HIGH POINTS OF A RANDOM MODEL OF THE RIEMANN-ZETA FUNCTION AND GAUSSIAN MULTIPLICATIVE CHAOS

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ABSTRACT. We study the total mass of high points in a random model for the Riemann-Zeta function. We consider the same model as in [8, 2], and build on the convergence to 'Gaussian' multiplicative chaos proved in [14]. We show that the total mass of points which are a linear order below the maximum divided by their expectation converges almost surely to the Gaussian multiplicative chaos of the approximating Gaussian process times a random function. We use the second moment method together with a branching approximation to establish this convergence.

1. INTRODUCTION

1.1. **The model.** Let $\mathcal{P}$ denote the set of all prime numbers. Let $(\theta_p)_{p\in\mathcal{P}}$ be independent identically distributed random variables, being uniformly distributed on $[0, 2\pi]$. For $N \in \mathbb{N}$, a good model for the large values of the logarithm of the Riemann-zeta function on a typical interval of length 1 of the critical line as proposed in [8] is

$$X_N(x) = \sum_{j=1}^{N} \frac{1}{\sqrt{p_j}} \Big(\cos(x \ln p_j)\cos(\theta_{p_j}) + \sin(x \ln p_j)\sin(\theta_{p_j})\Big) \quad x \in [0, 1] \,. \tag{1.1}$$

By Theorem 7 in [14], the process X_N can be well approximated by a log-correlated Gaussian field $G_N(x), x \in [0, 1]$. Namely, take

$$G_N(x) = \sum_{j=1}^{N} \frac{1}{2\sqrt{p_j}} \Big(W_j^{(1)} \cos(x \ln p_j) + W_j^{(2)} \sin(x \ln p_j)\Big), \tag{1.2}$$

where $(W_j^{(i)})_{j\in\mathbb{N}, i\in\{1,2\}}$ are i.i.d. standard normal distributed. It is shown in [14] that

$$X_N(x) - G_N(x) \equiv E_N(x), \quad x \in [0, 1], \tag{1.3}$$

where $E_N(x)$ converges almost surely uniformly to a random function $E(x)$. Moreover, the error $E_N(x)$ has uniform exponential moments

$$\mathbb{E}\left(e^{\lambda \sup_{N\geq 1, x\in[0,1]} E_N(x)}\right) < \infty, \tag{1.4}$$

where $\mathbb{E}$ denotes expectation with respect to the θ_p's.

Some of the behavior of the large values of the process $X_N(x)$, $x \in [0, 1]$ is captured by the random measure

$$M_{\alpha,N}(dx) = \frac{e^{\alpha X_N(x)}}{\mathbb{E} e^{\alpha X_N(x)}} dx \,. \tag{1.5}$$

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By the independence of the θ_p's, it is not hard to see that $M_{\alpha,N}$ converges almost surely as $N \to \infty$. By Theorem 4 in [14], the almost sure weak limit of $M_{\alpha,N}(dx)$ is non-trivial for $0 < \alpha < 2$. We denote the limit of the total mass by M_α

$$M_\alpha = \lim_{N\to\infty} \int_0^1 M_{\alpha,N}(dx) \;\; a.s. \tag{1.6}$$

For log-correlated Gaussian field the analogous limiting measure is called Gaussian multiplicative chaos and M_α corresponds to the total mass of the limiting measure. For Gaussian multiplicative chaos it was first proven by [9] that the limit is nontrivial for small α and was recently revisited (see [13, 12]). Note that in our case the limit of $M_{\alpha,N}(dx)$ is almost a Gaussian multiplicative measure (see [14]). The connection between the Riemann-zeta function and Gaussian multiplicative chaos has been further analysed in [15].

The fact that the Riemann-zeta function (or a random model of it) can be well approximated by a log-correlated field have recently been used to study the extremes on a random interval [4, 11, 2].

1.2. **Main result.** Consider the Lebesgue measure of α-high points:

$$W_{\alpha,N} = \text{Leb}\{X_N(x) > \frac{\alpha}{2} \ln\ln N\} \tag{1.7}$$

The main result of this note is to relate the limit M_α to the Lebesgue measure of high points building on the ideas of [7]:

Theorem 1.1. *For any* $0 < \alpha < 2$ *and* M_α *as in (1.6), we have*

$$\frac{W_{\alpha,N}}{\mathbb{E}\left(W_{\alpha,N}\right)} \to M_\alpha, \tag{1.8}$$

in probability as $N \to \infty$.

It was proved in [2] that the maximum of $X_N(x)$ on $[0,1]$ is $\ln\ln N - (3/4 \pm \epsilon)\ln\ln\ln N$ with large probability. In view of this and of Theorem 1.1, it is not surprising to see that the M_α is non-trivial for $\alpha < 2$. The critical case where $\alpha \to 2$ is interesting as it is related to the fluctuations of the maximum of X_N. It is reasonable to expect that our approach can be adapted to the method of [5] to prove the critical case. Another upshot of the proof is that it highlights the fact that M_α depends on small primes, cf. Lemma 2.1.

The problem for the Riemann-zeta function is trickier. We expect that the equivalent of Theorem 1.1 still holds:

Conjecture 1.2. *Let* τ *be a uniform random variable on* $[T, 2T]$. *Let* $W_{\alpha,T} = \text{Leb}\{h \in [0,1] : \ln|\zeta(1/2 + i(\tau + h))| > \frac{\alpha}{2}\ln\ln T\}$. *Then we have*

$$\lim_{T\to\infty} \frac{W_{\alpha,T}}{\mathbb{E}[W_{\alpha,T}]} = \lim_{T\to\infty} \frac{\int_0^1 |\zeta(1/2 + i(\tau+h))|^\alpha}{\mathbb{E}[|\zeta(1/2 + i\tau|^\alpha]} \quad a.s.$$

This would be consistent with the conjecture of Fyodorov & Keating for the Lebesgue measure of high points, see Section 2.5 in [6] One issue is that it is not obvious that a result akin to Equation (1.3) holds, mainly because of the singularities of $\ln\zeta$ at the zeros. One way around this would be to restrict to Gaussian comparison to one-point and two-point large deviation estimates. This seems doable in view of Lemmas 3.2 and 3.3 and the Gaussian comparison theorem proved for the zeta function in [1].

1.3. **Outline of the proof.** The proof of Theorem 1.1 is based on a first and a second moment estimate and follow the global strategy proposed in [7] for branching Brownian motion. First, we prove convergence of a conditional first moment to the desired limiting object in Lemma 2.1. Its proof builds on results on the Gaussian comparison and convergence to Gaussian multiplicative chaos established in [14]. Next, a localisation result is established in Lemma 2.2. Finally, we turn to the proof of Proposition 3.4 which is based on a second moment computation. We use a branching approximation similar to the one employed in [2]. Using the obtained first and second moment estimates we are finally in the position to prove Theorem 1.1.

