Numerik von Differentialgleichungen
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Prof. Dr. Bastian von Harrach
M. Sc. Tim Jahn

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

Aufgabe 11.3 - Lösungsvorschlag

Sei $x \in \Omega \subset \mathbb{R}^n, n = 1$ oder $n = 2$. Sei $u \in C(\bar{\Omega})^3$. Dann existiert $C > 0$, so dass für $h > 0$ und $h_O, h_W, h_N, h_S \leq h$ (die hinreichend klein seien, so dass die Auswertungen von u definiert sind)

(a) für $n = 1$

$$\left| \frac{2}{h_O(h_O + h_W)} u_O + \frac{2}{h_W(h_O + h_W)} u_W - \frac{2}{h_O h_W} u_Z - u''(x) \right| \leq C|u|_{C(\bar{\Omega})^3} h,$$

wobei $u_Z := u(x), u_W := u(x - h_W)$ und $u_O := u(x + h_O)$.

(b) für $n = 2$

$$\left| \frac{2}{h_O(h_O + h_W)} u_O + \frac{2}{h_W(h_O + h_W)} u_W + \frac{2}{h_N(h_S + h_N)} u_N \right.$$
$$\left. + \frac{2}{h_S(h_S + h_N)} u_S - \frac{2}{h_O h_W} u_Z - \frac{2}{h_S h_N} u_Z - \Delta u(x) \right| \leq C|u|_{C(\bar{\Omega})^3} h,$$

wobei $u_Z := u(x), u_W := u(x - h_W e_1), u_O := u(x + h_O e_1), u_S := u(x - h_S e_2)$ und $u_N := u(x + h_N e_2)$.

Lösungsvorschlag:

(1.) Im Fall $n = 1$ ist zu zeigen

$$\left| \frac{2}{h_O(h_O + h_W)} u_O - \frac{2}{h_O h_W} u_Z + \frac{2}{h_W(h_O + h_W)} u_W - u''(x) \right| \leq C|u|_{C^3(\overline{\Omega})} h, \tag{1}$$

mit $u_Z = u(x), u_W = u(x - h_W), u_O = u(x + h_O)$.
Da $u \in C^3(\overline{\Omega})$ erhalten wir durch Taylorentwicklung

$$\begin{aligned} \left| u(x + h_O) - \left(u(x) + h_O u'(x) + \frac{h_O^2}{2} u''(x) \right) \right| &\leq C|u|_{C^3(\overline{\Omega})} h_O^3, \\ \left| u(x - h_W) - \left(u(x) - h_W u'(x) + \frac{h_W^2}{2} u''(x) \right) \right| &\leq C|u|_{C^3(\overline{\Omega})} h_W^3. \end{aligned}$$

Somit gilt:

$$I = \frac{2}{h_O(h_O + h_W)} u_O - \frac{2}{h_O h_W} u_Z + \frac{2}{h_W(h_O + h_W)} u_W - u''(x)$$

$$
\begin{aligned}
&= \frac{2}{h_O(h_O+h_W)}\left[u(x+h_O)-\left(u(x)+h_O u'(x)+\frac{h_O^2}{2}u''(x)\right)\right]-\frac{2}{h_O h_W}u(x)\\
&\quad+\frac{2}{h_W(h_O+h_W)}\left[u(x-h_W)-\left(u(x)-h_W u'(x)+\frac{h_W^2}{2}u''(x)\right)\right]-u''(x)\\
&\quad+\frac{2}{h_O(h_O+h_W)}\left(u(x)+h_O u'(x)+\frac{h_O^2}{2}u''(x)\right)\\
&\quad+\frac{2}{h_W(h_O+h_W)}\left(u(x)-h_W u'(x)+\frac{h_W^2}{2}u''(x)\right)\\
&= \frac{2}{h_O(h_O+h_W)}\left[u(x+h_O)-\left(u(x)+h_O u'(x)+\frac{h_O^2}{2}u''(x)\right)\right]\\
&\quad+\frac{2}{h_W(h_O+h_W)}\left[u(x-h_W)-\left(u(x)-h_W u'(x)+\frac{h_W^2}{2}u''(x)\right)\right].
\end{aligned}
$$

Somit erhalten wir die folgende Abschätzung

$$
\begin{aligned}
|I| &\le \frac{2}{h_O(h_O+h_W)}\left|u(x+h_O)-\left(u(x)+h_O u'(x)+\frac{h_O^2}{2}u''(x)\right)\right|\\
&\quad+\frac{2}{h_W(h_O+h_W)}\left|u(x-h_W)-\left(u(x)-h_W u'(x)+\frac{h_W^2}{2}u''(x)\right)\right|\\
&\le \frac{2}{h_O(h_O+h_W)}C|u|_{C^3(\overline{\Omega})}h_O^3+\frac{2}{h_W(h_O+h_W)}C|u|_{C^3(\overline{\Omega})}h_W^3\\
&\le C|u|_{C^3(\overline{\Omega})}\frac{h_O^2+h_W^2}{h_O+h_W}\le C|u|_{C^3(\overline{\Omega})}\frac{h(h_O+h_W)}{h_O+h_W}\le C|u|_{C^3(\overline{\Omega})}h.
\end{aligned}
$$

