PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

BLOW-UP OF SEMILINEAR PDE'S AT THE CRITICAL DIMENSION. A PROBABILISTIC APPROACH

MATTHIAS BIRKNER, JOSÉ ALFREDO LÓPEZ-MIMBELA,
AND ANTON WAKOLBINGER

Abstract. We present a probabilistic approach which proves blow-up of solutions of the Fujita equation $\partial w/\partial t = -(-\Delta)^{\alpha/2}w + w^{1+\beta}$ in the critical dimension $d = \alpha/\beta$. By using the Feynman-Kac representation twice, we construct a subsolution which locally grows to infinity as $t \to \infty$. In this way, we cover results proved earlier by analytic methods. Our method also applies to extend a blow-up result for systems proved for the Laplacian case by Escobedo and Levine [2] to the case of α-Laplacians with possibly different parameters α.

1. Introduction and overview

Consider the semilinear equation

$$
\begin{aligned}
\frac{\partial w_t}{\partial t} &= \Delta_\alpha w_t + \gamma w_t^{1+\beta}, \\
w_0 &= \varphi,
\end{aligned} \tag{1.1}
$$

in $\mathbb{R}^d$, where $\Delta_\alpha := -\left(-\Delta^{\alpha/2}\right)$, $0 < \alpha \le 2$, denotes the α-Laplacian, β and γ are positive numbers and the initial condition φ is a nonnegative function on $\mathbb{R}^d$.

In Fujita's pioneering work [4] it was shown (originally for the case $\alpha = 2$) that $d = \alpha/\beta$ is the critical dimension for blow-up of (1.1): if $d > \alpha/\beta$ then (1.1) admits a global solution for all sufficiently small initial conditions, whereas if $d < \alpha/\beta$, then for any non-vanishing initial condition the solution is infinite for suitably large t.

For the case $d = \alpha/\beta$ it was proved by Sugitani [12] by subtle analytic arguments that (1.1) blows up. Using different, partly probabilistic methods, this was also proved by Portnoy ([9, 10]) for the special case $\alpha = 2$, $\beta = 1$. Related results on systems where the space variable is restricted to a bounded domain in $\mathbb{R}^d$ can be found in the recent paper of Wang [13] and the references therein.

In this note we give a short probabilistic proof for blow-up at the critical dimension, using the Feynman-Kac representation. Here is an outline.

Recall that the solution w of the initial value problem on $[0, T) \times \mathbb{R}^d$

$$
\begin{aligned}
\frac{\partial w_t}{\partial t} &= \Delta_\alpha w_t + w_t v_t, \\
w_0 &= \varphi,
\end{aligned} \tag{1.2}
$$

2000 *Mathematics Subject Classification.* 60H30, 35K57, 35B35, 60J57.

Key words and phrases. Blow-up of semilinear systems, Feynman-Kac representation, symmetric stable processes.

with $v : [0,T) \times \mathbb{R}^d \mapsto \mathbb{R}_+$ locally bounded has by the Feynman-Kac formula (cf. Stroock [11], §4.3, Freidlin [3], Thm. 2.2, or Dynkin [1], Thm. 9.7) a probabilistic interpretation as the density (with respect to Lebesgue measure on $\mathbb{R}^d$) of the measure

$$\int \mathbb{E}_x\left[\mathbf{1}\,(W_t \in dy) \exp \int_0^t v_s(W_s)\,ds\right] \varphi(x)\,dx = w_t(y)\,dy \tag{1.3}$$

where $\mathbb{E}_x$ denotes expectation with respect to the symmetric α-stable process (W_t) started at $W_0 = x$. This shows in particular that any solution $\widetilde{w}$ of (1.2) with v replaced by $\widetilde{v} \le v$ and $\widetilde{w}_0 = w_0$ fulfills $\widetilde{w} \le w$.

Consider for $i = 0, 1, 2$ the initial value problems

$$\begin{aligned} \frac{\partial w_{t,i}}{\partial t} &= \Delta_\alpha w_{t,i} + \gamma w_{t,i} w_{t,i-1}^\beta \\ w_{0,i} &= \varphi \end{aligned} \tag{1.4}$$

where $w_{t,-1} = 0$. Then $f_t := w_{t,0}$, $g_t := w_{t,1}$ and $h_t := w_{t,2}$ are all subsolutions of (1.1). Since $f_t(y) = \mathbb{E}_y[\varphi(W_t)]$, where (W_t) is a symmetric α-stable process, $f_t(y)$ decays like const $\cdot\, t^{-d/\alpha}$ (see Section 2). Since "typically" $f_s(W_s)$ should be bounded from below by const $\cdot\, s^{-d/\alpha}$, and also $\mathbb{P}_x\{W_t \in dy\} \ge \text{const} \cdot t^{-d/\alpha}\,dy$ as long as $\|y - x\| \le t^{1/\alpha}$, one should expect (using (1.3) with $v_s = f_s^\beta$ to express the solution of (1.4) for $i = 1$) that

$$\begin{aligned} g_t(y) &= \int \mathbb{E}_x\left[\exp \int_0^t f_s(W_s)^\beta\,ds \,\middle|\, W_t = y\right] \varphi(x)\,dx \\ &\ge ct^{-d/\alpha} \exp\left(\text{const} \int_1^t s^{-d\beta/\alpha} ds\right) \\ &= ct^{-d/\alpha} \exp\left(\text{const} \cdot \log t\right) \ge ct^{-d/\alpha+\varepsilon} \end{aligned} \tag{1.5}$$

as long as $\|y\| \le t^{1/\alpha}$. This intuition can be turned into a proof basically by applying Jensen's inequality and scaling arguments.

