Theory Probab. Appl. Translated from Russian Journal
Vol. 47, No. 3

BRANCHING SYSTEMS WITH LONG-LIVING PARTICLES AT THE CRITICAL DIMENSION*

K. FLEISCHMANN†, V. A. VATUTIN‡, and A. WAKOLBINGER§

(*Translated by V. A. Vatutin*)

Abstract. A spatial branching process is considered in which particles have a lifetime law with a tail index smaller than one. It is shown that at the critical dimension, unlike classical branching particle systems the population does not suffer local extinction when started from a spatially homogeneous Poissonian initial population. In fact, persistent convergence to a mixed Poissonian particle system is shown. The random intensity of the limiting process is characterized in law by the random density in a space point of a related age-dependent superprocess at a fixed time. The proof relies on a refined study of the system starting from asymptotically large but finite initial populations.

Key words. branching particle system, critical dimension, limit theorem, long-living particles, absolute continuity, random density, superprocess, persistence, mixed Poissonian particle system, residual lifetime process, stable subordinator

DOI. 10.1137/S0040585X97979809

1. Introduction and statement of results.

1.1. Motivation and purpose. The study of spatially homogeneous (critical) branching particle systems $Z = \{Z_t\colon t \geqq 0\}$ in $\mathbf{R}^d$ with long-living particles was initiated in [24] and [26] (see also [20]). Here "*long-living*" means that the lifetime distribution of a particle has a tail of index $\gamma \in (0,1)$, implying that the mean lifetime is infinite. A new phenomenon had been revealed in the mentioned papers: starting from a homogeneous Poissonian system of particles, in *supercritical* dimensions the persistent limit in law is homogeneous *Poissonian* again, opposed to the usual non-Poissonian limits of systems of particles with finite expected lifetimes. (See also Proposition 1(b) below.) This has the following intuitive reason: due to the long lifetimes, all the siblings of a particle in the limit population were born so long ago that they moved out of the finite window of observation. Therefore, only "completely mixed" populations can be observed in the limit.

Let us manifest this by some *heuristic calculation* based on backward tree considerations. Assume the particles' motion index is $\alpha \in (0,2]$, and the index of the branching is $1+\beta$ with $\beta \in (0,1]$ (see Hypothesis 1 below). By renewal theory (see [9, sections 11.5 and 11.3]), the number of branching points along an ancestral line of a particle, called "ego," picked at time t "at random" is asymptotically in law (as $t \uparrow \infty$) of the form κt^{γ} for some random variable $\kappa > 0$, and the time between t and *any* of these earlier branching time points is of order t (as opposed to the case of exponentially distributed branching times). Consider such a branching

*Received by the editors January 30, 2002. This work was supported by the DFG; the second author was also supported by grants RFBR 00-15-96136, 99-01-00012, and INTAS 99-01317.
http://www.siam.org/journals/tvp/47-3/97980.html

†Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, D-10117 Berlin, Germany (fleischmann@wias-berlin.de, http://www.wias-berlin.de/~fleischm).
‡Steklov Mathematical Institute RAN, Gubkin St. 8, 119991 Moscow, Russia (vatutin@mi.ras.ru).
§Fachbereich Mathematik (12), Johann Wolfgang Goethe–Universität, D-60054 Frankfurt am Main, Germany (wakolbin@math.uni-frankfurt.de).

time point. The number $\daleth$, say, of particles additionally generated at this branching point (that is, being no direct descendants of the "ego") has generating function[1] $\Phi(s) = 1 - c\,(1-s)^\beta$ (differentiate the critical offspring generating function f in (3) below to get this Palm offspring generating function Φ). Consider *any* of these $\daleth$ offspring. Let q denote the probability that one of its descendants at time t hits a (fixed) ball B around "ego." The arguments in [26, Lemmas 2 and 3] yield that q decays at least with the order $t^{-\theta}$, where $\theta := (d/\alpha+\gamma)/(1+\beta)$. Therefore, the probability that none of the $\daleth$ particles has an offspring that populates B at time t is asymptotically $\Phi(1-q) = 1 - c\,q^\beta \geqq 1 - c\,t^{-(d/\alpha+\gamma)\beta/(1+\beta)}$. Consequently, the probability that none of the relatives of "ego" populates a ball around "ego" asymptotically equals

$$\begin{aligned}\mathbf{E}\big(1 - ct^{(-d/\alpha-\gamma)\beta/(1+\beta)}\big)^{\kappa t^\gamma} &\approx \mathbf{E}\exp\big[-\ c\kappa t^{\gamma-(d/\alpha+\gamma)\beta/(1+\beta)}\big]\\ &= \mathbf{E}\exp\big[-c\kappa t^{(\gamma-d\beta/\alpha)/(1+\beta)}\big].\end{aligned} \tag{1}$$

This indicates that the number $d_{\mathrm{c}} := \alpha\gamma/\beta$ should be critical for the dimension d of space. Indeed, for $d > d_{\mathrm{c}}$, the expression in (1) converges to 1, showing that in supercritical dimensions "ego" asymptotically has no relatives around. In other words, relationships of particles in our observation window vanish as $t \uparrow \infty$, which corresponds to a convergence of the original populations towards a Poissonian system (with deterministic intensity).

If $d < d_{\mathrm{c}}$ instead, then (1) converges to 0. In other words, provided that $t^{-\theta}$ is the correct order of decay of q, then for $d < d_{\mathrm{c}}$ each small ball around "ego" is met by a relative of "ego" with asymptotic probability one. This can be taken as an indication of the usual clumping of the original particle system in subcritical dimensions and is in fact in line with the local extinction of the branching particle system for $d < d_{\mathrm{c}}$, which was proved in [26] (see also Proposition 1(a) below).

At the critical dimension d_{c}, however, (1) suggests that at late times t the probability that relatives of "ego" show up in B might be strictly between 0 and 1. Since the time distance to the latest common ancestor of all these relatives is of order t, the spatial correlations of these relatives should vanish as $t \uparrow \infty$. In fact the population should tend to a homogeneous mixed Poissonian system, where the randomness of the intensity comes from the randomness of the branching at very early times.

In this paper we prove that such a picture is indeed true and hereby essentially strengthen a result of [26], where it was shown that at the critical dimension there exist nontrivial limit points as $t \uparrow \infty$. The problem of long-term convergence, however, was left open in [26]. The *main motivation* for the present work was to settle this problem. Indeed we now prove uniqueness of the limit distribution and identify it. According to our *main result* the following convergence statement holds for the process $Z = \{Z_t\colon\ t \geqq 0\}$ under consideration. At the critical dimension, Z_t converges in law as $t \uparrow \infty$ to a limit Z_∞, which is in fact a *mixed homogeneous Poissonian* particle system whose random intensity i_∞ is nondegenerate and of full expectation (*persistence*). In particular, the limit is *not* (*spatially*) *ergodic*. (See Theorem 2 below.)

To gain the needed information on the random intensity i_∞, we need to relate our branching particle system Z with a superprocess. This first of all requires passing to a Markovian description. To get this, we take into account the *residual lifetimes* of the particles. Then the refined process $\widehat{Z}$ has states in the set of counting measures on $\mathbf{R}_+ \times \mathbf{R}^d$, instead of only on $\mathbf{R}^d$. A rescaling $\widehat{Z}^{(n)}$ of $\widehat{Z}$ then leads to a *superprocess* $\widehat{X}$ on $\mathbf{R}_+ \times \mathbf{R}^d$. The scaling here is as follows: time is sped up by

[1] With c we always denote a positive constant which may vary from place to place.

the factor $n^{\beta/\gamma}$, whereas mass is renormalized by $1/n$, which also requires increasing the original particles' initial state by a factor of n (high density scaling). Now one has indeed the convergence in law $\widehat{Z}^{(n)} \to \widehat{X}$ as $n \uparrow \infty$ (see Proposition 2 below). Going back to the marginal measures on $\mathbf{R}^d$, the superprocess $\widehat{X}$ is turned into a measure-valued process X. This process appeared in [20] under the name "projected superprocess"; one should note, however, that X is not Markov, and hence is not a superprocess in the usual terminology. Adapting a criterion of [19] to our needs, we will show that the process X has *absolutely continuous states* $X_t(\mathrm{d}a) = X_t(a)\,\mathrm{d}a$ in all dimensions $d < \alpha/\beta$, covering the critical dimension d_c (see Proposition 3 below). This corresponds to the intuition that the long lifetimes of the infinitesimal particles combined with the spatial spread have an additional smoothing effect on the mass distribution (compared with the case of lifetimes with finite mean). At the same time, at the critical dimension d_c, starting from a marked Lebesgue initial state, $\widehat{X}$ has a *self-similarity* property (Proposition 4(b) below). For $d = d_\mathrm{c}$, this turns the absolute continuity of the states of X into the existence of a large time limit X_∞ which is a random multiple of the Lebesgue measure $X_\infty = i_\infty \ell$, and where, by self-similarity, i_∞ coincides in law with $X_1(0)$, the random density of X_1 at the origin (see Corollary 2(b) below). Finally, it is well known that in the case of exponentially distributed lifetimes the branching particle systems starting from Poisson initial states arise from the corresponding superprocesses via a Poissonization (cf. [14, p. 277]): the branching particle state at time t coincides in law with a mixed Poisson system, where the "Poisson's random intensity measure" is given by the superprocess state at time t. In our non-Markovian case of long-living particles, an *asymptotic local Poissonization* result still holds, starting from a large pile of particles at a distant position (Corollary 3 below): the limit is mixed Poisson with random intensity measure $X_\infty = i_\infty \ell$. Some efforts are needed to make this key idea work.

It turns out that at the critical dimension the branching systems Z with long-living particles and their continuum versions X share crucial features with some *catalytic* branching models; see [13] and [11]. (For recent surveys on catalytic models, see [5], [6], [18].) These features are absolutely continuous states and self-similarity in the continuum versions leading to uniform limits $X_\infty = i_\infty \ell$ with nondegenerate random intensity $i_\infty = X_1(0)$. These continuum model results are then used to handle the more complicated particle models by showing that a *sparse branching* as $t \uparrow \infty$ (caused in one case by the asymptotic rareness of collisions with the catalyst and in the other case by the particles' long lifetimes) leads to an asymptotic local Poissonization, that is, to mixed Poissonian limits with random intensity measure $X_\infty = i_\infty \ell$.

Let us now introduce the mentioned (refined) branching particle system $\widehat{Z}$ in more detail.

1.2. The $(d, \alpha, \beta, \gamma)$-branching particle system $\widehat{Z}$. We are dealing with a model of (critical) branching in $\mathbf{R}^d$ which is a spatial generalization of the so-called age-dependent (critical) branching process or, more precisely, (critical) Bellman–Harris branching process. It is based on the following ingredients which, for convenience, we expose as a hypothesis.

HYPOTHESIS 1 (ingredients of the branching particle system).

(a) (Particle motion process ξ.) Fix a constant $\alpha \in (0, 2]$. Consider the symmetric α-*stable* process $(\xi, P_a, a \in \mathbf{R}^d)$ in $\mathbf{R}^d$, (cf. [2, p. 317] or [1, Chap. VIII]), that is, the (time-homogeneous) Markov process with generator $\Delta_\alpha := -(-\Delta)^{\alpha/2}$, the fractional Laplacian [27, p. 260]), and with càdlàg paths. Denote by $\mathrm{p} = \{\mathrm{p}_t(b)\colon t > 0, b \in \mathbf{R}^d\}$ the continuous transition densities of this *particle motion process* ξ.

(b) (Particles' lifetime τ.) Introduce the nonlattice distribution function G of a random variable $\tau > 0$ with tail

$$P\{\tau > u\} = 1 - G(u) \sim c_G\, u^{-\gamma} \qquad \text{as} \quad u \uparrow \infty \tag{2}$$

for some index $\gamma \in (0,1)$ and a constant $c_G > 0$. That is, the *lifetime distribution* G of particles is in the normal domain of attraction of a stable law of index γ. In particular, τ has infinite expectation $E\tau = \infty$.

(c) (Critical branching mechanism.) We fix our attention on the *offspring generating function*

$$f(s) := Es^{\zeta} = s + c_f(1-s)^{1+\beta}, \qquad 0 \leqq s \leqq 1, \tag{3}$$

of the random number ζ of offspring of a particle, with constants $\beta \in (0,1]$ and $c_f \in (0, 1/(1+\beta)]$. That is,

$$P\{\zeta = k\} = \delta_{1,k} + c_f\,(-1)^k \binom{1+\beta}{k}, \qquad k \geqq 0,$$

with $\delta_{1,k}$ the Kronecker symbol. Consequently, $E\zeta = 1$ (*criticality*), and we consider a branching mechanism in the normal domain attraction of a stable law of index $1+\beta$. Note that $E\zeta^2 < \infty$ if and only if $\beta = 1$.