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2. First moment estimates

For $R \leq N$, we define $\mathcal{F}_R$ to be the σ-algebra generated by $(\theta_p)_{p\leq R}$. We will often condition on $\mathcal{F}_R$ to fix the dependence on the small primes. The variance of $G_N(x)-G_R(x)$, $x \in [0,1]$ is by definition

$$\sigma_R^2(N) \equiv \mathrm{Var}(G_N(x) - G_R(x)) = \frac{1}{2}\sum_{R<p\leq N} p^{-1} \tag{2.1}$$

The prime number theorem, see e.g. [10], implies that the density of the primes goes like $(\ln p)^{-1}$. More precisely, we have

$$\sigma_R^2(N) = \left|\sum_{R<p\leq N}^{N} p^{-1} - \frac{1}{2}(\ln\ln N - \ln\ln R)\right| = o(1) \text{ as } N\to\infty \text{ and } R\to\infty. \tag{2.2}$$

It turns out that the non-trivial contribution to Theorem 1.1 comes from the small primes.

Lemma 2.1. *For $W_{\alpha,N}$ as in (1.7), we have for* $0<\alpha<2$

$$\lim_{R\to\infty}\lim_{N\to\infty}\frac{\mathbb{E}\left(W_{\alpha,N}|\mathcal{F}_R\right)}{\mathbb{E}\left(W_{\alpha,N}\right)} = M_\alpha \quad a.s. \tag{2.3}$$

Proof. We start by computing $\mathbb{E}\left(W_{\alpha,N}|\mathcal{F}_R\right)$. Using Fubini's Theorem we can write the left-hand side of (2.3) as

$$\begin{aligned} &\int_0^1 \mathbb{P}\left(X_N(x) > \frac{\alpha}{2}\ln\ln N\middle|\mathcal{F}_R\right)dx \\ &= \int_0^1 \mathbb{P}\left(G_N(x)-G_R(x)+(E_N(x)-E_R(x)) > \frac{\alpha}{2}\ln\ln N - G_R(x) - E_R(x)\middle|\mathcal{F}_R\right)dx, \end{aligned} \tag{2.4}$$

where we used (1.3). Moreover, again for each $\epsilon>0$ there is R_0 such that for all $R\geq R_0$ $|E_R(x)-E_N(x)|<\epsilon$ almost surely and uniformly in x. Hence, we can again upper bound (2.4) by

$$\int_0^1 \mathbb{P}\left(G_N(x)-G_R(x) > \frac{\alpha}{2}\ln\ln N - G_R(x) - E_R(x) - \epsilon)\middle|\mathcal{F}_R\right)dx, \tag{2.5}$$

and a corresponding lower by replacing ϵ by $-\epsilon$. Next, observe that by definition of $X_N(x)$ and $E_N(x)$, $G_N(x)-G_R(x)$ are independent of $\mathcal{F}_R$. We have that the probability in (2.5) is bounded from above by

$$\frac{\sigma_R(N)}{\sqrt{2\pi}(\alpha \ln\ln N - G_R(x) - E_R(x) - \epsilon)} \exp\left(-\frac{(\frac{\alpha}{2}\ln\ln N - G_R(x) - E_R(x) - \epsilon)^2}{2\sigma_R^2(N)}\right)$$

$$= \frac{\sigma_R(N)}{\sqrt{2\pi}(\alpha \ln\ln N)} \exp\left(-\frac{\alpha^2(\ln\ln N)^2}{8\sigma_R^2(N)} + \alpha(E_R(x) + G_R(x) + \epsilon)\right)(1 + o(1)), \tag{2.6}$$

Next, we turn to $\mathbb{E}\left(W_{\alpha,N}\right)$. We have that

$$\mathbb{E}\left(W_{\alpha,N}\right) = \mathbb{E}\left(\mathbb{E}\left(W_{\alpha,N}|\mathcal{F}_R\right)\right) \leq \frac{\sigma_r(N)}{\sqrt{2\pi}(\frac{\alpha}{2}\ln\ln N)} \exp\left(-\frac{\alpha^2(\ln\ln N)^2}{8\sigma_R^2(N)}\right) \times \int_0^1 \mathbb{E}\left(\exp\left(\alpha(E_R(x) + G_R(x) - \epsilon)\right)\right) dx (1 + o(1)) \tag{2.7}$$

A corresponding lower bound we obtain by replacing ϵ by $-\epsilon$. Taking the quotient of (2.6) and and (2.7) and integrating with respect to x we get

$$\frac{\int_0^1 \exp\left(\alpha(E_R(x) + G_R(x) + \epsilon)\right)}{\int_0^1 \mathbb{E}\left(\exp\left(\alpha(E_R(x) + G_R(x) - \epsilon)\right)\right) dx}(1 + o(1)) \leq \frac{\mathbb{E}\left(W_{\alpha,N}|\mathcal{F}_R\right)}{\mathbb{E}\left(W_{\alpha,N}\right)} \leq \frac{\int_0^1 \exp\left(\alpha(E_R(x) + G_R(x) - \epsilon)\right)}{\int_0^1 \mathbb{E}\left(\exp\left(\alpha(E_R(x) + G_R(x) + \epsilon)\right)\right) dx}(1 + o(1)), \tag{2.8}$$

Pulling the terms involving ϵ out of the integral and noting the normalization of $M_{\alpha,R}$ is chosen such that $\mathbb{E}M_{\alpha,R} = 1$ and noting that

$$\mathbb{E}\left(\frac{\int_0^1 \exp\left(\alpha(E_R(x) + G_R(x))\right) dx}{\int_0^1 \mathbb{E}\left(\exp\left(\alpha(E_R(x) + G_R(x))\right)\right) dx}\right) = 1, \tag{2.9}$$

we can rewrite (2.7) as

$$M_{\alpha,R}e^{2\alpha\epsilon}(1 + o(1)) \leq \frac{\mathbb{E}\left(W_{\alpha,N}|\mathcal{F}_R\right)}{\mathbb{E}\left(W_{\alpha,N}\right)} \leq M_{\alpha,R}e^{-2\alpha\epsilon}(1 + o(1)). \tag{2.10}$$

Note that (2.10) holds for all $\epsilon > 0$. When taking $N, R \uparrow \infty$ $M_{\alpha,R}$ converges a.s. to M_α hence we have a.s.