(Die Konstante C kann sich von Zeile zu Zeile ändern)
(2.) Im Fall $n=2$ ist zu zeigen

$$
\begin{aligned}
&\left|\frac{2}{h_O(h_O+h_W)}u_O+\frac{2}{h_W(h_O+h_W)}u_W+\frac{2}{h_N(h_N+h_S)}u_N\right.\\
&\left.+\frac{2}{h_S(h_N+h_S)}u_S-\left(\frac{2}{h_O h_W}+\frac{2}{h_N h_S}\right)u_Z-\Delta u(x)\right|\le C|u|_{C^3(\overline{\Omega})}h, \qquad (2)
\end{aligned}
$$

mit $u_Z=u(x_1,x_2), u_W=u(x_1-h_W,x_2), u_O=u(x_1+h_O,x_2), u_N=u(x_1,x_2-h_N)$, und $u_S=u(x_1,x_2+h_S)$.
Da $u\in C^3(\overline{\Omega})$ erhalten wir durch Taylorentwicklung

$$
\begin{aligned}
\left|u(x_1+h_O,x_2)-\left(u(x_1,x_2)+h_O\frac{\partial}{\partial x_1}u(x_1,x_2)+\frac{h_O^2}{2}\frac{\partial^2}{\partial x_1^2}u(x_1,x_2)\right)\right| &\le C|u|_{C^3(\overline{\Omega})}h_O^3,\\
\left|u(x_1-h_W,x_2)-\left(u(x_1,x_2)-h_W\frac{\partial}{\partial x_1}u(x_1,x_2)+\frac{h_W^2}{2}\frac{\partial^2}{\partial x_1^2}u(x_1,x_2)\right)\right| &\le C|u|_{C^3(\overline{\Omega})}h_W^3,\\
\left|u(x_1,x_2-h_N)-\left(u(x_1,x_2)-h_N\frac{\partial}{\partial x_2}u(x_1,x_2)+\frac{h_N^2}{2}\frac{\partial^2}{\partial x_2^2}u(x_1,x_2)\right)\right| &\le C|u|_{C^3(\overline{\Omega})}h_N^3,\\
\left|u(x_1,x_2+h_S)-\left(u(x_1,x_2)+h_S\frac{\partial}{\partial x_2}u(x_1,x_2)+\frac{h_S^2}{2}\frac{\partial^2}{\partial x_2^2}u(x_1,x_2)\right)\right| &\le C|u|_{C^3(\overline{\Omega})}h_S^3.
\end{aligned}
$$

Somit gilt:

$$
II = \frac{2}{h_O(h_O+h_W)}u_O+\frac{2}{h_W(h_O+h_W)}u_W+\frac{2}{h_N(h_N+h_S)}u_N
$$

