Branching systems with long living particles at the critical dimension

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January 23, 2002

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1991 Mathematics Subject Classification 60J80, 60G70, 60J15.

Key words and phrases branching particle system, critical dimension, limit theorem, long living particles, absolute continuity, random density, superprocess, persistence, mixed Poissonian particle system, residual lifetime process, stable subordinator

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*) Supported by the DFG.
†) Supported by the grants RFBR 00-15-96136, 99-01-00012, INTAS 99-01317, and the DFG.
‡) Supported by the DFG.

Abstract

A spatial branching process is considered in which particles have a life time law with a tail index smaller than one. At the critical dimension, unlike classical branching particle systems the population does not suffer local extinction when started from a spatially homogenous Poissonian initial population. In fact, persistent convergence to a mixed Poissonian particle system is shown. The random limiting intensity is characterized in law by the random density in a space point of a related age-dependent superprocess at a fixed time. The proof relies on a refined study of the system starting from asymptotically large but finite initial populations.

1 Introduction and statement of results

1.1 Motivation and purpose

The study of spatially homogeneous (critical) branching particle systems in R^d with long living particles has been initiated in [SW93] and [VW99] (see also [KS98]). Here *'long living'* means that the lifetime distribution of a particle has a tail of index $\gamma \in (0,1)$, implying that the mean lifetime is infinite. A new phenomenon had been revealed: Starting from a homogeneous Poissonian system of particles, in *supercritical* dimensions the persistent limit in law is homogeneous *Poissonian* again, opposed to the usual non-Poissonian limits of systems of particles with finite expected lifetimes. (See also Proposition 3(b) below.) This has the following intuitive reason: Due to the long lifetimes, all the siblings of a particle in the limit population were born so long ago that they moved out of the finite window of observation. Therefore, only "completely mixed" populations can be observed in the limit.

Let us manifest this by some *heuristic calculation* based on backward tree considerations. Assume the particles' motion index is $\alpha \in (0,2]$, and the index of the branching is $1+\beta$ with $\beta \in (0,1]$ (see Hypothesis 1 below). By renewal theory ([Fel71, § 11.5 and § 11.3]), the number of branching points along an ancestral line of a particle, called "ego", picked at time t "at random" is asymptotically in law (as $t \uparrow \infty$) of the form κt^γ for some random variable $\kappa > 0$, and the time between t and *any* of these earlier branching time points is of order t (as opposed to the case of exponentially distributed branching times). Consider such a branching time point. The number $\beth$, say, of particles (additionally) generated at this branching point has generating function[1)] $\Phi(s) = 1 - c\,(1-s)^\beta$ (differentiate the critical offspring generating function f in (3) below to get this Palm offspring generating function Φ). Consider *any* of these $\beth$ offspring. Let q denote

[1)] With c we always denote a positive constant which may vary from place to place.

the probability that some of its descendants at time t hits a (fixed) ball B around "ego". The arguments in ([VW99, Lemmas 2 and 3]) yield that q decays at least with the order $t^{-\theta}$, where $\theta := (d/\alpha+\gamma)/(1+\beta)$. Therefore, the probability that none of the $\beth$ particles has an offspring that populates B at time t is asymptotically $\Phi(1-q) = 1 - c\,q^{\beta} \geq 1 - c\,t^{-(d/\alpha+\gamma)\beta/(1+\beta)}$. Consequently, the probability that none of the relatives of "ego" populates a ball around "ego" asymptotically equals

$$\begin{aligned} \mathbf{E}\big(1 - c\,t^{(-\frac{d}{\alpha}-\gamma)\frac{\beta}{1+\beta}}\big)^{\kappa t^{\gamma}} \;&\approx\; \mathbf{E}\exp\big[-\,c\,\kappa t^{\gamma-(\frac{d}{\alpha}+\gamma)\frac{\beta}{1+\beta}}\big] \\ &=\; \mathbf{E}\exp\big[-\,c\,\kappa t^{(\gamma-\frac{d\beta}{\alpha})\frac{1}{1+\beta}}\big]. \end{aligned} \tag{1}$$

This indicates that the number $d_{\mathrm{c}} := \alpha\gamma/\beta$ should be critical for the dimension d of space. Indeed, for $d > d_{\mathrm{c}}$, the expression in (1) converges to 1, showing that in supercritical dimensions "ego" asymptotically has no relatives around. In other words, relationships of particles in our observation window vanish as $t \uparrow \infty$, which corresponds to a convergence of the original populations towards a Poissonian system (with deterministic intensity).

