOI MR **7, 8, 11, 12**
- [12] W. KRYŃSKI, Paraconformal structures, ordinary differential equations and totally geodesic manifolds, *J. Geom. Phys.* **103** (2016), 1–19. DOI MR **8, 11**
- [13] S. V. MANAKOV and P. M. SANTINI, Solvable vector nonlinear Riemann problems, exact implicit solutions of dispersionless PDEs and wave breaking, *J. Phys. A* **44** (2011), 345203, 19. DOI MR **9**
- [14] S. V. MANAKOV and P. M. SANTINI, Integrable dispersionless PDEs arising as commutation condition of pairs of vector fields, *Journal of Physics: Conference Series* **482** (2014), 012029. **9**
- [15] P. NUROWSKI, Comment on GL(2,  $\mathbb{R}$ ) geometry of fourth-order ODEs, *J. Geom. Phys.* **59** (2009), 267–278. DOI MR **7**
- [16] A. D. SMITH, Integrable GL(2) geometry and hydrodynamic partial differential equations, *Comm. Anal. Geom.* **18** (2010), 743–790. DOI MR **7**

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, UL. ŚNIADECKICH 8, 00-656 WARSZAWA, POLAND

*Email address:* krynski@impan.pl

INSTITUT FÜR MATHEMATIK, GOETHE-UNIVERSITÄT FRANKFURT, ROBERT-MAYER-STR. 10, 60325 FRANKFURT AM MAIN, GERMANY

*Email address:* metttler@math.uni-frankfurt.de