After dealing in this way in Proposition 2.1 with the case $i = 1$, we then turn to the case $i = 2$ in (1.4). Like g_t, also $h_t = w_{t,2}$ has a Feynman-Kac representation, but now with f_s^β replaced by g_s^β in the exponent. By (1.5), the integrand $g_s(W_s)^\beta$ in this exponent should "typically" remain bounded from below by const $\cdot\, s^{-1+\varepsilon\beta}$. Thus we expect that

$$h_t(y) \ge \text{const} \cdot t^{-d/\alpha} \exp\left(-c \int_0^t s^{-1+\varepsilon\beta} ds\right),$$

and in fact we will prove this in Proposition 2.3. In particular, h_t is a subsolution of (1.1) which locally grows to infinity. This fact suffices to show blow up, as we will recall in Section 3.

Section 4 comments briefly on the case of subcritical dimensions, and Section 5 on Portnoy's method. In Section 6 we give some extensions. Apart from re-proving Sugitani's result, we show that blow-up of (1.1) with a certain *time-dependent* nonlinearity, which was recently proved by Gedda and Kirane [5], arises as an easy corollary of our probabilistic approach.

In Section 7 we obtain conditions for blow-up of a class of semilinear *systems.* We are able to extend a blow-up result of Escobedo and Levine [2] and show blow-up at the critical dimensions of a system which we were able to analyze before only in the case of sub– and supercritical dimensions [7, 8].

2. Constructing subsolutions by the Feynman-Kac formula

In this and the following section we consider $d = \alpha/\beta$ and prove that (1.1) blows up in this case. Furthermore assume without loss of generality that the initial condition φ of (1.1) does not vanish a.s. on the unit ball. Let $p_t(x)$ denote the transition density of the symmetric α-stable process, and write

$$f_t(y) := \int p_t(y-x)\varphi(x)\,dx = \mathbb{E}_y\left[\varphi(W_t)\right]. \tag{2.1}$$

For all $t \geq 1$ we have the inequality

$$f_t(y) \;\geq\; c_0 t^{-d/\alpha}\mathbf{1}_{B_1}(t^{-1/\alpha}y)\int_{B_1}\varphi(x)\,dx \tag{2.2}$$

for some $c_0 > 0$, where B_r denotes the ball in $\mathbb{R}^d$ with radius r centered at the origin. Indeed, let $y \in B_{t^{1/\alpha}}$. Then we have by the scaling property of W_t

$$\begin{aligned} f_t(y) &= \mathbb{E}_0\left[\varphi(W_t+y)\right] = \mathbb{E}_0\left[\varphi\left(t^{1/\alpha}(W_1+t^{-1/\alpha}y)\right)\right] \\ &\geq \int_{B_1} p_1(x-t^{-1/\alpha}y)\varphi(t^{1/\alpha}x)\,dx \geq c_0\int_{B_1}\varphi(t^{1/\alpha}x)\,dx = c_0 t^{-d/\alpha}\int_{B_{t^{1/\alpha}}}\varphi(x)\,dx. \end{aligned}$$

This argument also shows that, for sufficiently large t

$$f_t(y) \;\geq\; c_0' t^{-d/\alpha}\mathbf{1}_{B_1}(t^{-1/\alpha}y). \tag{2.3}$$

for some $c_0' > 0$.

2.1. The first iteration: a subsolution with a slow decay.

We are going to obtain a lower bound for the solution g_t of

$$\begin{aligned} \frac{\partial g_t}{\partial t} &= \Delta_\alpha g_t + \gamma g_t f_t^{\beta}, \\ g_0 &= \varphi, \end{aligned} \tag{2.4}$$

where f_t is defined in (2.1). Since f_t is a subsolution of (1.1), g_t is a subsolution of (1.1) as well.