If $d < d_{\mathrm{c}}$ instead, then (1) converges to 0. In other words, provided that $t^{-\theta}$ is the correct order of decay of q, then for $d < d_{\mathrm{c}}$ each small ball around "ego" is met by a relative of "ego" with asymptotic probability one. This can be taken as an indication for the usual clumping of the original particle system in subcritical dimensions, and is in fact in line with the local extinction of the branching particle system for $d < d_{\mathrm{c}}$ which was proved in [VW99] (see also Proposition 3(a) below).

At the critical dimension d_{c} however, (1) suggests that at late times t the probability that relatives of "ego" show up in B might be strictly between 0 and 1. Since the latest common ancestor of any of these relatives lived way back in the past of order t, the spatial correlations of these relatives should vanish as $t \uparrow \infty$. In fact the population should tend to a homogeneous mixed Poissonian system, where the randomness of the intensity comes from the randomness of the branching at very early times.

In this paper we prove that such a picture indeed is true, and hereby essentially strengthen a result of ([VW99]) where it was shown that at the critical dimension there exist non-trivial limit points as $t \uparrow \infty$. Here we now prove uniqueness of the limit point and identify it. According to our *main result* the following convergence statement holds for the process $Z = \{Z_t : t \geq 0\}$ under consideration. At the critical dimension, Z_t converges in law as $t \uparrow \infty$, to a limit Z_∞ which is in fact a *mixed homogeneous Poissonian* particle system whose random intensity i_∞ is non-degenerate and of full expectation (*persistence*). In particular, the limit is *not (spatially) ergodic*. (See Theorem 11 below.)

To gain the needed information on the random intensity i_∞, we need to relate our branching particle system Z with a superprocess. This first of all requires to pass to a Markovian description. To get this, we take into account the *residual lifetimes* of the particles. Then the refined process $\hat{Z}$ has states in the set of counting measures on $\mathsf{R}_+ \times \mathsf{R}^d$, instead only on R^d.

A rescaling $\hat{Z}^{(n)}$ of $\hat{Z}$ then leads to a *superprocess* $\hat{X}$ on $\mathsf{R}_+ \times \mathsf{R}^d$. The scaling here is as follows: Time is speeded up by the factor $n^{\beta/\gamma}$, whereas mass is renormalized by $1/n$, which also requires to increase the original particles' initial state by a factor of n (high density scaling). Now one has indeed the convergence in law $\hat{Z}^{(n)} \to \hat{X}$ as $n \uparrow \infty$ (see Proposition 6 below). Going back to the marginal measures on R^d, the superprocess $\hat{X}$ is turned into a measure-valued process X. This process appeared in [KS98] under the name "projected superprocess"; one should note however, that X is not Markov, hence not a superprocess in the usual terminology. Adapting a criterion of [Kle00b] to our needs, we will show that the process X has *absolutely continuous states* $X_t(\mathrm{d}a) = X_t(a)\,\mathrm{d}a$ in all dimensions $d < \alpha/\beta$, covering the critical dimension d_{c} (Proposition 7 below). This corresponds to the intuition that the long lifetimes of the infinitesimal particles combined with the spatial spread have an additional smoothing effect on the mass distribution (compared with the case of lifetimes with finite mean). At the same time, at the critical dimension d_{c}, starting from a marked Lebesgue initial state, $\hat{X}$ has a *self-similarity* property (Proposition 8(b) below). For $d = d_{\mathrm{c}}$, this turns the absolute continuity of the states of X into the existence of a large time limit X_∞ which is a random multiple $i_\infty \ell$ of the Lebesgue measure, and where by self-similarity, i_∞ coincides in law with $X_1(0)$, the random density of X_1 at the origin (see Corollary 10(b) below). Finally, it is well-known that in the case of exponentially distributed lifetimes the branching particle systems starting from Poisson initial states arise from the corresponding superprocesses via a Poissonization (cf. [GRW90, p.277]): The branching particle state at time t coincides in law with a mixed Poisson system, where the "Poisson's random intensity measure" is given by the superprocess state at time t. In our non-Markovian case of long living particles, an *asymptotic local Poissonization* result still holds, starting from a large pile of particles at a distant position (Corollary 14 below): The limit is mixed Poisson with random intensity measure $X_\infty = i_\infty \ell$. Some efforts are needed to make this key idea working.

The *main motivation* for the present work was to settle the problem of long-term convergence in the critical dimension which was left open in [VW99]. Moreover, the limit had to be identified. This required first of all to develop a superprocess setting related to the particle model.