(d) (Phase space E.) A point $e = (u,a) \in \mathsf{E} := \mathbf{R}_+ \times \mathbf{R}^d$ describes the *residual lifetime* u and the *position* a of a particle. With the metric

$$\mathrm{d}_{\mathsf{E}}(e_1, e_2) := 1 \wedge |u_1 - u_2| + |a_1 - a_2|,$$

$e_i = (u_i,\, a_i) \in \mathsf{E}$, $i = 1,2$, we get a Polish space $(\mathsf{E}, \mathrm{d}_{\mathsf{E}})$.

(e) (Test functions.) Fix a constant $p \in (d, d+\alpha]$ (recall that α is the motion index), and introduce the reference function $\Phi_p(a) := (1+|a|^2)^{-p/2}$, $a \in \mathbf{R}^d$. Let $\widehat{\mathcal{C}}_p = \widehat{\mathcal{C}}_p(\mathsf{E})$ denote the set of all continuous functions $\psi\colon\ \mathsf{E} \to \mathbf{R}$ such that

$$\|\psi\| := \sup_{(u,a)\in\mathsf{E}} \left|\frac{\psi(u,a)}{\phi_p(a)}\right| < \infty,$$

and such that the map

$$(u,a) \;\mapsto\; \frac{\psi(u,a)}{\phi_p(a)}$$

on E can be continuously extended to a function on $\dot{\mathbf{R}}_+ \times \dot{\mathbf{R}}^d$, where $\dot{\mathbf{R}}_+$ and $\dot{\mathbf{R}}^d$ are the one-point compactifications of $\mathbf{R}_+$ and $\mathbf{R}^d$, respectively. Then $(\widehat{\mathcal{C}}_p, \|\cdot\|)$ is a separable Banach space.

(f) (State space $\mathcal{N}_p^{\circ}$.) Let $\mathcal{M}_p = \mathcal{M}_p(\mathsf{E})$ denote the set of all *p-tempered measures* on $\mathsf{E} = \mathbf{R}_+ \times \mathbf{R}^d$, that is, measures μ on E such that the integral

$$\langle \mu, \phi_p\rangle := \int_{\mathsf{E}} \mu\bigl(\mathrm{d}(u,a)\bigr)\, \phi_p(a)$$

is finite. Introduce the weakest topology in $\mathcal{M}_p$ such that for each $\psi \in \widehat{\mathcal{C}}_p$ the mapping

$$\mu \longmapsto \langle \mu, \psi\rangle := \int_{\mathsf{E}} \mu(\mathrm{d}e)\, \psi(e)$$

is continuous. Note that $\nu \times \ell$ belongs to $\mathcal{M}_p$ for each finite measure ν on $\mathbf{R}_+$ (since $\langle \ell, \phi_p\rangle < \infty$). Write $\mathcal{N}_p = \mathcal{N}_p(\mathsf{E})$ for the subset of all those measures μ in $\mathcal{M}_p$

with values in $\{0, 1, \ldots, \infty\}$ (counting measures), and $\mathcal{N}_p^\circ = \mathcal{N}_p^\circ(\mathsf{E})$ if additionally $\mu(\{0\} \times \mathbf{R}^d) = 0$. We let both sets $\mathcal{N}_p$ and $\mathcal{N}_p^\circ$ inherit the topologies of $\mathcal{M}_p$. The set $\mathcal{N}_p^\circ$ will serve as a *state space* of our branching particle system. In particular, the Dirac delta measure $\delta_{(u,a)} \in \mathcal{N}_p^\circ$ describes a single particle having residual lifetime $u > 0$ and sitting at position $a \in \mathbf{R}^d$.

DEFINITION 1 (branching particle system $\widehat{Z}$ and process Z). *Abstaining from a more detailed definition, the process $\widehat{Z} = \{\widehat{Z}_t \colon t \geqq 0\}$ we are interested in can now be described by the following properties:*

(i) *Given a particle $\delta_{(u,a)} \in \mathcal{N}_p^\circ$ at time $r \geqq 0$, its further path in $(0, \infty) \times \mathbf{R}^d$ is $t \mapsto (u - (t - r),\, \xi_{t-r})$, $r \leqq t < r + u$, where ξ is distributed according to P_a.*

(ii) *As soon as a particle reaches the residual lifetime $0+$, it dies, and on the same location an offspring is born whose number is an independent probabilistic copy of ζ.*

(iii) *Newly born particles get independent residual lifetimes, all distributed as $\tau > 0$, and proceed independently.*

(iv) *At time $t = 0$, we start with a system of particles described by a measure $\mu \in \mathcal{N}_p^\circ$.*

(v) *Write $\mathbf{P}_\mu$ for the law of $\widehat{Z}$. It is considered as a measure on the set $\mathcal{D}(\mathbf{R}_+, \mathcal{N}_p)$ of all $\mathcal{N}_p$-valued* càdlàg *paths ω satisfying additionally $\omega_t \in \mathcal{N}_p^\circ$, $t \geqq 0$.*

For convenience, we call process $(\widehat{Z}, \mathbf{P}_\mu,\, \mu \in \mathcal{N}_p^\circ)$ a $(d, \alpha, \beta, \gamma)$-branching particle system. *Observe that $\widehat{Z}$ is a time-homogeneous Markov process.*

Note also that we make maximal independence assumptions in defining the model. The main dependence assumption is that newly born particles start their evolution from an ancestor's death place.

Integrating out the residual lifetimes, we return to the non-Markovian process $Z_t = \widehat{Z}_t(\mathbf{R}_+ \times (\cdot))$, $t \geqq 0$, already mentioned in subsection 1.1.

We are interested in the long-term behavior of the marginal process Z of this $(d, \alpha, \beta, \gamma)$-branching particle system $\widehat{Z}$. For simplicity, we take Poissonian particle systems with intensity measure[2] $i_0 G(\mathrm{d}u)\ell(\mathrm{d}a)$ as initial states, where $i_0 > 0$ is a fixed constant. Note that in this case all the initial residual lifetimes are independent copies of the random variable τ, which means that all initial particles are in effect newborns (but do not have siblings).

Recall that a random counting measure π on E is called a *Poissonian* particle system with intensity measure $\mu \in \mathcal{M}_p$ if it has the log-Laplace transform

$$-\log \mathbf{E} \exp\langle \pi, -\psi \rangle = \left\langle \mu, 1 - \mathrm{e}^{-\psi} \right\rangle, \qquad \psi \in \widehat{\mathcal{C}}_p^+$$

(the index $+$ on a set refers to all of its nonnegative members, as we already used $\mathbf{R}_+ = [0, \infty)$). We write π_μ for such a Poissonian particle system with intensity measure μ. If $\mu = i_0 \nu \times \ell$ with a constant $i_0 \geqq 0$, a probability law ν on $(0, \infty)$ (and ℓ the Lebesgue measure on $\mathbf{R}^d$), then $\pi_{i_0 \nu \times \ell}$ is a *homogeneous* Poissonian particle system with intensity i_0 and residual lifetimes distributed according to ν. If i_0 is additionally random, then $\pi_{i_0 \nu \times \ell}$ is said to be a *mixed homogeneous Poissonian* particle system with random intensity i_0. In other words, if σ denotes the law of i_0, then

$$\mathbf{P}\left\{\pi_{i_0 \nu \times \ell} \in (\cdot)\right\} = \int_0^\infty \sigma(\mathrm{d}i)\, \mathbf{P}\left\{\pi_{i \nu \times \ell} \in (\cdot)\right\}. \tag{4}$$

A similar terminology is used for Poissonian particle systems on $\mathbf{R}^d$.

[2]We do not distinguish in notation between the distribution function G and its related probability measure $G(\mathrm{d}t)$ on $(0, \infty)$.

1.3. Lifetimes with finite mean and the extinction-persistence dichotomy. Let us assume that $\widehat{Z}_0$ is the homogeneous Poissonian particle system $\pi_{i_0 G\times\ell}$ with (deterministic) intensity $i_0 > 0$ and residual lifetime distributed according to G (recall that in this case only newborns without siblings are considered in the beginning).

First we contrast our marginal process Z of a (d,α,β,γ)-branching particle system $\widehat{Z}$ with the case where assumption (2) on a γ-tail is replaced with that of a (nonlattice) lifetime distribution function G with *finite* expectation. Then there is a dichotomy between persistent convergence and local extinction depending on whether $d > \alpha/\beta$ holds or is violated (cf. [26]). This kind of picture is of course known from other variants of spatially homogeneous (critical) branching processes; see, for instance, [21], [3], [16], [4], [12], and [15]. An intuitive reason for this dichotomy is the following: By the critical branching, the offspring of any considered finite subpopulation always goes to extinction. Because of the smaller mobility in low dimensions, this effect leads finally to a local extinction of the infinite population. In large dimensions, however, the higher mobility allows some of the particles (coming from far away) to enter the finite window of observation and thus to show up in the limit population. Consequently, in low dimensions the local fluctuations (coming from the critical branching) prevail, whereas in higher dimensions the effect of the motion is dominating. Altogether, it depends on the dimension of space and the stable law indices of branching and motion which of the competing features (local mass fluctuation or spatial dispersion) wins in the long run.

The same statement holds true for (d,α,β,γ)-branching particle systems: below the critical dimension $d_{\mathrm{c}} = \alpha\gamma/\beta$ there is local extinction, and above it there is persistence (see subsection 1.4 below). However, at the critical dimension itself, the asymptotic behavior is essentially different from that of systems with finite expected lifetime of particles (see subsection 1.9).

1.4. Long-living particles in noncritical dimensions. Let us return to our (d,α,β,γ)-branching particle system $\widehat{Z}$ according to Definition 1. Before coming to our main case (the critical dimension), we will briefly recall the picture in noncritical dimensions. The following proposition is proved in [24] and [26] as far as the spatial marginal process X is concerned. Here we state an extension which includes the residual lifetimes. The proof of this extension can be easily obtained by repeating the line of arguments of the mentioned papers, and thus is omitted.

PROPOSITION 1. *Consider the (d,α,β,γ)-branching particle system $\widehat{Z}$ starting from a homogeneous Poissonian particle system with intensity $i_0 > 0$ and with residual lifetimes distributed according to G, that is, $\widehat{Z}_0 = \pi_{i_0 G\times\ell}$. For a constant $K > 0$, introduce the time-scaled process $\widehat{Z}^{(K)}$:*

$$\widehat{Z}_t^{(K)}\big((\,\cdot\,)\times(\,\cdot\,)\big) := \widehat{Z}_{Kt}\big(K(\,\cdot\,)\times(\,\cdot\,)\big), \qquad t \geqq 0, \tag{5}$$

and fix $t > 0$. Then the following statements hold.

(a) (Local extinction in subcritical dimensions; see [26].) *If $d < \alpha\gamma/\beta$, then $\widehat{Z}_t^{(K)}$ suffers local extinction as $K \uparrow \infty$, that is, $\widehat{Z}_t^{(K)} \to 0$ in probability.*

(b) (Persistent convergence in supercritical dimensions; see [24], [26].) *On the other hand, if $d > \alpha\gamma/\beta$, then $\widehat{Z}_t^{(K)}$ converges in law as $K \uparrow \infty$ to a homogeneous Poissonian limit population $\widehat{Z}_t^{(\infty)} = \pi_{i_0 G_\infty^t \times \ell}$ of full intensity i_0 and with residual* *lifetimes distributed according to*

$$G_\infty^t(\mathrm{d}u) := \frac{\sin \pi\gamma}{\pi} \frac{t^\gamma \mathrm{d}u}{u^\gamma\,(t+u)}, \qquad u > 0 \tag{6}$$

(for the occurrence of G_∞^t; see the respective theorem in [9, section 14.3]).

As pointed out already in subsection 1.1, the striking new feature relative to the case of a lifetime distribution with finite mean (subsection 1.3) is that in supercritical dimensions the local dependencies between relatives are lost in the limit. This is caused by the heavy tail of the lifetime distribution G of the particles — siblings are too far away in the long run.

According to a result of [26], at the critical dimension $d_{\mathrm{c}} = \alpha\gamma/\beta$ local extinction in law of Z_t as $t \uparrow \infty$ does *not* hold (recall our discussion in subsection 1.1). Our purpose is to clarify this situation. In order to describe this in detail, we first need to introduce a superprocess variant $\widehat{X}$ of the (d,α,β,γ)-branching particle system $\widehat{Z}$, since the detailed description of the long-term behavior of Z depends on a quantity derived from $\widehat{X}$.