Proposition 2.1. *There exist ε, $c > 0$ such that, for all $t \geq 2$ and all $y \in \mathbb{R}^d$ obeying $\|y\| \leq t^{1/\alpha}$,*

$$g_t(y) \geq\; c\, t^{-d/\alpha+\varepsilon}. \tag{2.5}$$

Proof. By the Feynman-Kac formula, g_t arises as the density of the measure defined in (1.3) (with v_s replaced by f_s^β). We therefore have, using (2.2) and Jensen's inequality,

$$\begin{aligned} g_t(y) &= \int \varphi(x)\,p_t(y-x)\,\mathbb{E}_x\!\left[\exp\int_0^t \gamma f_s(W_s)^\beta\,ds\,\middle|\,W_t=y\right]dx \\ &\geq \int \varphi(x)\,p_t(y-x)\,\mathbb{E}_x\!\left[\exp\int_1^{t/2} c_2 s^{-\beta d/\alpha}\mathbf{1}_{B_{s^{1/\alpha}}}(W_s)\,ds\,\middle|\,W_t=y\right]dx \\ &\geq \int_{B_1}\varphi(x)\,p_t(y-x)\exp\left(c_2\int_1^{t/2} s^{-\beta d/\alpha}\,\mathbb{P}_x\left\{W_s\in B_{s^{1/\alpha}}\,|\,W_t=y\right\}ds\right)dx \\ &\geq c_3 t^{-d/\alpha}\exp\left(c_4\int_1^{t/2} s^{-\beta d/\alpha}ds\right) \end{aligned} \tag{2.6}$$

where the last estimate relies on Lemma 2.2 below. (Here and below c_i, $i = 1, 2, \dots$ denote "locally defined" positive constants). The assertion now follows from our assumption $d = \alpha/\beta$. □

The intuition behind the following assertion is clear: conditioning on some "typical" state at time t does not much affect the behavior of (W_t) between times 0 and $t/2$.

Lemma 2.2. *There exists a $c > 0$ such that for all $t \geq 2$, $y \in B_{t^{1/\alpha}}$, $x \in B_1$ and $s \in [1, t/2]$,*

$$\mathbb{P}_x \left\{ W_s \in B_{s^{1/\alpha}} \,|\, W_t = y \right\} \;\geq\; c. \tag{2.7}$$

Proof. First note that (2.7) is equivalent to

$$\int_{B_{s^{1/\alpha}}} p_s(z-x) p_{t-s}(y-z)\, dz \;\geq\; c_5 p_t(y-x). \tag{2.8}$$

Next, let us state the following facts, which are easy consequences of the scaling property of (W_t):

(i) For all $z \in B_{s^{1/\alpha}}$ and $r := t - s$

$$\begin{aligned} p_r(y-z)\, dz &= \mathbb{P}_0 \left\{ r^{1/\alpha} W_1 + y \in dz \right\} \geq \inf_{a \in B_{2^{1/\alpha}}} \mathbb{P}_0 \left\{ W_1 \in 2^{1/\alpha} t^{-1/\alpha}\, dz - a \right\} \\ &\geq c_5 t^{-d/\alpha} dz. \end{aligned}$$

(ii) Similarly, for all $z \in B_{s^{1/\alpha}}$, $p_s(z-x) \geq c_6 s^{-d/\alpha}$.

Combining (i) and (ii) we see that the LHS of (2.8) is bounded from below by $c_7 t^{-d/\alpha}$. Since $p_t(\cdot)$ is bounded above by $\mathrm{const} \cdot t^{-d/\alpha}$ the claim is proved. □

2.2. **The second iteration: a subsolution growing to infinity.** We are now aiming at a lower estimate for the solution h_t of

$$\begin{aligned} \frac{\partial h_t}{\partial t} &= \Delta_\alpha h_t + h_t g_t^\beta, \\ h_0 &= \varphi \end{aligned} \tag{2.9}$$

where g_t is the subsolution of (1.1) constructed in the previous subsection. Clearly, also h_t is a subsolution of (1.1).

Proposition 2.3. $\inf \{ h_t(y) \,|\, \|y\| \leq 1 \} \;\to\; \infty$ *as $t \to \infty$, more specifically there exist constants $\varepsilon, c', c'' > 0$ such that*

$$h_t(y) \geq c' t^{-d/\alpha} \exp(c'' t^{\varepsilon\beta}) 1_{B_1}(y).$$

Proof. We proceed as in the proof of Proposition 2.1. First we note that the Feynman-Kac formula gives

$$h_t(y) = \int \varphi(x) p_t(y-x)\, \mathbb{E}_x \left[\exp \int_0^t \gamma g_s(W_s)^\beta ds \,\middle|\, W_t = y \right] dx. \tag{2.10}$$

Using Jensen's inequality and (2.5), we see that the RHS of (2.10) is bounded from below by

$$\begin{aligned}
&\int \varphi(x)p_t(y-x)\exp\left(\gamma\int_2^{t/2}\mathbb{E}_x\big[g_s(W_s)^\beta\big|\,W_t=y\big]\;ds\right)dx\\
&\geq \int_{B_1}\varphi(x)p_t(y-x)\\
&\qquad\cdot\exp\left(\gamma\int_2^{t/2}cs^{-\beta d/\alpha+\varepsilon\beta}\,\mathbb{P}_x\left\{W_s\in B_{s^{1/\alpha}}\,|\,W_t=y\right\}\;ds\right)dx \qquad (2.11)\\
&\geq c_8t^{-d/\alpha}\exp(c_9t^{\varepsilon\beta}). \qquad (2.12)
\end{aligned}$$

Here, we used Lemma 2.2 and the assumption $d=\alpha/\beta$ in the last inequality. □

3. Completion of the proof of blow-up

From Proposition 2.3 we know that

$$K(t):=\inf_{x\in B_1}w_t(x)\to\infty \text{ as } t\to\infty \qquad (3.1)$$

where B_1 denotes the unit ball. In fact this is enough to guarantee blow-up. Here is an easy argument which is borrowed from [6] §4, and which we include for convenience.