$$
\begin{aligned}
&+\frac{2}{h_S(h_N+h_S)}u_S - \left(\frac{2}{h_O h_W}+\frac{2}{h_N h_S}\right)u_Z - \Delta u(x) \\
=\ & \frac{2}{h_O(h_O+h_W)}\left[u(x_1+h_O,x_2) - \left(u(x_1,x_2) + h_O\frac{\partial}{\partial x_1}u(x_1,x_2) + \frac{h_O^2}{2}\frac{\partial^2}{\partial x_1^2}u(x_1,x_2)\right)\right] \\
&+\frac{2}{h_W(h_O+h_W)}\left[u(x_1-h_W,x_2) - \left(u(x_1,x_2) - h_W\frac{\partial}{\partial x_1}u(x_1,x_2) + \frac{h_W^2}{2}\frac{\partial^2}{\partial x_1^2}u(x_1,x_2)\right)\right] \\
&+\frac{2}{h_N(h_N+h_S)}\left[u(x_1,x_2-h_N) - \left(u(x_1,x_2) - h_N\frac{\partial}{\partial x_2}u(x_1,x_2) + \frac{h_N^2}{2}\frac{\partial^2}{\partial x_2^2}u(x_1,x_2)\right)\right] \\
&+\frac{2}{h_S(h_N+h_S)}\left[u(x_1,x_2+h_S) - \left(u(x_1,x_2) + h_S\frac{\partial}{\partial x_2}u(x_1,x_2) + \frac{h_S^2}{2}\frac{\partial^2}{\partial x_2^2}u(x_1,x_2)\right)\right] \\
&+\frac{2}{h_O(h_O+h_W)}\left(u(x_1,x_2) + h_O\frac{\partial}{\partial x_1}u(x_1,x_2) + \frac{h_O^2}{2}\frac{\partial^2}{\partial x_1^2}u(x_1,x_2)\right) \\
&+\frac{2}{h_W(h_O+h_W)}\left(u(x_1,x_2) - h_W\frac{\partial}{\partial x_1}u(x_1,x_2) + \frac{h_W^2}{2}\frac{\partial^2}{\partial x_1^2}u(x_1,x_2)\right) \\
&+\frac{2}{h_N(h_N+h_S)}\left(u(x_1,x_2) - h_N\frac{\partial}{\partial x_2}u(x_1,x_2) + \frac{h_N^2}{2}\frac{\partial^2}{\partial x_2^2}u(x_1,x_2)\right) \\
&+\frac{2}{h_S(h_N+h_S)}\left(u(x_1,x_2) + h_S\frac{\partial}{\partial x_2}u(x_1,x_2) + \frac{h_S^2}{2}\frac{\partial^2}{\partial x_2^2}u(x_1,x_2)\right) \\
&-\left(\frac{2}{h_O h_W}+\frac{2}{h_N h_S}\right)u(x_1,x_2) - \left(\frac{\partial^2}{\partial x_1^2}u(x_1,x_2) + \frac{\partial^2}{\partial x_2^2}u(x_1,x_2)\right) \\
=\ & \frac{2}{h_O(h_O+h_W)}\left[u(x_1+h_O,x_2) - \left(u(x_1,x_2) + h_O\frac{\partial}{\partial x_1}u(x_1,x_2) + \frac{h_O^2}{2}\frac{\partial^2}{\partial x_1^2}u(x_1,x_2)\right)\right] \\
&+\frac{2}{h_W(h_O+h_W)}\left[u(x_1-h_W,x_2) - \left(u(x_1,x_2) - h_W\frac{\partial}{\partial x_1}u(x_1,x_2) + \frac{h_W^2}{2}\frac{\partial^2}{\partial x_1^2}u(x_1,x_2)\right)\right] \\
&+\frac{2}{h_N(h_N+h_S)}\left[u(x_1,x_2-h_N) - \left(u(x_1,x_2) - h_N\frac{\partial}{\partial x_2}u(x_1,x_2) + \frac{h_N^2}{2}\frac{\partial^2}{\partial x_2^2}u(x_1,x_2)\right)\right] \\
&+\frac{2}{h_S(h_N+h_S)}\left[u(x_1,x_2+h_S) - \left(u(x_1,x_2) + h_S\frac{\partial}{\partial x_2}u(x_1,x_2) + \frac{h_S^2}{2}\frac{\partial^2}{\partial x_2^2}u(x_1,x_2)\right)\right].
\end{aligned}
$$

Somit erhalten wir die folgende Abschätzung

$$
\begin{aligned}
|II|\ \leq\ & \frac{2}{h_O(h_O+h_W)}\left|u(x_1+h_O,x_2) - \left(u(x_1,x_2) + h_O\frac{\partial}{\partial x_1}u(x_1,x_2) + \frac{h_O^2}{2}\frac{\partial^2}{\partial x_1^2}u(x_1,x_2)\right)\right| \\
&+\frac{2}{h_W(h_O+h_W)}\left|u(x_1-h_W,x_2) - \left(u(x_1,x_2) - h_W\frac{\partial}{\partial x_1}u(x_1,x_2) + \frac{h_W^2}{2}\frac{\partial^2}{\partial x_1^2}u(x_1,x_2)\right)\right| \\
&+\frac{2}{h_N(h_N+h_S)}\left|u(x_1,x_2-h_N) - \left(u(x_1,x_2) - h_N\frac{\partial}{\partial x_2}u(x_1,x_2) + \frac{h_N^2}{2}\frac{\partial^2}{\partial x_2^2}u(x_1,x_2)\right)\right| \\
&+\frac{2}{h_S(h_N+h_S)}\left|u(x_1,x_2+h_S) - \left(u(x_1,x_2) + h_S\frac{\partial}{\partial x_2}u(x_1,x_2) + \frac{h_S^2}{2}\frac{\partial^2}{\partial x_2^2}u(x_1,x_2)\right)\right| \\
\leq\ & \frac{2}{h_O(h_O+h_W)}C|u|_{C^3(\overline{\Omega})}h_O^3 + \frac{2}{h_W(h_O+h_W)}C|u|_{C^3(\overline{\Omega})}h_W^3 \\
&+\frac{2}{h_N(h_N+h_S)}C|u|_{C^3(\overline{\Omega})}h_N^3 + \frac{2}{h_S(h_N+h_S)}C|u|_{C^3(\overline{\Omega})}h_S^3 \\
\leq\ & C|u|_{C^3(\overline{\Omega})}\frac{h_O^2+h_W^2}{h_O+h_W} + C|u|_{C^3(\overline{\Omega})}\frac{h_N^2+h_S^2}{h_N+h_S} \leq C|u|_{C^3(\overline{\Omega})}\frac{h(h_O+h_W)}{h_O+h_W} + C|u|_{C^3(\overline{\Omega})}\frac{h(h_N+h_S)}{h_N+h_S} \\
\leq\ & C|u|_{C^3(\overline{\Omega})}h.
\end{aligned}
$$