$$M_\alpha e^{2\alpha\epsilon}(1 + o(1)) \leq \liminf_{N,r\to\infty} \frac{\mathbb{E}\left(W_{\alpha,N}|\mathcal{F}_R\right)}{\mathbb{E}\left(W_{\alpha,N}\right)} \leq \limsup_{N,r\to\infty} \frac{\mathbb{E}\left(W_{\alpha,N}|\mathcal{F}_R\right)}{\mathbb{E}\left(W_{\alpha,N}\right)} \leq M_\alpha e^{-2\alpha\epsilon}(1 + o(1)). \tag{2.11}$$

As (2.11) does not depend on r and N anymore, we can take the limit as $\epsilon \to 0$ and obtain

$$\lim_{R\to\infty}\lim_{N\to\infty} \frac{\mathbb{E}\left(W_{\alpha,N}|\mathcal{F}_R\right)}{\mathbb{E}\left(W_{\alpha,N}\right)} = M_\alpha\ . \tag{2.12}$$

□

Next, we want to control

$$W^>_{\alpha,N} = \text{Leb}\{x \in [0,1] : X_N(x) \geq \alpha \ln\ln N; \exists k \in [R, N] :\ X_k(x) > (\alpha + \epsilon)\ln\ln k\}\ . \tag{2.13}$$

The idea is that, at high points, the value $X_N(x)$ is most likely shared equally by the increments as defined in (3.1) below.

Lemma 2.2. *For all $\epsilon > 0$ there exists R_0 such that for all $R = o(N)$ and $R, N > R_0$ such that for all $c > 0$*

$$\mathbb{P}\left(W^{>}_{\alpha,N} > c\mathbb{E}W_{\alpha,N}\right) \le e^{-\epsilon r}, \tag{2.14}$$

where $r = \ln\ln R$.

Proof. We want to use Markov's inequality to bound the probability on (2.14). Hence, we need to bound $\mathbb{E}W^{>}_{\alpha,N}$ from above. First, we bound $\mathbb{E}\left(W^{>}_{\alpha,N}|\mathcal{F}_R\right)$ from above by

$$\begin{aligned}\int_0^1 \mathbb{P}\Big(&\{G_N(x) - G_R(x) \ge \alpha \ln\ln N - X_R(x) - \epsilon'\} \\ &\cap \{\exists K \in [R, N] :\ G_K(x) - G_R(x) > (\alpha + \epsilon) \ln\ln K - X_R(x) - \epsilon'\} \Big| \mathcal{F}_R\Big),\end{aligned} \tag{2.15}$$

where we used (1.3) and the fact that $E_R(x)$ converges a.s. uniformly to a continuous function $E(x)$. Hence, for all $\epsilon' > 0$ there is R_0 such that for all $K \ge R_0$ and all x we have $|E_K(x) - E_R(x)| < \epsilon'$.

Similarly as in (2.1), the variable $G_K(x) - G_R(x)$ is Gaussian with mean 0 and variance

$$\sigma_R^2(K) = \frac{1}{2} \sum_{R<p\le K} p^{-1} .$$

Let

$$B_K(x) = G_K(x) - G_R(x) - \frac{\sigma_R^2(K)}{\sigma_R^2(N)}(G_N(x) - G_R(x)), \tag{2.16}$$

then $(B_K(x))_{K=1}^N$ are points on a time-changed brownian bridge from zero to zero in time $\sigma_R(N)^2$. As a Brownian bridge is independent from its endpoint, Equation (2.15) is equal to

$$\begin{aligned}\int_0^1 \int_{\frac{\alpha}{2}\ln\ln N - X_R(x)-\epsilon'}^{\infty} &\mathbb{P}(G_N(x) - G_R(x) \in dy) \\ &\times \mathbb{P}\left(\exists K \in [R, N] :\ B_K(x) > \frac{\alpha+\epsilon}{2} \ln\ln K - X_R(x) - \epsilon' - \frac{\sigma_R^2(K)}{\sigma_R^2(N)} y \,\Big|\mathcal{F}_R\right) dx\end{aligned} \tag{2.17}$$

as $|\sum_{p\le K} p^{-1} - \ln\ln K| < C$. Let $r = \ln\ln R$ and $n = \ln\ln N$. Next, let us control the probability that $X_R(x)$ is too large.

$$\mathbb{P}\left(X_R(x) \ge \frac{\epsilon r}{3}\right) \le \mathbb{P}\left(G_R(x) \ge \frac{\epsilon r}{4}\right) + \mathbb{P}\left(E_R(x) \ge \frac{\epsilon r}{12}\right). \tag{2.18}$$

The second probability in (2.18) is bounded by $Ce^{-\frac{\epsilon r}{12}}$ by (1.4). For the first probability in (2.18) is bounded by $Ce^{-\frac{\epsilon r}{32}}$ by Gaussian tail asymptotics and the variance estimate for $G_R(x)$ for r large enough and uniformly in x. On the event that $\{X_R(x) \le \frac{\epsilon r}{3}\}$ we can bound the second probability in (2.17) from above by

$$\mathbb{P}\left(\exists s \in \left[0, \sigma_r^2(N)\right] :\ b(s) > (\alpha + \epsilon)\left((s + \sigma_0(R)^2) - C\right) - \frac{\epsilon r}{3} - \epsilon' - \frac{s}{\sigma_r^2(N)} y\right), \tag{2.19}$$

where $b(s)$ is a Brownian bridge from zero to zero in time $\sigma_r^2(N)$. Consider the line l from $(0, \frac{\epsilon}{6}r)$ to $\left(\sigma_r^2(N), (\alpha + \epsilon)(n/2 - C) - \epsilon' - y\right)$. One checks that $l(s) \le (\alpha+\epsilon)\left((s + \sigma_0(R)^2) - C\right)-$

$\frac{\epsilon r}{3} - \epsilon' - \frac{s}{\sigma_r^2(N)} y$ for all r large enough. The probability of Brownian bridge to stay under a linear function is well known, see e.g., Lemma 2.2 in [3],

$$\mathbb{P}\left(\exists s \in \left[0, \sigma_r^2(N)\right] : \ b(s) > l(s)\right) = \exp\left(-2\frac{l(0) l\left(\sigma_r^2(N)\right)}{\sigma_r^2(N)}\right) \tag{2.20}$$

Hence, on the event we can bound the expectation of (2.17) by

$$\int_0^1 \mathbb{E} \int_{\frac{\alpha}{2}\ln\ln N - X_r(x)}^{\infty} \mathbb{P}(G_N(x) - G_R(x) \in dy) e^{-\frac{\epsilon r(\alpha/2+\epsilon/2)n - (\alpha+\epsilon)C - \epsilon' - y)}{3/2(n-r)}} (1 + o(1)) dx + C e^{-\frac{\epsilon r}{32}} \mathbb{E}(W_{\alpha,N}) \tag{2.21}$$