We are going to re-start (1.1) with the initial condition w_{t_0}, with a suitable choice of t_0 given below. Writing $u_t:=w_{t_0+t}$ we first recall the integral form of (1.1)

$$u_t(x)=\int p_t(y-x)u_0(y)\,dy+\int_0^t\gamma\,ds\int p_{t-s}(y-x)u_s(y)^{1+\beta}\,dy. \qquad (3.2)$$

Noting that $\zeta:=\min_{x\in B_1}\min_{0\leq s\leq 1}\mathbb{P}_x\{W_s\in B_1\}$ is strictly positive, we obtain for all $t\in[0,1]$ from (3.1) the estimate

$$\min_{x\in B_1}u_t(x)\;\geq\;\zeta K(t_0)+\gamma\zeta\int_0^t\left(\min_{y\in B_1}u_s(y)\right)^{1+\beta}ds. \qquad (3.3)$$

Now choose t_0 so big that the blow-up time of the equation

$$v(t)=\zeta K(t_0)+\gamma\zeta\int_0^t v(s)^{1+\beta}\,ds \qquad (3.4)$$

is smaller than 1. Then, *a fortiori*, $\min_{x\in B_1}u_1(x)=\infty$, which shows blow-up of w.

4. Subcritical dimensions: one iteration suffices

In the case $d<\alpha/\beta$, (2.6) shows that already the first subsolution g_t (constructed in Section 2.1) grows to infinity on the unit ball B_1 in the sense that $\inf\{g_t(y)|\ \|y\|\leq 1\}\to\infty$ as $t\to\infty$. Thus, in view of the previous section, for subcritical dimensions a single application of the Feynman-Kac formula suffices to show blow-up of (1.1).

5. A remark on Portnoy's method

Portnoy [9] studies the iteration scheme

$$\begin{aligned} v_{n+1}(x) &= (\Pi_1 v_n)(x) + (\Pi_1 v_n)^2(x) \\ v_0 &= \varphi \geq 0 \end{aligned} \tag{5.1}$$

where Π_1 is a transition probability on $\mathbb{R}^d$. He shows that under suitable assumptions on Π_1 (which include the case of a standard Brownian transition probability), (5.1) admits no bounded solution for $d = 1$ and $d = 2$ provided φ does not a.s. vanish.

A closer look on his proofs shows that he achieves this by analyzing subsolutions $v_n^{(i)}$ of (5.1) which are given by the scheme

$$\begin{aligned} v_{n+1}^{(0)} &= \Pi_1 v_n^{(0)} = \Pi_{n+1}\varphi \\ v_{n+1}^{(i)} &= \Pi_1 v_n^{(i)} + \left(\Pi_1 v_n^{(i)}\right)\left(\Pi_1 v_n^{(i-1)}\right), \quad i = 1, 2. \end{aligned} \tag{5.2}$$

The analysis of (5.2) is carried through probabilistically in terms of random walks, which is much in the spirit of a discrete time Feynman-Kac approach.

It can be extracted from Portnoy's arguments that, for the Brownian case, say,

$$v_n^{(1)} \text{ grows to infinity for } d = 1, \tag{5.3}$$

and

$$v_n^{(2)} \text{ grows to infinity for } d = 2. \tag{5.4}$$

An easy application of Jensen's inequality plus induction shows that w_n is bounded from below by v_n (where w_t is the solution of (1.1) with $\beta = 1$). Indeed,

$$\begin{aligned} w_n &= \Pi_1 w_{n-1} + \int_0^1 \Pi_s w_{n-s}^2 \, ds \geq \Pi_1 w_{n-1} + \left(\int_0^1 \Pi_s w_{n-s} \, ds \right)^2 \\ &\geq \Pi_1 w_{n-1} + \left(\int_0^1 \Pi_s \Pi_{1-s} w_{n-1} \, ds \right)^2 \geq \Pi_1 v_{n-1} + (\Pi_1 v_{n-1})^2 = v_n. \end{aligned}$$

Together with the argument in Section 3 above, (5.3) and (5.4) thus imply blow-up of w for $\beta = 1$ and $\alpha = 2$ in one and two dimensions. (In [10], a more complicated argument is used to show $w_n \geq v_n$ and the blow-up of w.)

6. Extensions

6.1. **Sugitani's condition.** Sugitani [12] considers instead of (1.1) the slightly more general equation

$$\begin{aligned} \frac{\partial w_t}{\partial t} &= \Delta_\alpha w_t + F(w_t) \\ w_0 &= \varphi, \end{aligned} \tag{6.1}$$

where $F : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and convex, and $F(u) \sim \gamma u^{1+\beta}$ as $u \to 0$. This requires only slight modifications in Section 2:

In (2.4) and below, $f_t(u)^\beta$ has to be replaced by $F(f_t(u))/f_t(u)$, which by assumption can be bounded from below by $c f_t(u)^\beta$.