1.5. The (d,α,β,γ)-superprocess $\widehat{X}$. Since Feller's paper [8], it is common to also ask for diffusion-type approximations of branching particle systems. For our present (d,α,β,γ)-model (starting with a finite initial system) such approximation has been provided by [20], that dropped, however, the residual lifetimes from the description. In this way one loses the Markov property—in fact, in both the particle systems and their high density limits (despite the independence assumptions between motion and aging). In other words, one never will end up with a Markov superprocess (not even with a time-inhomogeneous one, contrasting a statement on p. 249 in [20]). This is the reason why we insist on keeping the residual lifetimes in our description also in the superprocess limit.

In order to introduce the limiting (d,α,β,γ)-superprocess $\widehat{X}$ needed for the further procedure, we will first describe the basic "motion" process for this superprocess.

Recall that we are working with the phase space $\mathsf{E} = \mathbf{R}_+ \times \mathbf{R}^d$. By means of the $\mathbf{R}^d$-component, we describe the particles' α-stable motion process $(\xi,\, P_a,\, a \in \mathbf{R}^d)$. Additionally we need an independent (limiting) residual lifetime process $t \mapsto \vartheta_t$. To introduce it, for our lifetime tail index $0 < \gamma < 1$, we start from an independent *γ-stable subordinator* $\eta = \{\eta_t\colon\ t \geqq 0\}$ in $\mathbf{R}_+$ which is a càdlàg time-homogeneous Markov process with stationary independent increments having log-Laplace transition function

$$-\log E\big\{\exp[-\theta\eta_t]\,\big|\,\eta_0 = u\big\} = u\theta + c_\eta\, t\, \theta^\gamma, \qquad \theta, t, u \geqq 0,$$

where it is convenient for us to normalize the constant $c_\eta > 0$ to

$$c_\eta := \frac{1}{\Gamma(1+\gamma)},$$

where Γ denotes the Gamma function. From this we deduce two other processes, the process

$$k_t := \inf\{s > 0\colon\ \eta_s > t\}, \qquad t \geqq 0,$$

inverse to η, and the (limiting) *residual lifetime process* $\vartheta := \vartheta_t$,

$$\vartheta_t := \inf\{\eta_s - t\colon\ \eta_s > t,\ s \geqq 0\} = \eta_{k_t} - t, \qquad t \geqq 0.$$

Note that ϑ is a càdlàg time-homogeneous strong Markov process and that $\mathrm{d}k_t$ is a continuous and time-homogeneous additive functional of ϑ. (See [1, p. 123] for background.) In particular, ϑ can be started from $\vartheta_0 = 0$. Also recall that in the special case $\gamma = \frac{1}{2}$ the process $t \mapsto k_{\eta_0+t}$ is just the local time at 0 of a Brownian motion started at 0, and ϑ describes the residual lifetimes of the Brownian excursions from 0.

As a result we get the pair $\hat{\xi} := (\vartheta, \xi)$ of independent processes whose laws we denote by $\widehat{P}_e$, $e \in \mathsf{E}$. This completes the construction of a càdlàg time-homogeneous Markov process $(\hat{\xi}, \widehat{P}_e, e \in \mathsf{E})$ in $\mathsf{E} = \mathbf{R}_+ \times \mathbf{R}^d$. It is this process $\hat{\xi}$ which will serve as the "motion" process of an intrinsic particle in the superprocess $\widehat{X}$ which we now introduce. The time-homogeneous continuous additive functional $\mathrm{d}k_t$ of $\hat{\xi}$ will serve as the branching functional for $\widehat{X}$.

Recall that $\mathcal{D} = \mathcal{D}(\mathbf{R}_+, \mathcal{M}_p)$ denotes the set of all càdlàg measure-valued paths $\omega\colon \mathbf{R}_+ \to \mathcal{M}_p = \mathcal{M}_p(\mathsf{E})$ endowed with the Skorokhod topology. Write $\mathcal{P}$ for the set of all probability laws on $\mathcal{D}$ furnished with the topology of weak convergence.

LEMMA 1 (unique existence of the $(d, \alpha, \beta, \gamma)$-superprocess $\widehat{X}$). *Fix constants*

$$0 < \alpha \leqq 2, \quad 0 < \beta \leqq 1, \quad 0 < \gamma < 1, \quad \textit{and} \quad \varrho > 0.$$

For each measure $\mu \in \mathcal{M}_p$, there is a unique law $\mathbb{P}_\mu \in \mathcal{P}$ of a time-homogeneous Markov process $\widehat{X}$ with log-Laplace transition functional

$$-\log \mathbb{E}_\mu \exp \langle \widehat{X}_t, -\psi \rangle = \langle \mu, \widehat{V}_t \psi \rangle, \qquad t \geqq 0, \quad \psi \in \widehat{\mathcal{C}}_p^+, \tag{7}$$

where the function $\widehat{v} = \widehat{V}\psi = \{\widehat{V}_t\psi(e)\colon t \geqq 0, e \in \mathsf{E}\} \geqq 0$ uniquely solves the integral equation

$$\widehat{v}_t(e) = \widehat{E}_e \biggl[\psi(\widehat{\xi}_t) - \varrho \int_0^t \mathrm{d}k_s\, \widehat{v}_{t-s}^{1+\beta}(\widehat{\xi}_s)\biggr], \qquad t \geqq 0, \quad e \in \mathsf{E}. \tag{8}$$

In particular, the following expectation formula holds:

$$\mathbb{E}_\mu \langle \widehat{X}_t, \psi \rangle = \int_{\mathsf{E}} \mu(\mathrm{d}e)\, \widehat{E}_e \psi(\widehat{\xi}_t), \qquad \mu \in \mathcal{M}_p, \quad t \geqq 0, \quad \psi \in \widehat{\mathcal{C}}_p^+ \tag{9}$$

(*criticality*).

We call $(\widehat{X}, \mathbb{P}_\mu, \mu \in \mathcal{M}_p)$ the $(d, \alpha, \beta, \gamma)$-*superprocess* with the branching rate ϱ. The construction of such a process is standard nowadays and we will abstain from this (see [23]).

A good interpretation of this measure-valued process $\widehat{X}$ is provided by the particle system approximation worked out in the next subsection.

Remark 1 (decoupling). Note that the integral in (8) can be rewritten as

$$\int_0^t \mathrm{d}k_s\, \widehat{v}_{t-s}^{1+\beta}(0, \xi_s),$$

since

$$\int_0^\infty \mathrm{d}k_s\, 1\{\vartheta_s > 0\} = 0, \qquad \widehat{P}_e\text{-a.s.}, \quad e \in \mathsf{E},$$

which follows just from the definitions of the processes k and ϑ. However, k and ξ are independent; hence from

$$\widehat{E}_e k_s = E_u k_s = 1_{[u,\infty)}(s)\,(s-u)^\gamma, \qquad e = (u, a), \quad s \geqq 0, \tag{10}$$

we conclude that (8) can be rewritten as

$$\widehat{v}_t(e) = \big[\widehat{E}_e\, \psi(\widehat{\xi}_t)\big] - \varrho \int_{u\wedge t}^{t} \mathrm{d}_s (s-u)^\gamma \; E_a \widehat{v}_{t-s}^{1+\beta}(0,\xi_s),$$

$t \geqq 0$, $e=(u,a) \in \mathsf{E}$. Here d_s indicates that the Stieltjes integral is formed with respect to the variable s (of the monotone function $s \mapsto (s-u)^\gamma$).

1.6. Approximation of $\widehat{X}$ by particle systems. As already indicated, $\widehat{X}$ arises from the (d,α,β,γ)-branching particle system $\widehat{Z}$ via a diffusion-type approximation, which we now want to make precise. The proof of the next proposition follows along the lines of [20, section 4] (see in particular Corollary 1 there).

PROPOSITION 2 (approximation of $\widehat{X}$ by particle systems). *For $n \geqq 1$, let $\widehat{Z}^{(n)}$ denote a (d,α,β,γ)-branching particle system but with lifetime distribution function G of Hypothesis* 1(b) *replaced with $G^{(n)}$:*

$$G^{(n)}(u) := G\big(n^{\beta/\gamma}\, u\big), \qquad u \geqq 0.$$

Moreover, assume that $\widehat{Z}^{(n)}$ start with (deterministic) initial populations $\widehat{Z}_0^{(n)} \in \mathcal{N}_p^\circ$ satisfying

$$\frac{1}{n}\,\widehat{Z}_0^{(n)} \underset{n\uparrow\infty}{\longrightarrow} \mu \quad \text{in } \mathcal{M}_p. \tag{11}$$

Then

$$\frac{1}{n}\,\widehat{Z}^{(n)} \underset{n\uparrow\infty}{\Longrightarrow} \widehat{X} \quad \text{in law,}$$

where $\widehat{X}$ is the (d,α,β,γ)-superprocess from Lemma 1 *starting from μ, and with special branching rate*

$$\varrho = \frac{\sin \pi\gamma}{\pi}\,\frac{c_f}{c_G}, \tag{12}$$

with c_f from (3) *and c_G from* (2).

In simple terms, if in the (d,α,β,γ)-branching particle system $\widehat{Z}$ the particles' lifetimes τ are replaced with the rescaled (and smaller) lifetimes $n^{-\beta/\gamma}\tau$, if their unit masses are replaced with the small masses $1/n$, and if the initial populations $\widehat{Z}_0^{(n)}$ are chosen such that the rescaled measures $n^{-1}\widehat{Z}_0^{(n)}$ converge as in (11), then the whole rescaled branching particle system $n^{-1}\,\widehat{Z}^{(n)}$ tends to the (d,α,β,γ)-superprocess $\widehat{X}$ of Lemma 1 with special branching rate ϱ as defined in (12).

1.7. The marginal measure process X. The residual lifetime process ϑ, starting from a nonzero state, is *deterministic* until it reaches the state 0. Thus, the "motion" process $\hat{\xi}$ does not have transition densities; hence the superprocess $\widehat{X}$ will also not have "smooth" states. However, integrating out the residual lifetimes, that is, passing to the spatial marginal process

$$X_t(\cdot) := \widehat{X}_t\big(\mathbf{R}_+ \times (\cdot)\big), \qquad t \geqq 0,$$

the situation changes, and this will be enough for our description of the long-term behavior of $\widehat{Z}$. In fact, in dimensions $d < \alpha/\beta$, the measure states that X_t are absolutely continuous. We want to expose this property as a proposition, which we will verify in subsection 2.2 below by a modification of methods borrowed from [19]. For this purpose, let $\chi \geqq 0$ denote a bounded continuous function on $\mathbf{R}^d$ with $\langle \ell, \chi \rangle = 1$, and consider the regularization

$$\chi_\varepsilon(a) := \varepsilon^{-d}\chi(\varepsilon^{-1}a), \qquad a \in \mathbf{R}^d, \quad \varepsilon > 0, \quad \varepsilon \downarrow 0, \tag{13}$$

of the δ-function δ_0 on $\mathbf{R}^d$.