Using the Gaussian tail asymptotics for $G_N(x) - G_R(x)$ together with (2.7), Equation (2.21) is bounded above by

$$\mathbb{E}(W_{\alpha,N}) e^{-r\frac{\epsilon^2}{6} + o(r)}. \tag{2.22}$$

This implies the claim of Lemma 2.2. □

3. Branching approximation and second moment estimates

3.1. **Definition of the increments.** The goal is to use a branching approximation similar to [2] to compute the necessary second moments. To this end, we define for $k \in \mathbb{N}$ and $x \in (0, 1)$

$$Y_k(x) = \sum_{e^{k-1} < \ln p \leq e^k} \frac{1}{2\sqrt{p_j}} \left(W_j^{(1)} \cos(x \ln p_j) + W_j^{(2)} \sin(x \ln p_j)\right). \tag{3.1}$$

By definition, we have

$$G_N(x) = \sum_{k=1}^{n} Y_k(x), \tag{3.2}$$

where for the rest of the section we set $n \equiv \ln\ln N$. The increments Y_k are such that

$$\rho_k(x, x') \equiv \mathbb{E}(Y_k(x) Y_k(x')) = \sum_{e^{k-1} < \ln p \leq e^k} \frac{1}{2p} \cos(|x - x'| \ln p_j). \tag{3.3}$$

The covariances can be computed again by the prime number theorem. This is done in Lemma 2.1 in [2]. It is convenient to state the result to introduce branching point of $x, x' \in (0, 1)$ by

$$x \wedge x' \equiv \lfloor \ln |x - x'|^{-1} \rfloor. \tag{3.4}$$

Lemma 3.1 (Lemma 2.1 in [2]). *For $k \geq 1$ and $x, x' \in (0, 1)$ we have*

$$\mathbb{E}(Y_k^2(x)) = \frac{1}{2} + O\left(e^{-c\sqrt{e^k}}\right), \tag{3.5}$$

and

$$\rho_k(x, x') = \begin{cases} \frac{1}{2} + O\left(e^{-2(x\wedge x' - k)}\right) + O\left(e^{-c\sqrt{e^k}}\right) & \textit{if } k \leq x \wedge x', \\ O\left(e^{-(k - x\wedge x')}\right) & \textit{if } k > x \wedge x' \end{cases} \tag{3.6}$$

There is a fast decoupling between the increments after the branching point where the distribution of $Y_k(x)$ and $Y_k(x')$ is very close to independent Gaussians with mean zero and variance $1/2$. We introduce a parameter Δ that gives some room before and after the branching point to ensure uniform estimates.

Lemma 3.2. *Let $\Delta > 0$. Let $x, x' \in (0, 1)$ and $m > x \wedge x' + \Delta$. Then we have*
(3.7)
$$\mathbb{P}\left(\sum_{k=m+1}^{n} Y_k(x) \in A, \sum_{k=m+1}^{n} Y_k(x') \in B\right) = \mathbb{P}\left(\sum_{k=m+1}^{n} Y_k \in A\right)\mathbb{P}\left(\sum_{k=m+1}^{n} Y_k \in B\right)\left(1 + O(e^{-c\delta}\right),$$
where $(Y_i)_{i\in\mathbb{N}}$ are iid Gaussians with mean zero and variance σ^2.

Proof. As$(\sum_{k=m+1}^{n} Y_k(x), \sum_{k=m+1}^{n} Y_k(x'))$ is a Gaussian process and its covariance is controlled in Lemma 3.1 it suffices to compare densities. This follows the same lines starting from Eq. (61) in [2] only that in our setting $\mu = 0$. □

Before the branching point we want to show that $Y_k(x)$ and $Y_k(x')$ are almost fully correlated. This is specified in the lemma below.

Lemma 3.3. *Let $\Delta > 0$. Let $x, x' \in (0, 1)$ and $r < m < x \wedge x' - \Delta$. Then we have*
$$\mathbb{P}\left(\sum_{k=r}^{m} Y_k(x) \in A, \sum_{k=r}^{m} Y_k(x') \in B\right) = \mathbb{P}\left(\sum_{k=r}^{m} Y_k \in A \cap B\right)\left(1 + O(e^{-c\Delta}\right), \tag{3.8}$$

Proof. As G is a Gaussian process this follows from the density estimates in Lemma 3.1. □

3.2. **Second moment computation.** The main result of this section is:

Proposition 3.4. *There exists $\kappa_\alpha > 0$ such that for $R = o(\ln\ln N)$ as $N \to \infty$ we have*
$$\mathbb{P}\left(\left|\frac{W_{\alpha,N} - \mathbb{E}\left(W_{\alpha,N}|\mathcal{F}_R\right)}{\mathbb{E}\left(W_{\alpha,N}\right)}\right| > c\right) \le (1 + o(1))Ce^{-k_\alpha r}, \tag{3.9}$$
where $r \equiv \ln\ln R$ and $C > 0$ a constant depending on c.

To prove Proposition 3.4 we essentially need to control the second moment of
$$W^{\le}_{\alpha,N} = \text{Leb}\Big\{x \in [0, 1] : \sum_{j\le n} Y_j(x) \ge \alpha n/2; \forall k \in [2r, n] : \sum_{j\le k} Y_j(x) \le (\alpha + \epsilon)k/2\Big\}.$$

Remark. Throughout the proof we restrict our computations to R and N such that $r = \ln\ln R$ and $n = \ln\ln N$ are natural numbers. The general case follows in the same way by considering the last resp. first summands in the representation in (3.2) of G_N separately. The desired estimates carry over by minor adjustments but would require a more involved notation. To keep the computations that follow as clear as possible and not to burden the reader with heavier notations we restrict ourselves to the case where $r, n \in \mathbb{N}$.