It turns out that at the critical dimension the branching systems Z with long living particles and there continuum versions X share some crucial features with some *catalytic* branching models, see [GKW99] and [FK99]. (For recent surveys on catalytic models, see [DF00a, DF00b, Kle00a].) These features are: absolutely continuous states and self-similarity in the continuum versions leading to uniform limits $X_\infty = i_\infty \ell$ with non-degenerate random intensity $i_\infty = X_1(0)$. These continuum model results are then used to handle the more complicated particle models by showing that a *sparse branching* as $t \uparrow \infty$ (by the escaping catalyst respectively the particles' long life times) leads to an asymptotic local Poissonization, that is to mixed Poissonian limits with random intensity measure $X_\infty = i_\infty \ell$.

Let us now introduce the mentioned (refined) branching particle system $\hat{Z}$ in more detail.

1.2 The $(d, \alpha, \beta, \gamma)$–branching particle system $\hat{Z}$

We are dealing with a model of (critical) branching in R^d which is a spatial generalization of the so-called age-dependent (critical) branching process or (critical) Bellman-Harris branching process. It is based on the following ingredients which, for convenience, we expose as a hypothesis.

Hypothesis 1 (Ingredients of the branching particle system)

(a) (Particle motion process ξ) Fix a constant $\alpha \in (0, 2]$. Consider the symmetric α–*stable* process $(\xi,\, P_a\,,\ a \in \mathsf{R}^d)$ in R^d, (cf. [Bre68, p.317] or [Ber96, Ch. VIII]), that is, the (time-homogeneous) Markov process with generator $\Delta_\alpha := -(-\Delta)^{\alpha/2}$, the fractional Laplacian ([Yos74, p.260]), and with càdlàg paths. Denote by $\mathrm{p} = \{\mathrm{p}_t(b) :\ t > 0,\ b \in \mathsf{R}^d\}$ the continuous transition densities of this *particle motion process* ξ.

(b) (Particles' lifetime τ) Introduce the non-lattice distribution function G of a random variable $\tau > 0$ with tail

$$P(\tau > u) \ = \ 1 - G\,(u) \ \sim \ c_G\, u^{-\gamma} \quad \text{as } \ u \uparrow \infty \tag{2}$$

for some index $\gamma \in (0, 1)$ and a constant $c_G > 0$. That is, the *lifetime distribution* G of particles is in the normal domain of attraction of a stable law of index γ. In particular, τ has infinite expectation $E\tau = \infty$.

(c) (Critical branching mechanism) Fix attention to the *offspring generating function*

$$f(s) \ := \ Es^{\,\zeta} \ = \ s + c_f\,(1-s)^{1+\beta}\,, \qquad 0 \le s \le 1, \tag{3}$$

of the random number ζ of offspring of a particle, with constants $\beta \in (0, 1]$ and $c_f \in \big(0, \frac{1}{1+\beta}\big]$. That is,

$$P(\zeta = k) \ = \ \delta_{1,k} \ + \ c_f\,(-1)^k \binom{1+\beta}{k}, \qquad k \ge 0, \tag{4}$$

with $\delta_{1,k}$ the Kronecker symbol. Consequently, $E\zeta = 1$ (*criticality*), and we consider a branching mechanism in the normal domain attraction of a stable law of index $1 + \beta$. Note that $E\zeta^2 < \infty$ if and only if $\beta = 1$.

(d) (Phase space E) A point $e = (u, a) \in \mathsf{E} := \mathsf{R}_+ \times \mathsf{R}^d$ describes the *residual lifetime* u and the *position* a of a particle. With the metric

$$\mathrm{d}_{\mathsf{E}}(e_1, e_2) \ := \ 1 \wedge |u_1 - u_2| \ + \ |a_1 - a_2|, \tag{5}$$

$e_i = (u_i\,, a_i) \in \mathsf{E},\ i = 1, 2,$ we get a Polish space $(\mathsf{E}, \mathrm{d}_{\mathsf{E}})$.