PROPOSITION 3 (marginal measure process X). *Let $d < \alpha/\beta$, and fix $t > 0$. Consider the marginal process $X = \widehat{X}(\mathbf{R}_+ \times (\cdot))$ of the $(d, \alpha, \beta, \gamma)$-superprocess $\widehat{X}$ with law $\mathbb{P}_\mu$, $\mu \in \mathcal{M}_p(\mathsf{E})$.*

(a) (Random density at a point.) *For all $a \in \mathbf{R}^d$, the limit in law*

$$\lim_{\varepsilon \downarrow 0} \langle X_t, \delta_a * \chi_\varepsilon \rangle =: X_t(a) = \langle X_t\,, \delta_a \rangle \tag{14}$$

exists. Moreover, the random density $X_t(a)$ of X_t at site a has the log-Laplace transform

$$-\log \mathbb{E}_\mu \exp\left[-\theta X_t(a)\right] = \left\langle \mu, \widehat{v}_t^{\,\theta,a} \right\rangle, \qquad \theta \geqq 0, \tag{15}$$

where (for $\theta \geqq 0$ and $a \in \mathbf{R}^d$ fixed)

$$\widehat{v}^{\,\theta,a} := \widehat{v} := \{\widehat{v}_t(e') \colon\ t > 0,\ e' = (u', a') \in \mathsf{E}\} \geqq 0 \tag{16}$$

uniquely solves the equation

$$\widehat{v}_t(e') = \theta\, \mathrm{p}_t(a - a') - \varrho \widehat{E}_{e'} \int_0^t \mathrm{d}k_s\, \widehat{v}_{t-s}^{\,1+\beta}(0, \xi_s), \tag{17}$$

$t > 0$, $e' = (u', a') \in \mathsf{E}$. *In particular, the following expectation formula holds*:

$$\mathbb{E}_\mu X_t(a) = \int_{\mathsf{E}} \mu\Big(\mathrm{d}\big(u', a'\big)\Big)\, \mathrm{p}_t(a - a')\Big). \tag{18}$$

(b) (Absolutely continuous measure states.) *$\mathbb{P}_\mu$-almost surely, X_t is absolutely continuous.*

The occurrence of the first term at the right-hand side of (17) is not surprising. In fact, proceeding formally, we have

$$X_t(a) = \langle X_t, \delta_a \rangle = \big\langle \widehat{X}_t\,, 1 \otimes \delta_a \big\rangle,$$

with δ_a the delta function at a and

$$1 \otimes \delta_a =: \overline{\delta_a}, \qquad a \in \mathbf{R}^d, \tag{19}$$

its constant extension to E. Moreover, replacing ψ in the log-Laplace equation (8) with the generalized function $\overline{\delta_a}$ on E, we get for $e' = (u', a') \in \mathsf{E}$,

$$\widehat{E}_{e'} \psi(\widehat{\xi}_t) = \widehat{E}_{(u',a')} 1(\vartheta_t)\, \delta_a(\xi_t) = E_{a'} \delta_a(\xi_t) = \mathrm{p}_t(a - a'), \tag{20}$$

since ξ has continuous transition density function p. Thus, (17) can be considered as a special case of (8), and

$$\widehat{v}^{\,\theta,a} = \widehat{V}\left(\theta \overline{\delta_a}\right).$$

Of course, part of statement (a) is that (17) is a well-posed equation. Note also that $\overline{\delta_a}$ is the generalized density function of the measure $\ell_+ \times \delta_a$ on E, where ℓ_+ denotes the Lebesgue measure on $\mathbf{R}$ restricted to $\mathbf{R}_+$, and we will not differentiate in notation between this measure and its generalized density function; that is, we write $\overline{\delta_a}$ for both.

Note also that the index γ does not enter into the dimension restriction $d < \alpha/\beta$ of this proposition. Roughly speaking, the enlargement of the lifetime of an "intrinsic particle" in the superprocess does not have an effect on the local issue of smoothness of the marginal measure states X_t. In particular, concerning absolute continuity of the states, there is no difference between the "classical" case corresponding to $\gamma = 1$ and our cases $\gamma < 1$. (Of course, one expects singularity of the marginal measure states if $d \geqq \alpha/\beta$.) Before returning to particle systems, we present some other interesting properties of the superprocess $\widehat{X}$.

1.8. Scaling properties of $\widehat{X}$ at the critical dimension. The next property follows from the log-Laplace representation in Lemma 1 by standard arguments. We skip the details.

PROPOSITION 4 (scaling properties of $\widehat{X}$). *Consider $d = \alpha\gamma/\beta$. Then the $(d, \alpha, \beta, \gamma)$-superprocess has the following properties.*

(a) (Scaling for finite initial masses.) *For each $a \in \mathbf{R}^d$ and $K, i_0 > 0$, the process*

$$\Big\{\big(K^{-d/\alpha}\widehat{X}_{Kt}\big(K(\,\cdot\,) \times K^{1/\alpha}(\,\cdot\,)\big)\big)_{t \geqq 0} \;\Big|\; \widehat{X}_0 = i_0\delta_{(0,a)}\Big\}$$

coincides in law with the process

$$\Big\{\widehat{X} \mid \widehat{X}_0 = K^{-d/\alpha} i_0 \delta_{(0, K^{-1/\alpha}a)}\Big\}.$$

(b) (Self-similarity.) *If $\widehat{X}_0 = i_0\delta_0 \times \ell$, then $\widehat{X}$ is self-similar*:

$$\Big\{K^{-d/\alpha}\widehat{X}_{Kt}\big(K(\,\cdot\,) \times K^{1/\alpha}(\,\cdot\,)\big)\colon\, t \geqq 0\Big\} \overset{\mathcal{L}}{=} \widehat{X}, \qquad K > 0.$$

Scaling properties are often a very useful tool. In the present case, we may combine them with Proposition 3 in order to get the following immediate result. Define $\widehat{X}^{(K)}$ just as we introduced $\widehat{Z}^{(K)}$ in (5). Recall the limiting residual lifetime distribution G^t_∞ introduced in (6).

COROLLARY 1 (limiting scaling behavior of $\widehat{X}$). *Let $d = \alpha\gamma/\beta$ and fix $t, i_0 > 0$. Then the $(d, \alpha, \beta, \gamma)$-superprocess $\widehat{X}$ has the following limiting scaling behavior.*

(a) (Asymptotics of $\widehat{X}^{(K)}$ for finite initial masses.) *Fix $a \in \mathbf{R}^d$. Assume that $\widehat{X}_0 = K^{d/\alpha} i_0 \delta_{(0, K^{1/\alpha}a)}$. Then*

$$\widehat{X}^{(K)}_t \underset{K\uparrow\infty}{\Longrightarrow} X'_t(0)\, G^t_\infty \times \ell \quad \textit{in law},$$

where $\widehat{X}'$ is the $(d, \alpha, \beta, \gamma)$-superprocess starting from $\widehat{X}'_0 = i_0\delta_{(0,a)}$ and $X'_t(0)$ is the random density at time t at the origin from the marginal process $X' := \widehat{X}'(\mathbf{R}_+ \times (\,\cdot\,))$, according to Proposition 3(a).

(b) (Persistent convergence of $\widehat{X}^{(K)}$.) *If $\widehat{X}_0 = i_0\delta_0 \times \ell$, then*

$$\widehat{X}^{(K)}_t \underset{K\uparrow\infty}{\Longrightarrow} X_t(0)\, G^t_\infty \times \ell \quad \textit{in law},$$

where $X_t(0)$ is the random density at time t at site 0 of the marginal process $X = \widehat{X}(\mathbf{R}_+ \times (\cdot))$ having full expectation i_0.

Specifying first to $t = 1$ and then to $K = t$ in the previous corollary and paying attention only to the spatial marginal measures lead to the following result.

Corollary 2 (long-term behavior of X). *Let $d = \alpha\gamma/\beta$. The marginal process X of the (d,α,β,γ)-superprocess $\widehat{X}$ has the following long-term behavior. Fix $i_0 > 0$.*

(a) (Asymptotics of X_t for t-dependent finite initial measures.) *Consider a whole family $\{{}^u\widehat{X}\colon\ u \geqq 0\}$ of (d,α,β,γ)-superprocesses starting from ${}^u\widehat{X}_0 = u^{d/\alpha} i_0 \delta_{(0,u^{1/\alpha}a)}$, $a \in \mathbf{R}^d$. Then the spatial marginal processes ${}^tX_t := {}^t\widehat{X}_t(\mathbf{R}_+ \times (\,\cdot\,))$ obey*

$$ {}^tX_t \underset{t\uparrow\infty}{\Longrightarrow} X_1'(0) \quad \textit{in law}, $$

where $X_1'(0)$ is the random density at time 1 at the origin of the marginal process $X' := \widehat{X}'(\mathbf{R}_+ \times (\,\cdot\,))$ (see Proposition 3(a)) of the (d,α,β,γ)-superprocess $\widehat{X}'$ starting from $\widehat{X}_0' = i_0\delta_{(0,a)}$.

(b) (Persistent convergence of X_t.) *If, instead, $\widehat{X}_0 = i_0\delta_0 \times \ell$, then*

$$ X_t := \widehat{X}_t\big(\mathbf{R}_+ \times (\,\cdot\,)\big) \underset{t\uparrow\infty}{\Longrightarrow} X_1'(0)\,\ell \quad \textit{in law}, $$

where $X_1'(0)$ is as in (a) but with $\widehat{X}_0' = i_0\delta_0 \times \ell$; hence the limiting state has full expectation $i_0\ell$.

Note that if the initial state of the superprocess is $\widehat{X}_0 = i_0\delta_0 \times \ell$ (as in (b)), it is easy to understand why, in the critical dimension, mixed Lebesgue measures occur in the spatial component in the limit as $K \uparrow \infty$ instead of the zero measure in the "classical" case of a (d,α,β)-superprocess. This behavior is caused by the coincidence that one has nice scaling properties *and* the existence of absolutely continuous states. While (d,α,β)-superprocesses have singular states at their critical dimension $d_{\mathrm{c}} = \alpha/\beta$, in contrast the spatial projections of (d,α,β,γ)-superprocesses have absolutely continuous states at their critical dimension $d_{\mathrm{c}} = \alpha\gamma/\beta$ $(< \alpha/\beta)$.

Recall also that the statement in Corollary 2(b) is quite analogous to a result on the super-Brownian reactant with a super-Brownian catalyst at the critical dimension $d = 2$; see [11]. In the last case an analogue of part (a) of Corollary 2 can be established.

1.9. Persistent convergence of Z at the critical dimension. Now we return to our marginal branching particle system $Z = \widehat{Z}(\mathbf{R}_+ \times (\cdot))$. First we state the analogue of Corollary 2(b). Note that the following theorem deepens the statement in [26] that nontrivial limit points exist.

Theorem 2 (persistent convergence to a mixed Poisson system). *Consider the (d,α,β,γ)-branching particle system $\widehat{Z}$ with $\widehat{Z}_0 = \pi_{i_0 G\times\ell}$. Assume that $d = \alpha\gamma/\beta$. Then*

$$ Z_t \underset{t\uparrow\infty}{\Longrightarrow} \pi_{X_1(0)\,\ell} \quad \textit{in law}, $$

where $X_1(0)$ is the random density at time 1 at site 0 of the marginal process $X = \widehat{X}(\mathbf{R}_+ \times (\cdot))$ of the (d,α,β,γ)-superprocess $\widehat{X}$ with $\widehat{X}_0 = i_0\delta_0 \times \ell$. In particular, the limiting state has full expectation $i_0\ell$ and is not (spatially) ergodic.

In other words, the limit population is mixed homogeneous Poisson, and its random intensity is just $X_1(0)$, the random density at time 1 at the origin of the marginal process X of the (d,α,β,γ)-superprocess $\widehat{X}$ starting from $i_0\delta_0 \times \ell$.

Theorem 2 will actually be derived from our next theorem, and this derivation will be provided at the end of the next subsection.

1.10. Refined asymptotics. The progress relative to [26], exhibited in Theorem 2, was possible by revealing a refined asymptotics as $K \uparrow \infty$ for the marginal process $Z^{(K)} := \widehat{Z}^{(K)}(\mathbf{R}_+ \times (\cdot))$ of the rescaled $(d, \alpha, \beta, \gamma)$-branching particle system $\widehat{Z}^{(K)}$ defined in (5) and starting from *finite* and asymptotically large initial populations (the particle analogue of Corollary 1(a)), which we now want to deal with. It can be counted as a spatial generalization of a result on age-dependent branching processes in [25].

For this purpose, we need to introduce further notation. Let $\mathcal{C}_p = \mathcal{C}_p(\mathbf{R}^d)$ denote the set of all continuous functions $\varphi \colon \mathbf{R}^d \to \mathbf{R}$ such that $|\varphi| \leqq c_\varphi \, \phi_p$ for some constant c_φ and such that φ/ϕ_p can continuously be extended to a function on the one-point compactification $\dot{\mathbf{R}}^d$. Note that the constant extension $\overline{\varphi} := \mathbf{1} \otimes \varphi$ of $\varphi \in \mathcal{C}_p$ belongs to $\widehat{\mathcal{C}}_p$.

The following notation, though potentially ambiguous at first glance, will be convenient.

NOTATION 1 (mixed laws). *We denote by* $\mathbf{P}_{\delta_{(\tau,a)}}$ *the ("annealed," deterministic) law* $\int_0^\infty \mathrm{d}G(u) \, \mathbf{P}_{\delta_{(u,a)}}$ *of* $\overline{Z}$ *started from a single particle with position* a *and residual lifetime* τ *distributed according to* G. *Also, in other cases of mixed laws we will use a similar notation. For instance, we write* $\widehat{P}_{(\tau,a)}$ *for* $\int_0^\infty \mathrm{d}G(u) \, \widehat{P}_{(u,a)}$.