Similarly, in (2.9) and below, $g_t(u)^\beta$ has to be replaced by $F(g_t(u))/g_t(u)$.

6.2. **A time dependent nonlinearity.** Recently, Guedda and Kirane [5] showed by analytic methods blow-up of the equation

$$\frac{\partial w_t}{\partial t} = \Delta_\alpha w_t + \gamma t^\sigma w_t^{1+\beta}, \quad w_0 = \varphi \ (\geq 0, \not\equiv 0) \tag{6.2}$$

for $\sigma \geq \beta d/\alpha - 1$. This result also follows quickly from our probabilistic approach. In fact, it suffices to consider the case $\sigma = \beta d/\alpha - 1$.

Lemma 6.1. *The solution of*

$$\begin{aligned} \frac{\partial w_t}{\partial t} &= \Delta_\alpha w_t + v_t w_t^{1+\beta}, \\ w_0 &= \varphi \ (\geq 0, \not\equiv 0) \end{aligned} \tag{6.3}$$

with $v : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}_+$, $v_t(x) \geq const \cdot t^{\beta d/\alpha - 1} 1_{B_1}(t^{-1/\alpha}x)$ *for* $t \geq 1$ *blows up in finite time.*

We briefly indicate the changes required in the arguments presented in sections 2 and 3 in order to prove Lemma 6.1.

1. Concerning the subsolution g_t, all what happens is that a factor $s^\sigma \mathbf{1}_{B_{s^{1/\alpha}}}(\cdot)$ enters into the exponentials in the Feynman-Kac representation in the RHS of (2.6). Since $s^{-\beta d/\alpha}$ in the RHS of (2.6) cancels against s^σ, the lower bound (2.6) remains unchanged, and so does the estimate (2.5).

2. Concerning the subsolution h_t, again a factor s^σ enters into the exponentials in (2.10) and (2.11). Since again $(s^{-d/\alpha})^\beta$ cancels against s^σ, the lower bound (2.12) remains unchanged, and so does the assertion in Proposition 2.3.

3. Concerning the argument in Section 3, from the space-time-inhomogeneity in (6.3) a factor $(t_0 + t)^\sigma$ enters in front of the integral in (3.3) (Observe that by our assumption $v_t \geq \text{const} \cdot t^\sigma$ uniformly on B_1 for $t \geq 1$). Still, since (2.12) guarantees a super-algebraic growth of $K(t)$, we can choose t_0 so big that the blow-up time of the equation

$$v(t) = \zeta K(t_0) + \gamma\zeta(t_0 + 1)^\sigma \int_0^t v(s)^{1+\beta} ds$$

is smaller than 1, so that the argument of Section 3 remains valid.

7. Blow-up of systems

In this section we apply our probabilistic approach to extend a blow-up result of Escobedo and Levine [2] (Theorem 7.1 and Remark 7.2). In Theorem 7.3 we show that a system which we investigated in [8] in high dimensions blows up at the critical dimension.

Theorem 7.1. *Assume that* (u, v) *solves*

$$\begin{aligned} \frac{\partial u_t}{\partial t} &= \Delta_{\alpha_1} u_t + u_t^{1+\beta_1} v_t^{\beta_2} \\ \frac{\partial v_t}{\partial t} &= \Delta_{\alpha_2} v_t + F(u_t, v_t) \\ & u_0 = \varphi_1, \ v_0 = \varphi_2, \end{aligned} \tag{7.1}$$

where $\alpha_1, \alpha_2 \in (0, 2]$, $\beta_1 > 0$, $\beta_2 \geq 0$, $F \geq 0$, $\varphi_1 \geq 0$, $\varphi_2 \geq 0$ *and both* φ_1 *and* φ_2 *do not a.s. vanish. Then* u *blows up if*

$$\alpha_2 \leq \alpha_1 \ \text{and} \ d \leq \left(\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2}\right)^{-1}. \tag{7.2}$$

Remark 7.2. For $\alpha_1 = \alpha_2 =: \alpha$, (7.2) turns into the condition $d \le \alpha/(\beta_1+\beta_2)$, which is also the condition for blow-up of the partial differential equation

$$\frac{\partial u}{\partial t} = \Delta_\alpha u + u^{1+\beta_1+\beta_2}.$$

For $\alpha = 2$, this specializes to one of the main results in Escobedo and Levine's paper [2]. They investigate by analytic tools the system

$$\frac{\partial u}{\partial t} = \Delta u + u^{1+\beta_1} v^{\beta_2}, \quad \frac{\partial v}{\partial t} = \Delta v + u^{\theta_1} v^{\theta_2}$$

and prove blow-up under the condition $d \le 2/(\beta_1+\beta_2)$.