(e) (Test functions) Fix a constant $p \in (d, d+\alpha]$ (recall that α is the motion index), and introduce the reference function

$$\phi_p(a) := \left(1+|a|^2\right)^{-p/2}, \qquad a \in \mathsf{R}^d. \tag{6}$$

Let $\hat{\mathcal{C}}_p = \hat{\mathcal{C}}_p(\mathsf{E})$ denote the set of all continuous functions $\psi : \mathsf{E} \to \mathsf{R}$ such that

$$\|\psi\| := \sup_{(u,a)\in\mathsf{E}} \left|\frac{\psi(u,a)}{\phi_p(a)}\right| < \infty, \tag{7}$$

and such that the map

$$(u,a) \mapsto \frac{\psi(u,a)}{\phi_p(a)} \quad \text{on } \mathsf{E} \tag{8}$$

can continuously be extended to a function on $\dot{\mathsf{R}}_+ \times \dot{\mathsf{R}}^d$, where $\dot{\mathsf{R}}_+$ and $\dot{\mathsf{R}}^d$ are the one-point compactifications of R_+ and R^d, respectively. Then $\left(\hat{\mathcal{C}}_p, \|\cdot\|\right)$ is a separable Banach space.

(f) (State space $\mathcal{N}_p^\circ$) Let $\mathcal{M}_p = \mathcal{M}_p(\mathsf{E})$ denote the set of all *p–tempered measures* on $\mathsf{E} = \mathsf{R}_+ \times \mathsf{R}^d$, that is measures μ on E such that the integral

$$\langle \mu, \phi_p \rangle := \int_{\mathsf{E}} \mu\big(\mathrm{d}(u,a)\big)\, \phi_p(a) \tag{9}$$

is finite. Introduce the weakest topology in $\mathcal{M}_p$ such that for each $\psi \in \hat{\mathcal{C}}_p$ the mapping

$$\mu \mapsto \langle \mu, \psi \rangle := \int_{\mathsf{E}} \mu(\mathrm{d}e)\, \psi(e) \tag{10}$$

is continuous. Note that $\nu \times \ell$ belongs to $\mathcal{M}_p$ for each finite measure ν on R_+ (since $\langle \ell, \phi_p \rangle < \infty$). Write $\mathcal{N}_p = \mathcal{N}_p(\mathsf{E})$ for the subset of all those measures μ in $\mathcal{M}_p$ with values in $\{0, 1, ..., \infty\}$ (counting measures), and $\mathcal{N}_p^\circ = \mathcal{N}_p^\circ(\mathsf{E})$ if additionally $\mu\left(\{0\} \times \mathsf{R}^d\right) = 0$. We let both sets $\mathcal{N}_p$ and $\mathcal{N}_p^\circ$ inherit the topologies of $\mathcal{M}_p$. The set $\mathcal{N}_p^\circ$ will serve as a *state space* of our branching particle system. In particular, the Dirac delta measure $\delta_{(u,a)} \in \mathcal{N}_p^\circ$ describes a single particle having residual lifetime $u > 0$ and sitting at position $a \in \mathsf{R}^d$. ◇

Definition 2 (Branching particle system $\hat{Z}$) Abstaining from a more detailed definition, the process $\hat{Z} = \left\{\hat{Z}_t : t \geq 0\right\}$ we are interested in can now be described by the following properties:

- Given a particle $\delta_{(u,a)} \in \mathcal{N}_p^\circ$ at time $r \geq 0$, its further path in $(0,\infty) \times \mathsf{R}^d$ is $t \mapsto \left(u-(t-r),\, \xi_{t-r}\right)$, $r \leq t < r+u$, where ξ is distributed according to P_a.

- As soon as a particle reaches the residual lifetime $0+$, it dies, and on the same location an offspring is born whose number is an independent copy of ζ.
- Newly born particles get independent residual lifetimes, all distributed as $\tau > 0$, and proceed independently.
- At time $t = 0$, we start with a system of particles described by a measure $\mu \in \mathcal{N}_p^\circ$.
- Write $\mathbf{P}_\mu$ for the law of $\hat{Z}$. It is considered as a measure on the set $\mathcal{D}(\mathsf{R}_+, \mathcal{N}_p)$ of all $\mathcal{N}_p$–valued *càdlàg* paths ω satisfying additionally $\omega_t \in \mathcal{N}_p^\circ$, $t \geq 0$.

For convenience, we call this process $(\hat{Z}, \mathbf{P}_\mu, \mu \in \mathcal{N}_p^\circ)$ a $(d, \alpha, \beta, \gamma)$–*branching particle system*. Note that $\hat{Z}$ is a time-homogeneous Markov process.

Note also that we put maximal independence assumptions in defining the model. The main dependence assumption is that newly born particles start their evolution from the ancestor's death place.