For $\varphi \in \mathcal{C}_p^+$, set

$$Q_t\varphi(a) := 1 - \mathbf{E}_{\delta_{(\tau,a)}} \exp\left\langle \widehat{Z}_t \, , -\overline{\varphi} \right\rangle = 1 - \mathbf{E}_{\delta_{(\tau,a)}} \exp\langle Z_t, -\varphi \rangle, \tag{21}$$

$t \geqq 0$, $a \in \mathbf{R}^d$. Note that $Q_t\varphi\,(a)$ occurs in the log-Laplace functional of the state $\widehat{Z}_t$ of the $(d, \alpha, \beta, \gamma)$-branching particle system starting from the homogeneous Poissonian particle system $\widehat{Z}_0 = \pi_{i_0 G \times \ell}$:

$$-\log \mathbf{E}_{\pi_{i_0 G \times \ell}} \exp\left\langle \widehat{Z}_t \, , -\overline{\varphi} \right\rangle = \langle i_0 \, \ell, Q_t\varphi \rangle.$$

Applying this to the rescaled process $\widehat{Z}^{(K)}$ introduced in (5), we get

$$\begin{aligned} -\log \mathbf{E}_{\pi_{i_0 G \times \ell}} \exp\left\langle \widehat{Z}_t^{(K)}, -\overline{\varphi} \right\rangle &= -\log \mathbf{E}_{\pi_{i_0 G \times \ell}} \exp\left\langle Z_t^{(K)}, -\varphi \right\rangle \\ &= i_0 \int_{\mathbf{R}^d} \mathrm{d}a \, K^{d/\alpha} \, Q_{Kt}\varphi\,(K^{1/\alpha}a). \end{aligned} \tag{22}$$

The asymptotics we are interested in concerns the quantity

$$V_t^{(K)}\varphi(a) := K^{d/\alpha} \, Q_{Kt}\varphi\,(K^{1/\alpha}a), \tag{23}$$

occurring in the integrand in (22). Here is our *key result.*

THEOREM 3 (refined asymptotics). *Assume that* $d = \alpha\gamma/\beta$. *Then, for each* $\varphi \in \mathcal{C}_p^+$, $t > 0$, *and* $a \in \mathbf{R}^d$,

$$V_t^{(K)}\varphi(a) \underset{K\uparrow\infty}{\longrightarrow} v_t^{\theta(\varphi)}(a), \tag{24}$$

where

$$\theta(\varphi) := \int_{\mathbf{R}^d} \mathrm{d}b \left[1 - \mathrm{e}^{-\varphi(b)}\right] \tag{25}$$

and

$$v_t^{\theta(\varphi)}(a) := \widehat{v}_t^{\;\theta(\varphi),0}(0, a) \tag{26}$$

with $\widehat{v}^{\theta(\varphi),0}$ *a unique solution to* (17) *in the case* $\theta = \theta(\varphi)$ *and* $a = 0$ *there, and with special* ϱ *as in* (12). *Consequently,* $v^{\theta(\varphi)} = v = \{v_t(a)\colon\ t > 0,\ a \in \mathbf{R}^d\} \geqq 0$ *solves*

$$v_t(a) = \theta(\varphi)\,\mathrm{p}_t(a) - \varrho \int_0^t \mathrm{d}(s^\gamma)\, E_a v_{t-s}^{1+\beta}(\xi_s), \tag{27}$$

$t > 0$, $a \in \mathbf{R}^d$.

The proof of this theorem will be given in subsection 2.6 below, after preparations in subsections 2.3–2.5. The proof will essentially use the assumption $\gamma < 1$ imposed in Hypothesis 1(b) for our entire paper. Observe that to get a refined asymptotics for the cases $\gamma = 1$ or $E\tau < \infty$, another approach is needed; these cases will be investigated elsewhere.

Let us also mention that for super-Brownian motions and branching Brownian motions with exponential lifetimes, refined asymptotic results in the critical dimension (in the sense of Theorem 3) were established in [17].

Now we want to explain how Theorem 3 is related to the mentioned asymptotics for the case of large but finite initial populations.

COROLLARY 3 (population starting from large finite populations). *Let* $d = \alpha\gamma/\beta$. *Assume*[3] $\widehat{Z}_0 = [K^{d/\alpha} i_0]\,\delta_{(\tau, K^{1/\alpha}a)}$ *with* $i_0 > 0$ *and* $a \in \mathbf{R}^d$. *Then for the marginal rescaled processes* $Z^{(K)} := \widehat{Z}^{(K)}(\mathbf{R}_+ \times (\cdot))$ *and* $t > 0$,

$$Z_t^{(K)} \underset{K\uparrow\infty}{\Longrightarrow} \pi_{X_t(0)\,\ell} \quad \textit{in law},$$

where $X_t(0)$ *is the random density at time* t *at site* 0 *of the marginal process* X *of the* (d,α,β,γ)*-superprocess* $\widehat{X}$ *with* $\widehat{X}_0 = i_0\delta_{(0,a)}$.

Clearly, the previous corollary can be restated in terms of a long-term behavior analogously to Corollary 2(a).

Proof of Corollary 3. Fix $t, i_0 > 0$, $a \in \mathbf{R}^d$, and $\varphi \in \mathcal{C}_p^+$. By the branching property and definition (5) of rescaling, for $K > 0$,

$$\log \mathbf{E}_{[K^{d/\alpha} i_0]\delta_{(\tau, K^{1/\alpha}a)}} \mathrm{e}^{-\langle Z_t^{(K)}, \varphi\rangle} = [K^{d/\alpha} i_0] \log \mathbf{E}_{\delta_{(\tau, K^{1/\alpha}a)}} \mathrm{e}^{-\langle Z_{Kt}, \varphi\rangle}.$$

Using definition (21) of $Q\varphi$, the limit as $K \uparrow \infty$ of the right-hand side of the latter equation equals

$$\lim_{K\uparrow\infty} [K^{d/\alpha} i_0] \log\left(1 - Q_{Kt}\varphi\,(K^{1/\alpha}a)\right) \;=\; -i_0 \lim_{K\uparrow\infty} V_t^{(K)}\varphi(a) = -i_0\, v_t^{\theta(\varphi)}(a),$$

where we used notation (23) and Theorem 3. But by definition (26) of $v_t^{\theta(\varphi)}(a)$, and the log-Laplace formula (15) in Proposition 3,

$$-i_0\, v_t^{\theta(\varphi)}(a) = \ln \mathbb{E}_{\,i_0\delta_{(0,a)}} \exp\left\{-\theta(\varphi)\, X_t(0)\right\}.$$

However, by formula (25), this is the log-Laplace transform of the mixed homogeneous Poissonian particle system $\pi_{X_t(0)\ell}$ as desired, where $\widehat{X}$ starts from $\widehat{X}_0 = i_0\,\delta_{(0,a)}$. This finishes the proof (up to the validity of Theorem 3).

Proof of Theorem 2. First, note that by definition (5),

$$Z_t = \widehat{Z}_1^{(t)}\left(\mathbf{R}_+ \times (\,\cdot\,)\right), \qquad t > 0.$$

[3] $[z]$ denotes the integer part of z.

We need to combine (22) and (24) (by using notation (23)) to conclude that

$$-\log \mathbf{E}_{\pi_{i_0 G\times\ell}} \exp\langle Z_t\,, -\varphi\rangle \underset{t\uparrow\infty}{\longrightarrow} i_0 \int_{\mathbf{R}^d} \mathrm{d}a\, v_1^{\theta(\varphi)}(a), \tag{28}$$

with $\varphi \in \mathcal{C}_p^+$ and $\theta(\varphi)$ as in (25), since the right-hand side in (28) equals the log-Laplace transform of the mixed homogeneous Poissonian particle system $\pi_{X_1(0)\ell}$ with $\widehat{X}$ starting from $\widehat{X}_0 = i_0\delta_0 \times \ell$. For this we only have to justify that in

$$\int_{\mathbf{R}^d} \mathrm{d}a\, t^{d/\alpha}\, Q_t\varphi\,(t^{1/\alpha}a) \tag{29}$$

from (22) the limit as $t \uparrow \infty$ can be provided under the integral. In fact, by definition (21) of Q,

$$Q_t\varphi(a) \leqq \mathbf{E}_{\delta_{(\tau,a)}} \langle Z_t\,, \varphi\rangle = E_a\varphi(\xi_t),$$

where we used the criticality of the process Z (see (46) below). Hence

$$t^{d/\alpha}\, Q_t\varphi\,(t^{1/\alpha}a) \leqq t^{d/\alpha} \int_{\mathbf{R}^d} \mathrm{d}b\, \mathrm{p}_t(b - t^{1/\alpha}a)\, \varphi(b). \tag{30}$$

However, by the *self-similarity*

$$K^{d/\alpha}\, \mathrm{p}_{Ks}(K^{1/\alpha}b) = \mathrm{p}_s(b), \qquad K, s > 0, \quad b \in \mathbf{R}^d, \tag{31}$$

of the stable transition densities of Hypothesis 1(a), the right-hand side in estimate (30) can be rewritten as

$$\int_{\mathbf{R}^d} \mathrm{d}b\, \mathrm{p}_1(t^{-1/\alpha}b - a)\, \varphi(b), \tag{32}$$

which converges to

$$\mathrm{p}_1(a) \int_{\mathbf{R}^d} \mathrm{d}b\, \varphi(b) =: \mathrm{p}_1(a)\, \|\varphi\|_1 \tag{33}$$

as $t \uparrow \infty$ (where we also used the symmetry of p_s). Finally, integrating the expressions in (32) and (33) with respect to da we get identically $\|\varphi\|_1$. Thus we can apply the extended dominated convergence theorem to justify that the limit in (29) can be provided under the integral. This finishes the proof (up to the validity of Theorem 3).

2. Remaining proofs. After some preparation, we will demonstrate in subsection 2.2 that X has random densities in each point. The main purpose, however, is to prove the refined asymptotics of Theorem 3, which is done in subsections 2.3–2.6.

2.1. Some notation. With c we will always denote a positive constant which might vary from formula to formula. The symbol c with some additional mark (for example, c_f or c_G) will, however, denote a specific constant. A constant of the form $c_{(\#)}$ means that this constant first occurred related to formula line (#).

Recall that p denotes the stable transition density function from Hypothesis 1(a). From the self-similarity (31) we immediately get the following simple estimate:

$$\mathrm{p}_r(a) \leqq c_{(34)}\, r^{-d/\alpha}, \qquad r > 0, \quad a \in \mathbf{R}^d. \tag{34}$$

On the other hand, we also have

$$\mathrm{p}_r(a) \leqq c_{(35)}\, r\, |a|^{-d-\alpha}, \qquad r > 0, \quad a \in \mathbf{R}^d; \tag{35}$$

see, for instance, [10, formula (A.8)].

We denote by $\widehat{S} = \{\widehat{S}_t\colon\ t \geqq 0\}$ the semigroup of the Markov process $\hat{\xi}$.

Recall also that the boldface letter $\mathbf{P}$ is related to a law of a particle system, the blackboard letter $\mathbb{P}$ to superprocesses, and the italic P to a law of a "basic" random object as ξ, τ, η. Corresponding expectations are expressed as $\mathbf{E}, \mathbb{E}, E$, respectively.

2.2. Absolutely continuous measure states of X (proof of Proposition 3). Here we will *prove Proposition* 3. The proof will be based on some modification of arguments in [19]. The main result there is a general criterion for the absolute continuity of the measure states of some catalytic superprocesses. For this it is assumed in [19] that the underlying motion process has transition densities, which, unfortunately, our motion process $\hat{\xi}$ for $\widehat{X}$ does not have. However, this is not an obstacle, since we only want to show that the projected process X has absolutely continuous states. To prove this, we will heavily exploit the methods of [19], and in this sense we also meet the invitation in [19] "to think about interesting new examples" for the main theorem there.

Recall that $d < \alpha/\beta$. Fix $\mu \in \mathcal{M}_p(\mathsf{E})$ and $t > 0$.

1° (*Absolute continuity of expectations.*) The expectation measure $\mathbb{E}_\mu X_t$ on $\mathbf{R}^d$ is absolutely continuous with continuous density function

$$a \longmapsto \int_{\mathsf{E}} \mu(\mathrm{d}(u,b))\ \mathrm{p}_t(b-a) =: \mu * \mathrm{p}_t\,(a)$$

(see expectation formula (9)).