Indeed, Markov's inequality and Lemma 2.2 imply
$$\mathbb{P}\left(\left|\frac{W_{\alpha,N} - \mathbb{E}\left(W_{\alpha,N}|\mathcal{F}_R\right)}{\mathbb{E}\left(W_{\alpha,N}\right)}\right| > c\right) \le \mathbb{P}\left(\frac{\left(W^{\le}_{\alpha,N} - \mathbb{E}\left(W^{\le}_{\alpha,N}|\mathcal{F}_R\right)\right)^2}{\mathbb{E}\left(W_{\alpha,N}\right)^2} > c^2/4\right) + Ce^{-Rc(\epsilon)}. \tag{3.10}$$

Clearly, we have
(3.11)
$$(W^{\le}_{\alpha,N})^2 = \text{Leb}^{\times 2}\{x, x' \in [0, 1] : \forall y \in \{x, x'\} \sum_{k\le n} Y_k(y) > \frac{\alpha}{2}n, \forall k \in [r, n] \sum_{j\le k} Y_j(y) \le \frac{\alpha+\epsilon}{2}k\}$$

Let $0 < \Delta < r$. We divide the right side into four terms depending on the branching point:

$(I) : x\wedge x' > n-\Delta \quad (II) : r+\Delta < x\wedge x' \le n-\Delta \quad (III) : r-\Delta < x\wedge x' \le r+\Delta \quad (IV) : x\wedge x' \le r-\Delta$.

The term (IV) is controlled in the following Lemma.

Lemma 3.5. *For $R = o(\ln\ln N)$ we have*

$$\lim_{\Delta\to\infty}\lim_{N\to\infty}\frac{\mathbb{E}((IV)|\mathcal{F}_R) - \left(\mathbb{E}\left(W^{\le}_{\alpha,N}|\mathcal{F}_R\right)\right)^2}{\mathbb{E}\left(W_{\alpha,N}\right)^2} = 0 \quad a.s. \tag{3.12}$$

Proof of Lemma 3.5. As $x\wedge x' < r-\Delta$ and by a similar rewriting in (2.5) we have by Lemma 3.2 that it is bounded from above by

$$\iint_{\{x\wedge x'\le r-\Delta\}}\prod_{y\in\{x,x'\}}\mathbb{P}\left(\sum_{k=r+1}^{n} Y_k > \frac{\alpha}{2}n - G_R(y) - E_R(y) - \epsilon\Big|\mathcal{F}_R\right)dx'dx\left(1+O\left(e^{-c\Delta}\right)\right) \tag{3.13}$$

$$\le \iint_{[0,1]^2}\prod_{y\in\{x,x'\}}\mathbb{P}\left(\sum_{k=r+1}^{n} Y_k > \frac{\alpha}{2}n - G_R(y) - E_R(y) - \epsilon\Big|\mathcal{F}_R\right)dx'dx\left(1+O\left(e^{-c\Delta}\right)\right)$$

We now compare (3.13) with $\left(\mathbb{E}\left(W^{\le}_{\alpha,N}|\mathcal{F}_R\right)\right)^2$ which is bounded below by

$$\iint_{[0,1]^2}\prod_{y\in\{x,x'\}}\mathbb{P}\left(G_N(y) - G_R(y) > \frac{\alpha}{2}n - G_R(y) - E_R(y) + \epsilon)|\mathcal{F}_R\right)dx'dx \tag{3.14}$$

for any $\epsilon > 0$. By 2.1 and the Gaussian approximation given in (3.1) the absolute value of the difference of (3.13) and (3.14) is bounded by

$$M^2_{\alpha,N}\mathbb{E}(W_{\alpha,N})^2 e^{-2\epsilon}\left(1+O\left(e^{-c\Delta}\right)\right) - M^2_{\alpha,N}e^{2\alpha\epsilon}. \tag{3.15}$$

Hence (3.15) divided by $\mathbb{E}(W_{\alpha,N})^2$ converges almost surely to

$$M^2_{\alpha}e^{-2\alpha\epsilon}\left(e^{-2\alpha\epsilon} - e^{2\alpha\epsilon} + e^{-2\alpha\epsilon}O\left(e^{-c\Delta}\right)\right) \tag{3.16}$$

for all $\epsilon,\Delta > 0$. Note that (3.15) converges to zero as $\epsilon\to 0$ and $\Delta\to\infty$. □

To control the terms (I), (II) and (III), we prove the following lemma.

Lemma 3.6. *Let $0 < \alpha < 2$. There exists $\kappa_\alpha > 0$ such that for $R = o(\ln\ln N)$ as $N\to\infty$ we have*

$$\mathbb{E}\left((I) + (II) + (III)\right) \le \mathbb{E}\left(W_{\alpha,N}\right)^2 e^{-\kappa_\alpha r}. \tag{3.17}$$

Proof of Lemma 3.6. We bound $\mathbb{E}((I)|\mathcal{F}_R)$ from above by

$$e^{-n+\Delta}\int_0^1\mathbb{P}\left(\sum_{k\le n} Y_k(x) > \frac{\alpha}{2}n\Big|\mathcal{F}_R\right)dx = e^{-n+\Delta}\mathbb{E}(W_{\alpha,N}|\mathcal{F}_R), \tag{3.18}$$

by (2.4). Hence,

$$\mathbb{E}((I)) \le \mathbb{E}(W_{\alpha,N})^2\frac{e^{-n+\Delta}}{\mathbb{E}(W_{\alpha,N})} = E(W_{\alpha,N})^2 o(1), \tag{3.19}$$

as $\mathbb{E}(W_{\alpha,N}) = cn^{-1/2}e^{-\frac{\alpha^2}{4}n}$ and $0 < \alpha < 2$.

Next, we turn to $\mathbb{E}((II)|\mathcal{F}_R)$. Using that uniformly in y for all R, N large enough $|E_N(y) - E_R(y)| \le \epsilon$, we can bound $\mathbb{E}((II)|\mathcal{F}_R)$ from above by

(3.20)

$$\iint_{\{r+\Delta\le x\wedge x'\le n-\Delta\}}\mathbb{P}\left(\forall_{y\in\{x,x'\}}\sum_{j=r+1}^{n} Y_j(y) > \frac{\alpha}{2}n - X_R(y) - \epsilon, \forall_{k\in[r,n]}\sum_{j=r}^{k} Y_j(y) \le \frac{\alpha+\epsilon}{2}k\Big|\mathcal{F}_R\right)dxdx'$$

Dropping the barrier constraint except at $x \wedge x' - \Delta$ and $x \wedge x' + \Delta$ we can bound the probability in (3.20) from above by

$$\mathbb{P}\left(\forall_{y\in\{x,x'\}} \sum_{j=r+1}^{n} Y_j(y) > \frac{\alpha}{2}n - X_R(y) - \epsilon,\ \forall_{k\in\{x\wedge x'-\Delta, x\wedge x'+\Delta\}} \sum_{j=1}^{k} Y_j(y) \le \frac{\alpha+\epsilon}{2}k \Big| \mathcal{F}_R\right). \tag{3.21}$$