Proof of Theorem 7.1. Let $f_{t,j}(y) := \int \varphi_j(x) p_{t,j}(y-x)\,dx,\ j = 1,2,$ where $p_{t,j}$ denotes the symmetric α_j-stable transition density. Obviously, $(f_{t,1}, f_{t,2})$ is a subsolution of (7.1), and from (2.2) we have for $t \ge 1$

$$f_{t,1}(y) \ \ge\ Ct^{-d/\alpha_1}\mathbf{1}_{B_1}(t^{-1/\alpha_1}y) \tag{7.3}$$

and

$$f_{t,2}(y) \ \ge\ Ct^{-d/\alpha_2}\mathbf{1}_{B_1}(t^{-1/\alpha_1}y), \tag{7.4}$$

where we used the assumption $\alpha_2 \le \alpha_1$ to obtain (7.4). Consequently for $t \ge 1$ and $||y|| \le t^{1/\alpha_1}$

$$v_t(y)^{\beta_2} \ge C' t^{-d\beta_2/\alpha_2} \ge C' t^{d\beta_1/\alpha_1 - 1}$$

where we used the assumption (7.2) in the last inequality. Now we infer blow-up of u using Lemma 6.1. □

Theorem 7.3. *Assume that (u,v) solves*

$$\begin{aligned} \frac{\partial u_t}{\partial t} &= \Delta_{\alpha_1} u_t + u_t v_t \\ \frac{\partial v_t}{\partial t} &= \Delta_{\alpha_2} v_t + u_t v_t \\ & \quad u_0 = \varphi_1,\ v_0 = \varphi_2, \end{aligned} \tag{7.5}$$

where $\alpha_1, \alpha_2 \in (0,2]$, $\varphi_1 \ge 0$, $\varphi_2 \ge 0$ and both φ_1 and φ_2 do not a.s. vanish. Then (u,v) blows up if $d \le \min(\alpha_1,\alpha_2)$.

Remark 7.4. It was shown in [8] that (7.5) admits global solutions if $d > \min(\alpha_1,\alpha_2)$ and φ_1 and φ_2 are sufficiently small.

Before proving Theorem 7.3, we prepare with a lemma which is an easy generalization of Lemma 2.2. Here and below, $(W_t^{(i)})$ denotes the symmetric stable process with index α_i and $p_{t,i}(x)$ its transition density, $i = 1,2$.

Lemma 7.5. *Assume that $\alpha := \alpha_2 \le \alpha_1$. There exists a $c > 0$ such that for all $t \ge 2$, $y \in B_{t^{1/\alpha}}$, $x \in B_1$ and $s \in [1, t/2]$,*

$$\mathbb{P}_x\left\{ W_s^{(2)} \in B_{s^{1/\alpha_1}} \,\middle|\, W_t^{(2)} = y \right\} \ \ge\ cs^{d/\alpha_1 - d/\alpha_2}$$

Proof. It suffices to show (2.8) with $cs^{d/\alpha_1 - d/\alpha_2}$ instead of c_5 and $p_{t,2}$ instead of p_t.

Again we have (i) and (ii) from the proof of Lemma 2.2, now with $(W_t^{(2)})$ instead of (W_t). Integrating the bound s^{-d/α_2} over $B_{s^{1/\alpha_1}}$ then gives the factor const $\cdot$ $s^{d/\alpha_1 - d/\alpha_2}$. □

Proof of Theorem 7.3. The proof proceeds in three steps. First we prove using the Feynman-Kac representation (see (1.3)) that (at least one component of) the solution (u, v) locally grows to ∞. In a second step we show that (u, v) can be bounded below uniformly in $B_1 \times B_1$ similarly as in Section 3 but this time by comparison with the solution of a suitable coupled pair of ODEs. Finally, in step 3 we show that this system of ODEs blows up.

1. From (2.3) we have

$$u_t \;\geq\; c_1 t^{-d/\alpha_1} \mathbf{1}_{B_{t^{1/\alpha_1}}} \tag{7.6}$$

and

$$v_t \;\geq\; c_2 t^{-d/\alpha_2} \mathbf{1}_{B_{t^{1/\alpha_2}}} \tag{7.7}$$

for all $t \geq t_0$ for some sufficiently large t_0. Let us now assume without loss of generality that $\alpha_2 \leq \alpha_1$. By the Feynman-Kac formula we have

$$u_t(y) = \int \varphi_1(x) p_{t,1}(y-x)\, \mathbb{E}_x \Big[\exp \int_0^t v_s(W_s^{(1)})\, ds \Big| W_t^{(1)} = y \Big] \; dx.$$

For $t \geq 2t_0$, by Jensen's inequality and (7.7), this can be bounded from below by

$$\int \varphi_1(x) p_{t,1}(y-x) \exp \left(\int_{t_0}^{t/2} c_2 s^{-d/\alpha_2}\, \mathbb{P}_x \left\{ W_s^{(1)} \in B_{s^{1/\alpha_2}} \Big| W_t^{(1)} = y \right\} ds \right) \; dx.$$

Noting that $B_{s^{1/\alpha_2}} \supseteq B_{s^{1/\alpha_1}}$ and using Lemma 2.2, we thus arrive at the lower bound

$$c_3 t^{-d/\alpha_1} \exp \left(c_4 \int_{t_0}^{t/2} s^{-d/\alpha_2} ds \right). \tag{7.8}$$

If $d < \alpha_2$, then this lower bound grows super-algebraically from which we will infer blow-up in steps 2 and 3.