2° (*Absolute continuity of states.*) Assume for the moment that

$$\lim_{\varepsilon \downarrow 0} \Big(-\log \mathbb{E}_\mu \exp \langle X_t\,,\ -\theta\, \delta_a * \chi_\varepsilon \rangle \Big) =:\ w(a,\theta) \tag{36}$$

exists for each $a \in \mathbf{R}^d$ and $\theta > 0$, and that

$$\frac{\partial}{\partial\theta}\, w(a,\theta)\,\big|_{\theta=0+} = \mu * \mathrm{p}_t\,(a), \qquad a \in \mathbf{R}^d. \tag{37}$$

Then, for each $a \in \mathbf{R}^d$, the random density $X_t(a) = \langle X_t, \delta_a \rangle$ exists as claimed in (14), has log-Laplace transform

$$-\log \mathbb{E}_\mu \exp\big[-\theta X_t(a)\big] = w(a,\theta), \qquad \theta > 0,$$

and "full" expectation $\mathbb{E}_\mu X_t(a) = \mu * \mathrm{p}_t\,(a)$. Hence, combined with step 1°, we recognize that the singular measure component of the random measure X_t must disappear a.s., that is, we get the absolute continuity claim (b) (see, for instance, [19, Lemma 2.2] for a more careful formulation).

3° (*Uniform regularity.*) To attack the proof of (36) and (37), we go back to the marginal measure $X_t = \widehat{X}_t(\mathbf{R}_+ \times (\cdot))$ of $\widehat{X}_t$ and use the log-Laplace representation (7). This means we have to study the log-Laplace equation (8) with the function ψ replaced by

$$\theta\, 1 \otimes \delta_a^\varepsilon =: \theta\, \overline{\delta_a^\varepsilon}\,, \qquad a \in \mathbf{R}^d, \tag{38}$$

where

$$\delta_a^\varepsilon(b) \;:=\; \delta_a * \chi_\varepsilon(b) = \chi_\varepsilon\,(b-a), \qquad b \in \mathbf{R}^d,$$

with χ_ε from (13), and we have to let $\varepsilon \downarrow 0$. Note that $\overline{\delta_a^\varepsilon}$ approaches the constantly extended δ-function $\overline{\delta_a} = 1 \otimes \delta_a$ on E from (19), which of course does not belong to the set $\widehat{\mathcal{C}}_p^+$ of test functions occurring in the log-Laplace equation (8). So we have to justify its usage in order to come to (17).

In line with (20) we define

$$\widehat{S}_t\overline{\delta_a} := 1 \otimes (\delta_a * \mathrm{p}_t), \qquad t > 0, \quad a \in \mathbf{R}^d. \tag{39}$$

We mention that $\widehat{S}_t\overline{\delta_a}$ could also be introduced by using the collision measure (see [7], [22]) between the transition measure related to $\widehat{S}_t$ which — by the independence of the components of the vector $(\vartheta, \xi) = \hat{\xi}$ — is a product measure with continuous density function p_t of the second component, and the measure $\overline{\delta_a} = \ell_+ \times \delta_a$.

First we show that the family $\{\overline{\delta_a}\colon\ a \in \mathbf{R}^d\}$ of measures on E has the following *uniform regularity* property: For all $(u,b) \in \mathsf{E}$ and $0 \leqq r \leqq t$,

$$\sup_{a\in\mathbf{R}^d} \widehat{E}_{(u,b)} \int_r^t \mathrm{d}k_s \left[\widehat{S}_{t-s}\overline{\delta_a}\,(\widehat{\xi}_s)\right]^{1+\beta} \tag{40}$$

$$\leqq c_{(41)}\, t^{-d/\alpha} \int_{\{u\vee r\leqq s\leqq t\}} \mathrm{d}_s(s-u)^\gamma\,(t-s)^{-\beta d/\alpha} < \infty. \tag{41}$$

In fact, by (39), and, since k and ξ are independent, by using (10), the expectation expression in (40) is bounded from above by

$$\begin{aligned} &\varrho \int_{\{u\vee r\leqq s\leqq t\}} \mathrm{d}_s(s-u)^\gamma\, E_b\, \mathrm{p}_{t-s}^{1+\beta}(\xi_s - a) \\ &\quad\leqq\ c\, t^{-d/\alpha} \int_{\{u\vee r\leqq s\leqq t\}} \mathrm{d}_s(s-u)^\gamma\,(t-s)^{-\beta d/\alpha}, \end{aligned} \tag{42}$$

where we first applied the simple estimate (34) to p^β, then the Chapman–Kolmogorov equation to the expectation on p, and finally again applied (34). But the integral in (42) is finite since γ and $\beta d/\alpha$ belong to $(0,1)$. Consequently, estimates (40)–(41) are true.

4° (*Uniform integrability in the regularization.*) Next we verify that for $u \geqq 0$ and $a, b \in \mathbf{R}^d$,

$$\lim_{r\uparrow t}\limsup_{\varepsilon\downarrow 0} \widehat{E}_{(u,b)} \int_r^t \mathrm{d}k_s \left[\widehat{S}_{t-s}\overline{\delta_a^\varepsilon}\,(\widehat{\xi}_s)\right]^{1+\beta} = 0 \tag{43}$$

(recall notation (38)). Indeed, from $\overline{\delta_a^\varepsilon} = 1 \otimes \delta_a^\varepsilon$ we get

$$\widehat{S}_t\overline{\delta_a^\varepsilon}\,(\widehat{\xi}_s) = S_t\delta_a^\varepsilon\,(\widehat{\xi}_s) = \int_{\mathbf{R}^d} \mathrm{d}a'\,\chi_\varepsilon(a')\,\mathrm{p}_t(a + a' - \xi_s)$$

(with S denoting the semigroup related to (38)). Hence, by Jensen's inequality,

$$\left[\widehat{S}_{t-s}\overline{\delta_a^\varepsilon}\,(\widehat{\xi}_s)\right]^{1+\beta} \leqq \int_{\mathbf{R}^d} \mathrm{d}a'\,\chi_\varepsilon(a')\left[\widehat{S}_{t-s}\overline{\delta_{a+a'}}\,(\widehat{\xi}_s)\right]^{1+\beta}$$

(recall (39)). Therefore, the expectation expression in (43) can be bounded from above by

$$\sup_{a\in\mathbf{R}^d} \widehat{E}_{(u,b)} \int_r^t \mathrm{d}k_s \left[\widehat{S}_{t-s}\overline{\delta_a}(\widehat{\xi}_s)\right]^{1+\beta},$$

which does not depend on ε and, moreover, tends to 0 as $r \uparrow t$ by the uniform regularity property (40)–(41). This gives (43).

5° (*Conclusions.*) According to step 3°, for a in $\mathbf{R}^d$ fixed, $\overline{\delta_a}$ is a regular measure on E in the sense of [19, Definition 1.1]. Here our $\widehat{S}_t\overline{\delta_a}$ takes the role of the density function of the motion process applied to the measure ν there, which corresponds to our $\overline{\delta_a}$. Now by Proposition 1.2 of [19] (or rather by carrying over its proof line by line with the obvious changes since we modified Definition 1.1 of [19]), (8) with ψ replaced by $\theta\overline{\delta_a}$, where $0 < \theta \leqq 1$, that is, (17), has exactly one solution $\widehat{v} = \widehat{v}^{\theta,a} = \widehat{V}(\theta\overline{\delta_a})$, and

$$\frac{\partial}{\partial\theta}\,\widehat{v}_t^{\theta,a}(e')\,\big|_{\theta=0+} = \mathrm{p}_t\,(a-a'), \qquad e' = (u',a') \in \mathsf{E}.$$

Moreover, as in the proof of formula line (2.12) in [19], by using (43),

$$\widehat{V}_t\big(\theta\,\overline{\delta_a^\varepsilon}\,\big)(e') \underset{\varepsilon\downarrow 0}{\longrightarrow} \widehat{V}_t(\theta\overline{\delta_a})(e') = \widehat{v}_t^{\theta,a}(e'), \qquad e' \in \mathsf{E}.$$

Hence, by the inequality

$$0 \leqq \widehat{V}_t\psi(e) \leqq \widehat{E}_e\psi(\widehat{\xi}_t), \qquad t \geqq 0, \quad e \in \mathsf{E}, \quad \psi \in \widehat{\mathcal{C}}_p^{\,+},$$

and the dominated convergence theorem, we obtain

$$\big\langle \mu, \widehat{V}_t(\theta\,\overline{\delta_a^\varepsilon}\,)\big\rangle \underset{\varepsilon\downarrow 0}{\longrightarrow} \big\langle \mu, \widehat{V}_t(\theta\overline{\delta_a})\big\rangle = \big\langle \mu, \widehat{v}_t^{\theta,a}\big\rangle.$$

This gives the needed statements (36) and (37) in step 2° with $w(a,\theta) = \langle \mu, \widehat{v}_t^{\theta,a}\rangle$, and at the same time claims (15)–(17), finishing the proof of Proposition 3.

2.3. Renewal-type equation and its scaling. Our starting point in proving the asymptotic properties of $V^{(K)}\varphi$ as needed for Theorem 3 is the fact that it satisfies an integral equation (Lemma 3 below). For this we need first some further notation.

Recall that τ with law G denotes the lifetime of a newly born particle in our branching particle system $\widehat{Z}$. Let $\tau_1, \tau_2, \ldots$ denote independent copies of τ, and set $\sigma_n := \sum_{1\leqq i\leqq n} \tau_i$, $n \geqq 0$. Setting

$$g_t := \sum_{n\geqq 1} \mathbf{1}_{\{\sigma_n \leqq t\}}, \qquad t \geqq 0, \tag{44}$$

we get the "*generation number process*" (see Figure 1), and letting

$$\vartheta_t^\circ := \sigma_{g_t+1} - t, \qquad t \geqq 0,$$

we get the *residual lifetime process* related to the evolution of a single particle in our branching particle system (see Figure 2). Recalling that ξ denotes the particles' motion process, we obtain the pair $\hat{\xi}^\circ := (\vartheta^\circ, \xi)$ of independent processes whose laws we denote by $\widehat{P}_e^\circ$, $e \in \mathsf{E}$. Note that $(\hat{\xi}^\circ, \widehat{P}_e^\circ, e \in \mathsf{E})$ is a càdlàg time-homogeneous Markov process in $\mathsf{E} = \mathbf{R}_+ \times \mathbf{R}^d$ and that the laws $\widehat{P}_e^\circ$, $e \in \mathsf{E}$, describe the renewal process σ and the generation number process g as well. Recall the notation $Q\varphi$ introduced in (21).

LEMMA 2 (renewal-type equation). *Fix* $\varphi \in \mathcal{C}_p^+$. *Then*

$$Q_t\varphi(a) = E_a\big[1 - \mathrm{e}^{-\varphi(\xi_t)}\big] - c_f\,\widehat{E}^\circ_{(\tau,a)} \int_{(0,t]} \mathrm{d}g_s\,\big[Q_{t-s}\varphi\,(\xi_s)\big]^{1+\beta}, \tag{45}$$

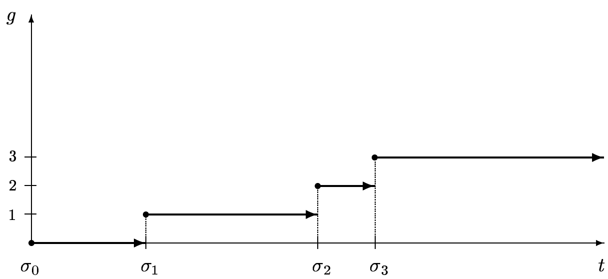

FIG. 1. *Generation number process g.*

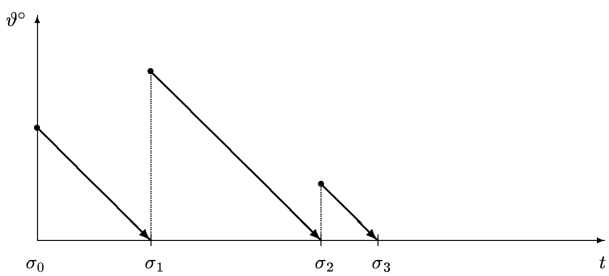

FIG. 2. *Residual lifetime process ϑ°.*

$t \geqq 0$, $a \in \mathbf{R}^d$.

Proof. To derive the desired equation it is necessary to follow the arguments used to prove [20, Lemma 3], with the obvious changes to adapt to our setting with the residual lifetimes in the description. In particular, first distinguish between $\tau \geqq t$ and $\tau < t$; then, in the latter case, further iterate.

Consequently, although the lifetime distribution is not exponential, one can exploit the renewal in the time points of death of a particle, and some invariance properties related to these points.