We evaluate the probability in the integral at a fixed $x \wedge x' = m$, and sum the contributions over m afterwards. We introduce an extra conditioning. Let $\mathcal{F}_k^Y = \sigma(Y_j, j \le k)$. We condition on $\mathcal{F}_{m+\Delta}^Y$, slightly after the branching point. Lemma 3.2 applied to (3.21) then yields

$$\begin{aligned}&\left(1+e^{-c\Delta}\right)\mathbb{E}\Bigg(\prod_{y\in\{x,x'\}} \mathbb{P}\left(\sum_{k=m+\Delta+1}^{n} Y_k(y) > \frac{\alpha}{2}n - X_R(y) - \epsilon - \sum_{r<j\le m+\Delta} Y_j(y) \Big| \mathcal{F}_{m+\Delta}^Y\right)\\ &\qquad ; \forall_{y\in\{x,x'\}, k\in\{m-\Delta, m+\Delta\}} \sum_{j\le k} Y_j(y) \le \frac{\alpha+\epsilon}{2}k \Big| \mathcal{F}_R\Bigg).\end{aligned} \tag{3.22}$$

We distinguish two cases. First, consider the case when for $y = x$ or $y = x'$,

$$\frac{\alpha}{2}n - X_R(y) - \epsilon - \sum_{r<j\le m+\Delta} Y_j(y) \le 0. \tag{3.23}$$

Note that due to the barrier in (3.22) this can only happen jointly with the barrier event if $m \ge \frac{\alpha}{\alpha+\epsilon}n - C'\epsilon$ for some constant $C' > 0$ independent of ϵ. In this case we bound the probabilities above by one and bound (3.22) from above by

$$\left(1+e^{-c\Delta}\right)\mathbb{P}\left(\frac{\alpha}{2}n - X_r(y) - \epsilon - \sum_{r<j\le m+\Delta} Y_j(y) \le 0 : \forall_{y\in\{x,x'\}} \sum_{j\le m+\Delta} Y_j(y) \le \frac{\alpha+\epsilon}{2}(m+\Delta) \Big| \mathcal{F}_R\right). \tag{3.24}$$

As for an upper bound we can drop all constraints in the expectation with respect x' (if $y = x$) and x otherwise, let us assume without loss of generality that $y = x$. We need to distinguish whether $\frac{\alpha}{2}n - X_R(x) - \epsilon > 0$ or not. On the event $\frac{\alpha}{2}n - X_R(x) - \epsilon \le 0$ we bound the expectation in (3.24) by one and obtain that the expectation of (3.24) from above by

$$\mathbb{P}\left(X_R(x) \ge \frac{\alpha}{2}n - \epsilon\right) \le \mathbb{E}\left(e^{\alpha X_R(x) - \alpha\left(\frac{\alpha}{2}n - \epsilon\right)}\right) \tag{3.25}$$

by the exponential Chebyshev inequality. Hence, integrating over x, x' in (3.28) we get

$$\begin{aligned} e^{-\frac{\alpha}{\alpha+\epsilon}n - C'\epsilon} \int_0^1 \mathbb{E}\left(e^{\alpha X_R(x) - \alpha\left(\frac{\alpha}{2}n-\epsilon\right)}\right) dx &\le \mathbb{E}\left(\int_0^1 e^{\alpha X_r(x)}\right) e^{-\alpha^2 n/2 - \alpha\epsilon}\\ &\le Cn\mathbb{E}\left(W_{\alpha,N}\right)^2 e^{-\frac{\alpha}{\alpha+\epsilon}n - C'\epsilon} e^{-\alpha r - \alpha\epsilon},\end{aligned} \tag{3.26}$$

by (2.7). When $\frac{\alpha}{2}n - X_R(x) - \epsilon > 0$, we bound (3.24) from above using Gaussian tail asymptotics by

$$\left(1+e^{-c\Delta}\right)\mathbb{P}\left(\sum_{r<j\le m+\Delta} Y_j(x) \ge \frac{\alpha}{2}n - X_R(y) - \epsilon \Big| \mathcal{F}_R\right) \le \left(1+e^{-c\Delta}\right) e^{-\frac{\left(\frac{\alpha}{2}n - X_R(x) - \epsilon\right)^2}{2\sigma_r(m+\Delta)}}. \tag{3.27}$$

The integral of (3.27) with respect to x and x' can be bounded from above by

$$\begin{aligned}
(3.28)\qquad & \left(1+e^{-c\Delta}\right)\sum_{\frac{\alpha}{\alpha+\epsilon}n-C'\epsilon\le m\le n-\Delta} e^{-m}\int_0^1 e^{-\frac{\left(\frac{\alpha}{2}n-X_r(x)-\epsilon\right)^2}{2\sigma_r(m+\Delta)}}dx\\
\le\ & \left(1+e^{-c\Delta}\right)\sum_{\frac{\alpha}{\alpha+\epsilon}n-C'\epsilon\le m\le n-\Delta} e^{-m}\int_0^1 e^{-(\alpha^2 n/4)-\frac{\alpha^2 n(n-2\sigma_r(m+\Delta))}{8\sigma_r(m+\Delta)}}e^{\alpha\frac{n\epsilon+nX_r(x)}{2\sigma_r(m+\Delta)}}dx\\
\le\ & \left(1+e^{-c\Delta}\right)\sum_{\frac{\alpha}{\alpha+\epsilon}n-C'\epsilon\le m\le n-\Delta} e^{-m}\int_0^1 e^{-(\alpha^2 n/2)-\frac{\alpha^2(n-2\sigma_r(m+\Delta))^2}{8\sigma_r(m+\Delta)}+\frac{\alpha^2}{4}(2\sigma_r(m+\Delta))}e^{\alpha\frac{n\epsilon+nX_r(x)}{2\sigma_r(m+\Delta)}}dx
\end{aligned}$$

Using that in the range of summation in (3.28) $\sigma_r(m+\Delta)$ is bounded form above and below by $\frac{1}{2}(m-r)+C$ resp. $\frac{1}{2}(m-r)-C$, for some constant large enough, we can bound (3.28) from above by

$$(3.29)\qquad \left(1+e^{-c\Delta}\right)\sum_{\frac{\alpha}{\alpha+\epsilon}n-C'\epsilon\le m\le n}\int_0^1 e^{-(\alpha^2 n/2)-\frac{\alpha^2(n-(m-r)-C)^2}{4(m-r+C)}}e^{\left(\frac{\alpha^2}{4}-1\right)m}e^{\alpha\frac{n(\epsilon+X_r(x))}{m+\Delta-r-C}+C\Delta}dx.$$

As $m\ge\frac{\alpha}{\alpha+\epsilon}n-C'\epsilon$, exponantial term in ϵ bounded by $e^{C\epsilon}$ and as $0<\alpha<2$ we have that on the one hand $\frac{\alpha^2}{4}-1<0$ and on the other hand we can choose together with (2.7) we can bound the corresponding expectation in (3.29) from above by

$$(3.30)\qquad \left(1+e^{-c\Delta}\right)E(W_{\alpha,N})^2e^{-cn}e^{cr},$$

for some $c>0$.