Integrating out the residual lifetimes, we get back the non-Markovian process

$$Z_t \;=\; \hat{Z}_t\big(\mathsf{R}_+ \times (\cdot)\big), \qquad t \geq 0, \tag{11}$$

already mentioned in Subsection 1.1. ◇

We are interested in the long-term behavior of the marginal process Z of this $(d, \alpha, \beta, \gamma)$–branching particle system $\hat{Z}$. For simplicity, we take Poissonian particle systems with intensity measure[2)] $i_0\, G(\mathrm{d}u)\ell(\mathrm{d}a)$ as initial states, where $i_0 > 0$ is a fixed constant. Note that in this case all the initial residual lifetimes are independent copies of the random variable τ, which means that all initial particles are in effect newborns (but do not have siblings).

Recall that a random counting measure π on E is called a *Poissonian* particle system with intensity measure $\mu \in \mathcal{M}_p$ if it has the log-Laplace transform

$$-\log \mathbf{E} \exp\langle \pi, -\psi \rangle \;=\; \big\langle \mu, 1 - \mathrm{e}^{-\psi} \big\rangle, \qquad \psi \in \hat{\mathcal{C}}_p^+, \tag{12}$$

[the index $+$ on a set refers to all of its non-negative members, as we already used $\mathsf{R}_+ = [0, \infty)$]. We write π_μ for such a Poissonian particle system with intensity measure μ. If $\mu = i_0 \nu \times \ell$ with a constant $i_0 \geq 0$, a probability law ν on $(0, \infty)$ (and ℓ the Lebesgue measure on R^d), then $\pi_{i_0 \nu \times \ell}$ is a *homogeneous* Poissonian particle system with intensity i_0 and residual lifetimes distributed according to ν. If i_0 is additionally random,

[2)] We do not distinguish in notation between the distribution function G and its related probability measure $G(\mathrm{d}t)$ on $(0, \infty)$.

then $\pi_{i_0\,\nu\times\ell}$ is said to be a *mixed homogeneous Poissonian* particle system with random intensity i_0. In other words, if σ denotes the law of i_0, then

$$\mathbf{P}\big(\pi_{i_0\,\nu\times\ell}\in(\cdot)\big) = \int_0^\infty \sigma(\mathrm{d}i)\,\mathbf{P}\big(\pi_{i\nu\times\ell}\in(\cdot)\big)\,. \tag{13}$$

A similar terminology is used for Poissonian particle systems on R^d.

1.3 Digression: Lifetimes with finite mean and the extinction-persistence dichotomy

Let us assume that $\hat{Z}_0$ is the homogeneous Poissonian particle system $\pi_{i_0\,G\times\ell}$ with (deterministic) intensity $i_0 > 0$ and residual lifetime distributed according to G (recall that in this case only newborns without siblings are considered in the beginning).

First we contrast our marginal process Z of a (d,α,β,γ)–branching particle system $\hat{Z}$ with the case where the assumption (2) on a γ–tail is replaced by that of a (non-lattice) lifetime distribution function G with *finite* expectation. Then there is a dichotomy between persistent convergence and local extinction depending on whether $d > \frac{\alpha}{\beta}$ holds or is violated (cf. [VW99]). This kind of picture is of course known from other variants of spatially homogeneous (critical) branching processes, see, for instance, [Lie69, Daw77, Kal77, DF85, Fle88], and [GW91]. An intuitive reason for this dichotomy is the following: By the critical branching, the offspring of any considered finite subpopulation always goes to extinction. Because of the smaller mobility in low dimensions, this effect leads finally to a local extinction of the infinite population. In large dimensions, however, the higher mobility allows some of the particles (coming from far away) to enter the finite window of observation and thus to show up in the limit population. Consequently, in low dimensions the local fluctuations (coming from the critical branching) prevail, whereas in higher dimensions the effect of the motion is dominating. Altogether, it depends on the dimension of space and the indices of branching and motion, which of the competing features (local mass fluctuation or spatial dispersion) wins in the long run.

The same statement holds true for (d,α,β,γ)–branching particle system: below the critical dimension $d_{\mathrm{c}} = \alpha\gamma/\beta$ there is local extinction, and above it there is persistence (see Subsection 1.4 below). However, at the critical dimension itself, the behavior is completely different from that of systems with finite expected lifetime of particles (see Subsection 1.9).

1.4 Digression: Long living particles in non-critical dimensions

Let us return to our (d,α,β,γ)–branching particle system $\hat{Z}$ according to Definition 2. Before coming to our main case, we will briefly recall the picture in non-critical dimensions. The following proposition is proved in

[SW93] and [VW99] as far as the spatial marginal process X is concerned. Here we state an extension which includes the residual life times. The proof of this extension is easy and will be omitted.