It is easy to see that (45) immediately yields the *expectation formula*

$$\mathbf{E}_{\delta_{(\tau,a)}}\langle \widehat{Z}_t, \overline{\varphi}\rangle = E_a \varphi(\xi_t), \qquad a \in \mathbf{R}^d, \quad t \geqq 0, \quad \varphi \in \mathcal{C}_p^+ \tag{46}$$

(to do this replace, for instance, φ with $\theta\varphi$ and differentiate (45) with respect to $\theta > 0$ at $\theta = 0+$).

Recall that we are interested in the rescaled quantity $V^{(K)}\varphi$ (where $K > 0$ and $\varphi \in \mathcal{C}_p^+$) introduced in (23). For $t \geqq 0$ and $a \in \mathbf{R}^d$, put

$$W_t^{(K)}\varphi(a) := K^{d/\alpha}\, E_{K^{1/\alpha}a}\left(1 - \exp\left[-\varphi(\xi_{Kt})\right]\right) \tag{47}$$

and introduce the (right-continuous) *renewal function* N corresponding to the lifetime distribution function G:

$$N_s := E g_s = \sum_{k=1}^{\infty} G^{*k}(s), \qquad s \geqq 0 \tag{48}$$

(recall notation (2.3)). Note that by assumption (2) in Hypothesis 1(b),

$$N_K \sim DK^\gamma \qquad \text{as} \quad K \uparrow \infty, \tag{49}$$

with the constant

$$D := \frac{\sin \pi\gamma}{\pi} \, \frac{1}{c_G} \,. \tag{50}$$

LEMMA 3 (scaled equation). *Fix $K > 0$ and $\varphi \in \mathcal{C}_p^+$. The functional $V^{(K)}\varphi$ defined in* (23) *satisfies the integral equation*

$$V_t^{(K)}\varphi(a) = W_t^{(K)}\varphi(a) - c_f \int_{(0,t]} \frac{\mathrm{d}_s N_{Ks}}{K^\gamma} \, E_a \big[V_{t-s}^{(K)}\varphi(\xi_s)\big]^{1+\beta}, \tag{51}$$

$t \geqq 0$, $a \in \mathbf{R}^d$.

Proof. Transforming (45) according to definition (23), observe that the left-hand side of (45) changes to $V_t^{(K)}\varphi\,(a)$, and the first part of the right-hand side of (45) changes to the $W^{(K)}$-term as in (51). To transform the remaining term, exploit the independence of the generation number process g and the motion process ξ, and apply definition (48) of N to get

$$\begin{aligned} &- c_f \, K^{d/\alpha} \, E_{K^{1/\alpha}a} \int_{(0,Kt]} \mathrm{d}N_s \left[Q_{Kt-s}\varphi\,(\xi_s)\right]^{1+\beta} \\ &\quad = \, - c_f \, K^{d/\alpha} \int_{(0,t]} \mathrm{d}_s N_{Ks} \; E_{K^{1/\alpha}a} \left[Q_{K(t-s)}\varphi\,(\xi_{Ks})\right]^{1+\beta}. \end{aligned}$$

Then by the *self-similarity* of the stable process ξ, that is,

$$E_{K^{1/\alpha}a}\big(K^{-1/\alpha}\xi_{Ks} \in \,\cdot\,\big) = E_a\,(\xi_s \in \,\cdot\,)\,, \tag{52}$$

$a \in \mathbf{R}^d$, $s \geqq 0$, $K > 0$, and the assumed critical parameter constellation $d\beta/\alpha = \gamma$, using once more (23), the remaining term of (51) also occurs, finishing the proof.

2.4. Convergence to the limiting equation. Our next task is to let $K \uparrow \infty$ in the integral equation (51), provided that $t > 0$. For this purpose, we first note that by the self-similarity as formulated in (52), $W^{(K)}$ defined in (47) can be written as

$$W_t^{(K)}\varphi(a) = \int_{\mathbf{R}^d} \mathrm{d}b \, \mathrm{p}_t\big(K^{-1/\alpha}b - a\big) \left[1 - \mathrm{e}^{-\varphi(b)}\right]. \tag{53}$$

By the dominated convergence theorem, this implies

$$W_t^{(K)}\varphi(a) \underset{K\uparrow\infty}{\longrightarrow} \theta(\varphi)\,\mathrm{p}_t(a), \qquad t > 0, \quad a \in \mathbf{R}^d, \tag{54}$$

with $\theta(\varphi)$ from (25). On the other hand, by (49), for $T > 0$ fixed,

$$\frac{N_{Ks}}{K^\gamma} = \frac{N_K}{K^\gamma} \, \frac{N_{Ks}}{N_K} \underset{K\uparrow\infty}{\longrightarrow} Ds^\gamma \qquad \text{uniformly in } s \in [0,T]. \tag{55}$$

To establish this fact, fix $\varepsilon \in (0,T)$ and deal with $s \leqq \varepsilon$ and $\varepsilon < s \leqq T$ separately. It remains to insert (54) and (55) into (51) by using the equality

$$c_f D = \varrho$$

(recall (50) and (12)) and check that

$$V_t^{(K)}\varphi(a) \underset{K\uparrow\infty}{\longrightarrow} v_t(a), \tag{56}$$

where v solves (27).

We begin the proof of (56) by showing its validity in the sense of some uniform convergence in the metric $L^1 = (L^1(\mathrm{d}a), \|\cdot\|_1)$. Since we later apply an analytic extension argument, we introduce an additional factor $\lambda \geqq 0$ in front of φ and consider the function

$$v^{(\lambda)} := v^{\theta(\lambda\varphi)} \tag{57}$$

instead of v. Then (27) changes to

$$v_t^{(\lambda)}(a) = \theta(\lambda\varphi)\,\mathrm{p}_t(a) - \varrho\int_0^t \mathrm{d}(s^\gamma)\, E_a\big[v_{t-s}^{(\lambda)}(\xi_s)\big]^{1+\beta}, \tag{58}$$

$t > 0$, $a \in \mathbf{R}^d$.

Lemma 4 (uniform L^1-convergence for small λ). *Fix $0 < L < T$ and $\varphi \in \mathcal{C}_p^+$. There exists a constant $\Lambda = \Lambda(T,\varphi) > 0$ such that for $V^{(K)}(\lambda\varphi)$ defined in (23) we have*

$$\lim_{K\uparrow\infty}\ \sup_{L\leqq t\leqq T}\ \sup_{\lambda\in[0,\Lambda]} \Big\| V_t^{(K)}(\lambda\varphi) - v_t^{(\lambda)}\Big\|_1 = 0, \tag{59}$$

where $v^{(\lambda)} \geqq 0$ solves (58) with the constant $\varrho = c_f D$.

To prepare for the proof of this lemma, by using (51) (with φ replaced by $\lambda\varphi$) and (58), we can estimate the L^1-norm expression in claim (59):

$$\Big\| V_t^{(K)}(\lambda\varphi) - v_t^{(\lambda)}\Big\|_1 \leqq A_1 + c_f A_2 + c_f A_3 + c_f A_4 + c_f D A_5. \tag{60}$$

Here, by using the identity

$$\int_{\mathbf{R}^d} \mathrm{d}a\, E_a F(\xi_s) = \|F\|_1\,, \qquad F \geqq 0 \text{ — measurable}, \quad s \geqq 0, \tag{61}$$

for fixed $t > 0$ and $\lambda \geqq 0$, we write the quantities entering the right-hand side as follows:

$$\begin{aligned}
A_1 &:= \Big\| W_t^{(K)}(\lambda\varphi) - \theta(\lambda\varphi)\,\mathrm{p}_t \Big\|_1,\\
A_2 &:= \int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{K^\gamma}\, \Big\| \big[V_{t-s}^{(K)}(\lambda\varphi)\big]^{1+\beta} - \big[v_{t-s}^{(\lambda)}\big]^{1+\beta} \Big\|_1,\\
A_3 &:= \frac{(N_{Kt} - N_{Kt-})}{K^\gamma} \Big\| \big[V_0^{(K)}(\lambda\varphi)\big]^{1+\beta} \Big\|_1,\\
A_4 &:= \left| \frac{N_K}{K^\gamma} - D \right| \left\| \int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{N_K} \big[v_{t-s}^{(\lambda)}\big]^{1+\beta} \right\|_1,\\
A_5 &:= \left| \int_{(0,t)} \left(\frac{\mathrm{d}_s N_{Ks}}{N_K} - \mathrm{d}(s^\gamma) \right) \Big\| \big[v_{t-s}^{(\lambda)}\big]^{1+\beta} \Big\|_1 \right|.
\end{aligned}$$

2.5. Uniform L^1-convergence. Here we prove Lemma 4. We proceed in several steps.

1° (*Error term* A_1.) First, we consider any constant $\Lambda > 0$ and all $\lambda \in [0, \Lambda]$. By (53),

$$\begin{aligned} A_1 &\leqq \int_{\mathbf{R}^d} \mathrm{d}a \int_{\mathbf{R}^d} \mathrm{d}\, b \left| \mathrm{p}_t\big(K^{-1/\alpha} b - a\big) - \mathrm{p}_t(a) \right| \left[1 - \mathrm{e}^{-\lambda\varphi(b)}\right] \\ &\leqq c_\varphi \Lambda \int_{\mathbf{R}^d} \mathrm{d}b\, \phi_p(b) \int_{\mathbf{R}^d} \mathrm{d}a \left| \mathrm{p}_t\big(K^{-1/\alpha} b - a\big) - \mathrm{p}_t(a) \right|. \end{aligned} \tag{62}$$

For any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that $\int_{|b| \geqq C} \mathrm{d}b\, \phi_p(b) \leqq \varepsilon$, whereas the internal integral in (62) is bounded by 2 since the p_s are probability density functions. On the other hand, by the uniform continuity of the stable transition density functions p on $[L, T] \times \mathbf{R}^d$, for $|b| \leqq C$ the internal integral in (62) is smaller than ε for K sufficiently large (depending on C). This shows that

$$\lim_{K \uparrow \infty} \sup_{\substack{L \leqq t \leqq T \\ \lambda \in [0, \Lambda]}} A_1 = 0. \tag{63}$$

2° (*Error term* A_3.) Using the simply bound $\|[V_0^{(K)}(\lambda\varphi)]^{1+\beta}\|_1 \leqq K^{\beta d/\alpha}\, \lambda \|\varphi\|_1$, which follows from

$$V_0^{(K)}(\lambda\varphi)\,(a) \leqq \lambda K^{d/\alpha} \varphi(K^{1/\alpha} a) \tag{64}$$

(by (51) and (47)), criticality $\beta d/\alpha = \gamma$, and the obvious relation

$$\lim_{K \uparrow \infty} \sup_{L \leqq t \leqq T} (N_{Kt} - N_{Kt-}) = 0 \tag{65}$$

(see, e.g., [9, relation (9.1.9)]), it is not difficult to see that also

$$\lim_{K \uparrow \infty} \sup_{\substack{L \leqq t \leqq T \\ \lambda \in [0, \Lambda]}} A_3 = 0. \tag{66}$$

3° (*Error term* A_4.) From (58) it follows that for $s > 0$,

$$0 \leqq v_s^{(\lambda)}(a) \leqq \theta(\lambda\varphi)\, \mathrm{p}_s(a) \leqq \Lambda\, \|\varphi\|_1\, \mathrm{p}_s(a). \tag{67}$$

Hence, using estimate (34) and the identity $d\beta/\alpha = \gamma$, we get

$$0 \leqq \left[v_s^{(\lambda)}(a)\right]^{1+\beta} \leqq c\, (\Lambda\, \|\varphi\|_1)^{1+\beta}\, \frac{\mathrm{p}_s(a)}{s^\gamma}. \tag{68}$$

However, the p_s are probability density functions. Therefore

$$A_4 \leqq c\, (\Lambda\, \|\varphi\|_1)^{1+\beta} \left| \frac{N_K}{K^\gamma} - D \right| \int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{N_K}\, \frac{1}{(t-s)^\gamma}. \tag{69}$$

The obvious substitution of variables shows that for $0 \leqq r < t$,

$$\int_r^t \mathrm{d}(s^\gamma)\, (t-s)^{-\gamma} \leqq \gamma\, t^{1-\gamma} \int_{r/t}^1 \frac{\mathrm{d}s}{s^{1-\gamma}\, (1-s)^\gamma} < \infty.$$