Finally, we turn to bound (3.22) for $\frac{\alpha}{2}n-X_r(y)-\epsilon-\sum_{r<j\le m+\Delta}Y_j(y)\ge 0$ we can bound (3.22) from above by a Gaussian tail bound and obtain

$$\begin{aligned}
(3.31)\qquad & \mathbb{E}\Bigg(\frac{(n-m-\Delta)/2}{2\pi\prod_{y\in\{x,x'\}}(\frac{\alpha}{2}(n-m-\Delta-\epsilon)-X_R(y)-\epsilon)}\mathbb{1}_{\forall_{y\in\{x,x'\},k\in\{m-\Delta,m+\Delta\}}\sum_{j\le k}Y_j(y)\le\frac{\alpha+\epsilon}{2}k}\\
&\times\exp\left(-\sum_{y\in\{x,x'\}}\frac{\left(\frac{\alpha}{2}n-X_R(y)-\epsilon-\sum_{r<j\le m+\Delta}Y_j(y)\right)^2}{(n-m-\Delta)}\right)\Big|\mathcal{F}_R\Bigg)
\end{aligned}$$

Next, we condition on $\mathcal{F}^Y_{m-\Delta}$. The terms depending on $\sum_{m-\Delta<j\le m+\Delta}Y_j$ can be bounded by the moment generating function:

$$(3.32)\qquad \mathbb{E}\left(e^{C\Delta\sum_{m-\Delta<j\le m+\Delta}Y_j(x)+Y_j(x')}\right)\le e^{C'\Delta^2}.$$

Hence, Equation (3.31) is bounded above by

(3.33)

$$e^{C'\Delta^2}\mathbb{E}\left(\frac{(n-m-\Delta)/2}{2\pi}\prod_{y\in\{x,x'\}}\mathbb{1}_{\sum_{j\le m-\Delta}Y_j(y)\le\frac{\alpha+\epsilon}{2}(m-\Delta)}\frac{\exp\left(-\frac{\left(\frac{\alpha}{2}n-X_R(y)-\epsilon-\sum_{j=r+1}^{m-\Delta}Y_j(y)\right)^2}{(n-m-\Delta)}\right)}{\frac{\alpha}{2}(n-m-\Delta-\epsilon)-X_R(y)-\epsilon}\Bigg|\mathcal{F}_R\right)$$

Using the fact that the variables $Y_j(x)$ and $Y_j(x')$ almost coincide for $j\le m-\Delta$ by Lemma 3.3, we have that (3.33) is bounded above by

(3.34)

$$e^{C'\Delta^2}\frac{(n-m-\Delta)/2}{2\pi(\frac{\alpha}{2}(n-m-\Delta-\epsilon)-X_R(x)-\epsilon)^2}\mathbb{E}\left(\mathbb{1}_{\sum_{j\le m-\Delta}Y_j(x)\le\frac{\alpha+\epsilon}{2}(m-\Delta)}e^{-\frac{2\left(\frac{\alpha}{2}n-X_R(x)-\epsilon-\sum_{j=r+1}^{m-\Delta}Y_j(x)\right)^2}{(n-m-\Delta)}}\Bigg|\mathcal{F}_R\right)(1+O(e^{-c\Delta}))$$

The expectation in (3.34) is equal to

$$\int_{-\infty}^{\frac{\alpha+\epsilon}{2}(m-\Delta)} e^{-\frac{2\left(\frac{\alpha}{2}n-X_r(x)-\epsilon-z\right)^2}{(n-m-\Delta)}} e^{-\frac{z^2}{(m-\Delta-r)}} \frac{dz}{\sqrt{\pi(m-\Delta-r)}}. \tag{3.35}$$

The integrand with respect to z is minimal for

$$z^* = \frac{2\left(\frac{\alpha}{2}n - X_R(x) - \epsilon\right)(m-\Delta-r)}{n+m-3\Delta-2r}. \tag{3.36}$$

When $\frac{\alpha+\epsilon}{2}m - z^* \ll 0$ which is the case when $m < (1-\delta)n$ for some $\delta > 0$, we can use Gaussian tail asymptotics to bound (3.35) from above by

$$\exp\left(-\frac{2\left(\frac{\alpha}{2}n - X_R(x) - \epsilon - \frac{\alpha+\epsilon}{2}m\right)^2}{(n-m-\Delta)} - \frac{\left(\frac{\alpha+\epsilon}{2}m\right)^2}{(m-\Delta-r)}\right). \tag{3.37}$$

Plugging this bound into (3.34), summing over $m < (1-\delta)n$, and computing the squares in the exponential, we obtain that (3.34) is bounded from above by

$$\sum_{l=r+\Delta}^{(1-\delta)n} e^{\left(\frac{\alpha^2}{4}-1\right)l} e^{C\Delta+C\epsilon} \mathbb{E}(W_{\alpha,N})^2(1+o(1)), \tag{3.38}$$

If $x \wedge x' < (1-\delta)n$ we can bound the Gaussian integral by one and get that (3.8) is bounded from above by

$$e^{-\frac{2\left(\frac{\alpha}{2}n-X_R(x)-\epsilon\right)^2}{(n-m-\Delta)}} e^{+\frac{(z^*)^2}{(m-\Delta-r)}} \frac{dy}{\sqrt{\pi(m-\Delta-r)}} e^{-\frac{2(z^*)^2}{(n-m-\Delta)}} \tag{3.39}$$

Using (3.36) we can bound the expectation of (3.39) for $m > (1-\delta)n$ by

$$e^{C(\Delta+\epsilon)} \mathbb{E}\left(W_{\alpha,N}\right)^2 \mathbb{E}\left(e^{\frac{2\left(\frac{\alpha}{2}n-X_R(x)-\epsilon\right)^2 m}{n(n+m)}}\right) \tag{3.40}$$

Plugging this into (3.34) we can bound the contribution from above

$$\sum_{m>(1-\delta)n} 2^{-m} e^{C(\Delta^2+\epsilon)} \mathbb{E}\left(W_{\alpha,N}\right)^2 \mathbb{E}\left(e^{\frac{2\left(\frac{\alpha}{2}n-X_r(x)-\epsilon\right)^2 m}{n(n+m)}}\right) \tag{3.41}$$

Noting that $2n - n\delta \leq n + m \leq 2n$ the above term can be bounded from above by

$$\sum_{m>(1-\delta)n} 2^{-m+\frac{2\frac{\alpha^2}{4}n^2}{n^2(2-\delta)l}} e^{C(\Delta^2+\epsilon)}. \tag{3.42}$$

Note that the exponent in (3.42) is negative for δ sufficiently small.