Let us now assume $d = \alpha_2$. Then (7.8) turns into the lower bound

$$u_t(y) \;\geq\; c_5 t^{-d/\alpha_1 + \varepsilon} \tag{7.9}$$

(uniformly in $y \in B_{t^{1/\alpha_1}}$ for t sufficiently large). Another application of the Feynman-Kac formula gives

$$v_t(y) = \int \varphi_2(x) p_{t,2}(y-x)\, \mathbb{E}_x \Big[\exp \int_0^t u_s(W_s^{(2)})\, ds \Big| W_t^{(2)} = y \Big] \; dx. \tag{7.10}$$

Using Jensen's inequality and (7.9), we can bound this from below by

$$\int \varphi_2(x) p_{t,2}(y-x) \exp \int_{t_0}^{t/2} c_1 s^{-d/\alpha_1 + \varepsilon}\, \mathbb{P}_x \left\{ W_s^{(2)} \in B_{s^{1/\alpha_1}} \Big| W_t^{(2)} = y \right\} ds \; dx.$$

In view of Lemma 7.5 we thus obtain as a lower bound for $v_t(y)$ (as long as t is sufficiently large and $y \in B_{t^{1/\alpha_2}}$):

$$\begin{aligned} c_6 t^{-d/\alpha_2} \exp \int_{t_0}^{t/2} c_7 s^{-d/\alpha_1 + \varepsilon} s^{d/\alpha_1 - d/\alpha_2} ds &= c_6 t^{-d/\alpha_2} \exp \int_{t_0}^{t/2} c_7 s^{-d/\alpha_2 + \varepsilon} ds \\ &= c_6 t^{-d/\alpha_2} \exp\left(c_8 t^{\varepsilon}\right). \end{aligned}$$

Thus in this case v grows (super-algebraically).

2. Rewriting (7.5) in integral form we obtain for $t, t_0 \geq 0$

$$
\begin{aligned}
u_{t+t_0}(x) &= \int dy\, p_{t,1}(y-x) u_{t_0}(y) + \int_0^t ds \int dy\, p_{t-s,1}(y-x) u_{t_0+s}(y) v_{t_0+s}(y) \\
v_{t+t_0}(x) &= \int dy\, p_{t,2}(y-x) v_{t_0}(y) + \int_0^t ds \int dy\, p_{t-s,2}(y-x) u_{t_0+s}(y) v_{t_0+s}(y).
\end{aligned}
$$

Let $\zeta := \min_{x\in B_1} \min_{0\leq s\leq 1} \Big(\mathbb{P}_x(W_s^{(1)} \in B_1) \wedge \mathbb{P}_x(W_s^{(2)} \in B_1)\Big) > 0$ and $\widetilde{u}(t) := \min_{x\in B_1} u_t(x)$, $\widetilde{v}(t) := \min_{x\in B_1} v_t(x)$. This allows us to estimate for $t \in [0,1]$

$$
\begin{aligned}
&\widetilde{u}(t_0+t) \geq \zeta \widetilde{u}(t_0) + \zeta \int_0^t ds\, \widetilde{u}(t_0+s)\widetilde{v}(t_0+s), \\
&\widetilde{v}(t_0+t) \geq \zeta \widetilde{v}(t_0) + \zeta \int_0^t ds\, \widetilde{u}(t_0+s)\widetilde{v}(t_0+s).
\end{aligned}
\tag{7.11}
$$

In step 1 we saw that $(\widetilde{u} \vee \widetilde{v})(t_0) \to \infty$ super-algebraically while $(\widetilde{u} \wedge \widetilde{v})(t_0)$ decays at most algebraically. Thus, t_0 can be chosen so big that the blow-up time of

$$
U(t) = \zeta \widetilde{u}(t_0) + \zeta \int_0^t ds\, U(s)V(s), \quad V(t) = \zeta \widetilde{v}(t_0) + \zeta \int_0^t ds\, U(s)V(s)
\tag{7.12}
$$

is less than 1 (see step 3). We conclude that (u, v) blows up.

3. It remains to study (7.12) which in ODE form is

$$U'(t) = \zeta U(t)V(t) = V'(t)$$

and WLOG assume that $U_0 := U(0) \geq V(0) =: V_0$. The solution is given by

$$
\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} = \begin{cases} \dfrac{U_0 - V_0}{1 - (V_0/U_0)\exp(\zeta(U_0 - V_0)t)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ V_0 - U_0 \end{pmatrix} & \text{if } U_0 > V_0 \\[2ex] \dfrac{1}{1/U_0 - \zeta t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } U_0 = V_0 \end{cases}
$$

for $0 \leq t < \tau$ with explosion time

$$
\tau = \begin{cases} \dfrac{\log U_0 - \log V_0}{\zeta(U_0 - V_0)} & \text{if } U_0 > V_0 \\[2ex] \dfrac{1}{\zeta U_0} & \text{if } U_0 = V_0. \end{cases}
$$