Combining this estimate with (55) it is easy to understand that, uniformly in $0 \leqq r < t \leqq T$,

$$\int_{[r,t)} \frac{\mathrm{d}_s N_{Ks}}{N_K} \frac{1}{(t-s)^\gamma} \underset{K\uparrow\infty}{\longrightarrow} \gamma \int_{r/t}^{1} \frac{\mathrm{d}s}{s^{1-\gamma}\,(1-s)^\gamma} \leqq \gamma\, B(\gamma, 1-\gamma) \tag{70}$$

with B denoting the Beta function. Indeed, approximate the integral by sums, and use monotonicity in time and continuity of the limit in (55). Thus, (55) leads us to the estimate

$$\lim_{K\uparrow\infty} \sup_{\substack{0<t\leqq T\\ \lambda\in[0,\Lambda]}} A_4 = 0. \tag{71}$$

4° (*Error term* A_5.) Let $\varepsilon \in (0,1)$. We split the integral in A_5 concerning the cases $(1-\varepsilon)\,t \leqq s < t$ and $0 \leqq s \leqq (1-\varepsilon)\,t$. In the first case, we first pass from the difference to the sum and use (68). Then we apply (70) with $r = (1-\varepsilon)\,t$ to see that we end up with an estimate vanishing as $\varepsilon \to 0$ uniformly in t, λ, and K. In the second case, we use the fact that the map $s \mapsto \|[v_{t-s}^{(\lambda)}]^{1+\beta}\|_1$ is bounded and continuous on the interval $[0, (1-\varepsilon)\,t]$ and the weak convergence

$$\frac{\mathrm{d}_s N_{Ks}}{N_K} \underset{K\uparrow\infty}{\longrightarrow} \frac{\gamma\,\mathrm{d}s}{s^{1-\gamma}}$$

on the same interval, uniformly in t. Putting both together, we conclude

$$\lim_{K\uparrow\infty} \sup_{\substack{L\leqq t\leqq T\\ \lambda\in[0,\Lambda]}} A_5 = 0. \tag{72}$$

5° (*A bound for* A_2.) Using the elementary inequality

$$\left|x^{1+\beta} - y^{1+\beta}\right| \leqq (1+\beta)\;|x-y|\;\left(x^\beta + y^\beta\right), \qquad x, y \geqq 0,$$

and the domination

$$0 \leqq V_t^{(K)}(\lambda\varphi)\,(a) \leqq W_t^{(K)}(\lambda\varphi)\,(a), \tag{73}$$

following from (51), we obtain $0 \leqq V_t^{(K)}(\lambda\varphi)\,(a) \leqq c\,\Lambda\,\|\varphi\|_1\, t^{-d/\alpha}$ (recall relations (53) and (34)). Now inequalities (67) and (34) allow us to conclude that, for $0 \leqq s < t$, the L^1-norm in A_2 is bounded by

$$c\,(\Lambda\,\|\varphi\|_1)^\beta\;(t-s)^{-\gamma}\,\left\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\right\|_1.$$

Thus,

$$A_2 \leqq c\,(\Lambda\,\|\varphi\|_1)^\beta \int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{K^\gamma}\, \frac{\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\|_1}{(t-s)^\gamma}. \tag{74}$$

6° (*A Gronwall-type inequality.*) Inserting (63), (66), (71), (72), and bound (74) into estimate (60) gives the following statement (recall that $\Lambda > 0$ is fixed and not yet specified): For each $\varepsilon > 0$, there is a constant $K_0 = K_0(\varepsilon, \Lambda) > 0$ such that

$$\begin{aligned} &\left\|V_t^{(K)}(\lambda\varphi) - v_t^{(\lambda)}\right\|_1 \\ &\quad\leqq \varepsilon + c_{(75)}\,(\Lambda\,\|\varphi\|_1)^\beta \int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{K^\gamma}\, \frac{\left\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\right\|_1}{(t-s)^\gamma} \end{aligned} \tag{75}$$

for all $K \geqq K_0$ and $t \in [L,T]$, as well as $\lambda \in [0,\Lambda]$. From the domination formulas (73) and (67) as well as the representation (53) of $W^{(K)}$, we obtain $\|V^{(K)}_{t-s}(\lambda\varphi) - v^{(\lambda)}_{t-s}\|_1 \leqq 2\Lambda\,\|\varphi\|_1$, uniformly in $K > 0$, $s < t$, and $\lambda \in [0,\Lambda]$. Hence,

$$\check{D}^{(K)}_t := \sup_{\substack{s\in[0,t)\\ \lambda\in[0,\Lambda]}} \big\|V^{(K)}_{t-s}(\lambda\varphi) - v^{(\lambda)}_{t-s}\big\|_1$$

is bounded in $K > 0$ and $t > 0$. Then from the Gronwall-type inequality in (75) we get

$$\check{D}^{(K)}_t \leqq \varepsilon + c_{(75)}\,(\Lambda\,\|\varphi\|_1)^\beta\ \check{D}^{(K)}_t\,\frac{N_K}{K^\gamma}\int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{N_K}\,\frac{1}{(t-s)^\gamma} \tag{76}$$

for all $K \geqq K_0$ and $t \in [L,T]$. But by the convergence and boundedness in (70),

$$\limsup_{K\uparrow\infty}\ \sup_{0<t\leqq T}\int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{N_K}\,\frac{1}{(t-s)^\gamma} := c_{(77)} < \infty. \tag{77}$$

Thus introducing the finite number

$$\check{D} := \limsup_{K\uparrow\infty}\ \sup_{L\leqq t\leqq T}\ \check{D}^{(K)}_t$$

from (76) and asymptotics (49) we obtain $\check{D} \leqq \varepsilon + c_{(75)}(\Lambda\,\|\varphi\|_1)^\beta\,\check{D}D\,c_{(77)}$. Since ε is arbitrary, we arrive at

$$\check{D} \leqq c_{(78)}\,(\Lambda\,\|\varphi\|_1)^\beta\ \check{D}. \tag{78}$$

Choosing now $\Lambda > 0$ so small that $c_{(78)}(\Lambda\,\|\varphi\|_1)^\beta < 1$, we obtain $\check{D} = 0$. Hence claim (59) is true. This finishes the proof of Lemma 4.

2.6. Pointwise convergence (proof of Theorem 3). We show now that the established L^1-convergence implies actually pointwise convergence.

Corollary 4 (pointwise convergence for small λ). *Fix $T > 0$, φ in $\mathcal{C}^+_p$, and take $\Lambda = \Lambda(T,\varphi) > 0$ as in Lemma 4. Then, for each $\lambda \in [0,\Lambda]$,*

$$V^{(K)}_t(\lambda\varphi)\,(a) \underset{K\uparrow\infty}{\longrightarrow} v^{(\lambda)}_t(a), \qquad t \in (0,T], \quad a \in \mathbf{R}^d. \tag{79}$$

Proof. The proof is very similar to that of Lemma 4 in the previous subsection, so we skip some details. In analogy to (60), for fixed $t \in (0,T]$, $a \in \mathbf{R}^d$, and $\lambda \in [0,\Lambda(T,\varphi)]$ we have

$$\big|V^{(K)}_t(\lambda\varphi)\,(a) - v^{(\lambda)}_t(a)\big| \leqq A'_1 + c_f\big[A'_2 + A'_3 + A'_4 + D\,A'_5\big],$$

where

$$\begin{aligned}
A'_1 &:= \Big|W^{(K)}_t(\lambda\varphi)\,(a) - \theta(\lambda\varphi)\,\mathrm{p}_t(a)\Big|,\\
A'_2 &:= \int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{K^\gamma}\int_{\mathbf{R}^d}\mathrm{d}b\,\mathrm{p}_s(b-a)\,\Big|\big[V^{(K)}_{t-s}(\lambda\varphi)\,(b)\big]^{1+\beta} - \big[v^{(\lambda)}_{t-s}(b)\big]^{1+\beta}\Big|,\\
A'_3 &:= \frac{(N_{Kt} - N_{Kt-})}{K^\gamma}\int_{\mathbf{R}^d}\mathrm{d}b\,\mathrm{p}_t(b-a)\,\big[V^{(K)}_0(\lambda\varphi)\,(b)\big]^{1+\beta},\\
A'_4 &:= \Big|\frac{N_K}{K^\gamma} - D\Big|\int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{N_K}\int_{\mathbf{R}^d}\mathrm{d}b\,\mathrm{p}_s(b-a)\,\big[v^{(\lambda)}_{t-s}(b)\big]^{1+\beta},\\
A'_5 &:= \Big|\int_{(0,t)}\Big(\frac{\mathrm{d}_s N_{Ks}}{N_K} - \mathrm{d}(s^\gamma)\Big)\int_{\mathbf{R}^d}\mathrm{d}b\,\mathrm{p}_s(b-a)\,\big[v^{(\lambda)}_{t-s}(b)\big]^{1+\beta}\Big|.
\end{aligned}$$

Clearly, $A_1' \to 0$ and $A_5' \to 0$ as $K \uparrow \infty$, by the dominated convergence theorem. The same is true for A_4'. In fact, we use (68) and the Chapman–Kolmogorov equation to get

$$A_4' \leqq c\,(\Lambda\,\|\varphi\|_1)^{1+\beta}\left|\frac{N_K}{K^\gamma} - D\right| p_t(a) \int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{N_K}\,\frac{1}{(t-s)^\gamma}, \tag{80}$$

so we may apply the estimates given immediately after (69). Also A_3' will disappear as $K \uparrow \infty$. Indeed, use (64), the dominated convergence theorem, criticality, and (65).

It remains to deal with the main term A_2'. Similar to the derivation of estimate (74),

$$A_2' \leqq c\,(\Lambda\,\|\varphi\|_1)^\beta \int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{K^\gamma (t-s)^\gamma} \int_{\mathbf{R}^d} \mathrm{d}b\,\mathrm{p}_s(b-a)\left|V_{t-s}^{(K)}(\lambda\varphi)\,(b) - v_{t-s}^{(\lambda)}(b)\right|. \tag{81}$$

For $0 < \varepsilon < 1$, we split the integral on $(0,t)$ into three parts: $0 < s < \varepsilon t$, $\varepsilon t \leqq s \leqq (1-\varepsilon)\,t$, and $(1-\varepsilon)\,t < s < t$. In the first and last cases, we pass from differences to sums in the integrands, use dominations (73) and (67), as well as (53), the Chapman–Kolmogorov equation, and estimate (34) to see that the internal integral of (81) is bounded in s and K. Since

$$\lim_{\varepsilon\downarrow 0}\limsup_{K\uparrow\infty} \int_{(0,\varepsilon t)\,\cup\,((1-\varepsilon)\,t,t)} \frac{\mathrm{d}_s N_{Ks}}{K^\gamma\,(t-s)^\gamma} = 0$$

by (55), to show the smallness of the expression in question it remains to deal with the integral in (81) restricted to $[\varepsilon t,\,(1-\varepsilon)\,t]$, for fixed $0 < \varepsilon < 1$. Here $\mathrm{p}_s(b-a) \leqq c$, and we are back to the L^1-norm,

$$\left\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\right\|_1 \leqq \sup_{u\in[\varepsilon t,\,(1-\varepsilon)\,t]} \left\|V_u^{(K)}(\lambda\varphi) - v_u^{(\lambda)}\right\|_1,$$

which by Lemma 4 converges to 0 as $K \uparrow \infty$. On the other hand, by (55)

$$\limsup_{K\uparrow\infty} \int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{K^\gamma\,(t-s)^\gamma} < \infty.$$

This combined with the previous arguments verifies (79). The proof of Corollary 4 is complete.

Completion of the proof of Theorem 3. Fix φ, t, a as in the theorem, and let $\lambda \geqq 0$. By (57) and (15),

$$v_t^{(\lambda)}(a) = -\log \mathbb{E}_{\delta_{(0,a)}} \exp\left[-\theta(\lambda\varphi)\,X_t(a)\right], \tag{82}$$

with $\theta(\lambda\varphi)$ from (25). On the other hand, by (23) and (21),

$$V_t^{(K)}(\lambda\varphi) = K^{d/\alpha}\left[1 - \mathbf{E}_{\delta_{(\tau,K^{1/\alpha}a)}} \exp\langle Z_{Kt},\,-\lambda\varphi\rangle\right]. \tag{83}$$

Note that the expressions in (82) and (83) are *analytic* functions in $\lambda > 0$ (or, to be more precise, in $\Re\lambda > 0$). Then from the convergence in Corollary 4 for small λ we get the desired convergence for all $\lambda \geqq 0$. This finishes the proof.

Acknowledgments. We are grateful to Achim Klenke and Jan Swart for comments which helped to improve the presentation of the paper.

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