Finally, we want to bound $\mathbb{E}((III))$. By Lemma 3.2 we have similar to (3.22) that $\mathbb{E}((III))$ is bounded from above by $\left(1+e^{-c\Delta}\right)$ times

$$\sum_{m=r-\Delta+1}^{r+\Delta} \iint_{\{x\wedge x'=m\}} \mathbb{E}\left(\prod_{y\in\{x,x'\}} \mathbb{P}\left(\sum_{k=m+\Delta+1}^{n} Y_k(y) > \frac{\alpha}{2}n - X_R(y) - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(y) \middle| \mathcal{F}^Y_{m+\Delta}\right)\right.$$
$$\left. ; \forall_{y\in\{x,x'\}} \sum_{j\leq m+\Delta} Y_j(y) \leq \frac{\alpha+\epsilon}{2}(m+\Delta)\right) dx dx' \tag{3.43}$$

If $\frac{\alpha}{2}n - X_R(y) - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(y) > 0$ for $y \in \{x, x'\}$ we can use Gaussian tail asymptotics for the probabilities in (3.43) to bound the expectation in (3.43) from above by

$$\mathbb{E}\left(\frac{(n-r)/2}{2\pi \prod_{y\in\{x,x'\}}(\frac{\alpha}{2}(n-r-2\Delta-\epsilon) - X_R(y) - \epsilon)} e^{-\frac{\left(\frac{\alpha}{2}n - X_R(x) - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(x)\right)^2}{(n-m-\Delta)}} e^{-\frac{\left(\frac{\alpha}{2}n - X_R(x') - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(x')\right)^2}{(n-m-\Delta)}} \right). \tag{3.44}$$

Noticing that the polynomial prefactor is bounded by C/n and otherwise proceeding as in (3.32) we can bound (3.43) from above by

$$e^{C'\Delta^2} \sum_{m=r-\Delta+1}^{r+\Delta} \iint_{\{x\wedge x'=m\}} \mathbb{E}\left(\frac{C}{n} e^{\frac{\left(\frac{\alpha}{2}n - X_R(x) - \epsilon\right)^2}{(n-m-\Delta)}} e^{\frac{\left(\frac{\alpha}{2}n - X_R(x') - \epsilon\right)^2}{(n-m-\Delta)}} |\mathcal{F}_R \right) dx' dx \left(1 + O\left(e^{-c\Delta}\right)\right)$$

$$\leq e^{C'\Delta^2 - C\Delta} e^{-r+\Delta} \mathbb{E}\left(\mathbb{E}\left(W_{\alpha,N}\right)^2 e^{2\alpha\epsilon} \left(1 + O\left(e^{-c\Delta}\right)\right)\right), \tag{3.45}$$

by (2.6) for any $\epsilon > 0$ and $\Delta > 0$. Note first that due to the barrier event in (3.43) the case $\frac{\alpha}{2}n - X_R(y) - \epsilon - \sum_{j=r+1}^{m+\Delta} Y_j(y) \leq 0$ for at least one $y \in \{x, x'\}$ can be excluded for $m \in \{r - \Delta, r + \Delta\}$.

This completes the control of (III) and hence also the proof of Theorem 3.6.

□

Proof of Proposition 3.4. We bound (3.10) from above by

$$\mathbb{P}\left(\frac{(I) + (II) + (III)}{\mathbb{E}\left(W_{\alpha,N}\right)^2} > c^2/8 \right) + \mathbb{P}\left(\frac{\mathbb{E}((IV)|\mathcal{F}_R) - \left(\mathbb{E}\left(W^{\leq}_{\alpha,N}|\mathcal{F}_R\right)\right)^2}{\mathbb{E}\left(W_{\alpha,N}\right)^2} > c^2/8 \right) + Ce^{-Rc(\epsilon)} \tag{3.46}$$

$$\leq \frac{8}{c^2}\mathbb{E}\left(\frac{(I) + (II) + (III)}{\mathbb{E}\left(W_{\alpha,N}\right)^2} \right) + \mathbb{P}\left(\frac{\mathbb{E}((IV)|\mathcal{F}_R) - \left(\mathbb{E}\left(W^{\leq}_{\alpha,N}|\mathcal{F}_R\right)\right)^2}{\mathbb{E}\left(W_{\alpha,N}\right)^2} > c^2/8 \right) + Ce^{-Rc(\epsilon)}$$

where we used Chebyshev's inequality. By Lemma 3.6 we can bound (3.10) from above by

$$\frac{8\mathbb{E}\left(W_{\alpha,N}\right)^2 e^{-\kappa_\alpha r}}{c^2\mathbb{E}\left(W_{\alpha,N}\right)^2} + \mathbb{P}\left(\frac{\mathbb{E}((IV)|\mathcal{F}_R) - \left(\mathbb{E}\left(W^{\leq}_{\alpha,N}|\mathcal{F}_R\right)\right)^2}{\mathbb{E}\left(W_{\alpha,N}\right)^2} > c^2/8 \right) \leq \frac{4}{c^2} e^{-\kappa_\alpha r} + Ce^{-Rc(\epsilon)}, \tag{3.47}$$

which yields Proposition 3.4 by possibly modifying the constants and noting that ϵ in (3.47) is arbitrary (but fixed) as the claim of Lemma 3.5 holds almost surely in the $N \to \infty$ limit. □

4. Proof of Theorem 1.1

Finally, we are in the position to prove Theorem 1.1 using Lemma 2.1 and Proposition 3.4.

Proof of Theorem 1.1. First, we rewrite

$$\frac{W_{\alpha,N}}{\mathbb{E}\left(W_{\alpha,N}\right)} = \frac{\mathbb{E}\left(W_{\alpha,N}|\mathcal{F}_R\right)}{\mathbb{E}\left(W_{\alpha,N}\right)} + \frac{W_{\alpha,N} - \mathbb{E}\left(W_{\alpha,N}|\mathcal{F}_R\right)}{\mathbb{E}\left(W_{\alpha,N}\right)}. \tag{4.1}$$

By Proposition 3.4 the second summand on the right hand side of (4.1) converges to zero in probability when first $N \to \infty$ and then $R \to \infty$. By Lemma 2.1 the term the first summand on the right hand side of (4.1) converges almost surely to M_α defined in (1.6). This completes the proof of Theorem 1.1. □

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