In our scenario we have $U_0 \geq \exp(\varepsilon_1 t_0)$, $V_0 \geq t_0^{-\varepsilon_2}$ for some $\varepsilon_1, \varepsilon_2 > 0$, which allows to chose t_0 big enough to enforce $\tau < 1$. Indeed if $V_0 \geq U_0/2$ we have $\tau \leq 2/(\zeta U_0)$, and if $1 \leq V_0 < U_0/2$ we can estimate $\tau \leq (2\log U_0)/(\zeta U_0)$. Finally, if $V_0 < 1$ we have $\tau \leq (\log U_0)/(\zeta(U_0 - 1)) + \varepsilon_2 \log t_0/(\zeta(\exp(\varepsilon_1 t_0) - 1))$. □

Remark 7.6. Consider instead of (7.5) the more general system

$$
\begin{aligned}
\frac{\partial u_t}{\partial t} &= \Delta_{\alpha_1} u_t + u_t v_t^{\beta_1} \\
\frac{\partial v_t}{\partial t} &= \Delta_{\alpha_2} v_t + u_t^{\beta_2} v_t \\
& u_0 = \varphi_1, \ v_0 = \varphi_2,
\end{aligned}
\tag{7.13}
$$

where α_1, α_2, φ_1, φ_2 are as in Theorem 7.3, and $\beta_1, \beta_2 > 0$. Assume that $\alpha_2 \leq \alpha_1$. Proceeding as in the proof of Theorem 7.3 but using the simple bound (7.6) instead of (7.9) in the Feynman-Kac representation corresponding to (7.10) one obtains quickly that (7.13) has a growing subsolution if

$$d < \max\left(\frac{\alpha_2}{\beta_1}, \left(\frac{\beta_2 - 1}{\alpha_1} + \frac{1}{\alpha_2}\right)^{-1}\right). \tag{7.14}$$

As before, from this one infers blow-up, this time by comparing with the ODE system $U'(t) = U(t)V^{\beta_1}(t)$, $V'(t) = V(t)U^{\beta_2}(t)$.

It remains an interesting question whether the RHS of (7.14) is the critical dimension for blow-up of (7.13) and whether there is blow-up at the critical dimension. We conjecture that this is the case at least for $\alpha_1 = \alpha_2 =: \alpha$, in which case the RHS of (7.14) turns into $\alpha/\min(\beta_1, \beta_2)$. Indeed, for the special case $\alpha = 2$, this was proved by Escobedo and Levine [2].

Acknowledgment We are grateful to a referee whose remarks helped to improve the presentation of the paper. The second named author appreciates the kind hospitality of Frankfurt University during his visit in summer 2000, and thanks CONACyT (Mexico) and DAAD (Germany) for partial support.

References

[1] Dynkin, E. B. (1965). Markov processes, vol. 1. Springer Verlag, Berlin.

[2] Escobedo, M. and Levine, H. (1995). Critical blowup and global existence numbers for a weakly coupled system of reaction-diffusion equations. *Arch. Rational Mech. Anal.* **129,** 47-100.

[3] Freidlin, M. (1985). Functional integration and partial differential equations. Princeton University Press.

[4] Fujita, H. (1966). On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I* **13,** 109-124.

[5] Guedda, M., and Kirane, M. (1999). A note on nonexistence of global solutions to a nonlinear integral equation. *Bull. Belg. Math. Soc. Simon Stevin* **6,** 491-497.

[6] Kobayashi, K., Sirao, T. and Tanaka, H. (1977). On the growing up problem for semilinear heat equations. *J. Math. Soc. Japan* **29,** 407-424.

[7] López-Mimbela, J.A. and Wakolbinger, A. (1998). Length of Galton-Watson trees and blow-up of semilinear systems. *J. Appl. Prob.* **35,** 802-811.

[8] López-Mimbela, J.A. and Wakolbinger, A. (2000). A probabilistic proof of non-explosion of a non-linear PDE system. *J. Appl. Prob.* **37,** 635-641.

[9] Portnoy, S. (1975). Transience and solvability of a non-linear diffusion equation. *Ann. Probab.* **3,** 465-477.

[10] Portnoy, S. (1976). On solutions to $u_t = \Delta u + u^2$ in two dimensions. *J. Math. Anal. Appl.* **55,** 291-294.

[11] Stroock, D. (1993). Probability theory, an analytic view. Cambridge University Press.

[12] Sugitani, S. (1975). On nonexistence of global solutions for some nonlinear integral equations. *Osaka J. Math.* **12,** 45-51.

[13] Wang, L. (2000). The blow-up for weakly coupled reaction-diffusion systems. *Proc. Amer. Math. Soc.*, electronically published.

FB Mathematik, J.W. Goethe Universität, D-60054 Frankfurt am Main, Germany.
E-mail address: birkner@math.uni-frankfurt.de

Centro de Investigación en Matemáticas, Apartado Postal 402, Guanajuato 36000, Mexico.
E-mail address: jalfredo@cimat.mx

FB Mathematik, J.W. Goethe Universität, D-60054 Frankfurt am Main, Germany.
E-mail address: wakolbinger@math.uni-frankfurt.de