Proposition 3 (Long-term behavior in non-critical dimensions) *Consider the $(d, \alpha, \beta, \gamma)$–branching particle system $\hat{Z}$ starting from a homogeneous Poissonian particle system with intensity $i_0 > 0$ and with residual lifetimes distributed according to G, that is, $\hat{Z}_0 = \pi_{i_0\, G\times\ell}$. For a constant $K > 0$, introduce the time-scaled process $\hat{Z}^{(K)}$:*

$$\hat{Z}^{(K)}_t\big((\,\cdot\,)\times(\,\cdot\,)\big) \;:=\; \hat{Z}_{Kt}\big(K(\,\cdot\,)\times(\,\cdot\,)\big)\,, \qquad t \geq 0, \tag{14}$$

and fix $t > 0$. Then the following statements hold.

(a) (Local extinction in subcritical dimensions) ([VW99]) *If $d < \frac{\alpha\gamma}{\beta}$, then $\hat{Z}^{(K)}_t$ suffers local extinction as $K \uparrow \infty$, that is, $\hat{Z}^{(K)}_t \to 0$ in probability.*

(b) (Persistent convergence in supercritical dimensions) ([SW93, VW99]) *On the other hand, if $d > \frac{\alpha\gamma}{\beta}$, then $\hat{Z}^{(K)}_t$ converges in law as $K \uparrow \infty$ to a homogeneous Poissonian limit population $\hat{Z}^{(\infty)}_t = \pi_{i_0\, G^t_\infty\times\ell}$ of full intensity i_0 and with residual lifetimes distributed according to*

$$G^t_\infty(\mathrm{d}u) \;:=\; \frac{\sin\pi\gamma}{\pi}\,\frac{t^\gamma\,\mathrm{d}u}{u^\gamma\,(t+u)}\,, \qquad u > 0, \tag{15}$$

(for the occurrence of G^t_∞, see [Fel71, *Theorem in* §14.3]).

As pointed out already in Subsection 1.1, the striking new feature relative to the case of a lifetime distribution with finite mean (Subsection 1.3) is that in supercritical dimensions the local dependencies between relatives are lost in the limit. This is caused by the heavy tail of the lifetime distribution G of the particles – siblings are too far away in the long run.

According to a result of [VW99], at the critical dimension $d_{\mathrm{c}} = \alpha\gamma/\beta$ local extinction in law of Z_t as $t \uparrow \infty$ does *not* hold (recall our discussion in Subsection 1.1). Our purpose is to enlighten this situation. In order to describe this in detail, we first need to introduce a superprocess variant $\hat{X}$ of the $(d, \alpha, \beta, \gamma)$–branching particle system $\hat{Z}$, since the detailed description of the long-term behavior of Z depends on a quantity derived from $\hat{X}$.

1.5 The $(d, \alpha, \beta, \gamma)$–superprocess $\hat{X}$

Since [Fel51], it is common to ask also for diffusion type approximations of branching particle systems. For our present $(d, \alpha, \beta, \gamma)$–model, at least starting with a finite initial system, such approximation had been provided

by [KS98], who dropped, however, the residual lifetimes from the description. In this way one looses the Markov property, in fact in both the particle systems as well as in their high density limits (despite the independence assumptions between motion and aging). In other words, one never will end up with a Markov superprocess (not even with a time-inhomogeneous one, contrasting a statement on p.249 in [KS98]). This is the reason why we insist in keeping the residual lifetimes in our description also in the superprocess limit.

In order to introduce the limiting $(d, \alpha, \beta, \gamma)$–superprocess $\hat{X}$, we first need to describe the basic "motion" process for this superprocess.

Recall that we are working with the phase space $\mathsf{E} = \mathsf{R}_+ \times \mathsf{R}^d$. In the R^d–component, we are keeping the particles' α–stable motion process $(\xi, P_a\,,\ a \in \mathsf{R}^d)$. Additionally we need an independent (limiting) residual lifetime process $t \mapsto \vartheta_t$. To introduce it, for our life time tail index $0 < \gamma < 1$, we start from an independent *γ–stable subordinator* $\eta = \{\eta_t : t \geq 0\}$ in R_+ which is a càdlàg time-homogeneous Markov process with stationary independent increments having log-Laplace transition function

$$-\log E\left\{\exp\left[-\theta\eta_t\right] \,\middle|\, \eta_0 = u\right\} \;=\; u\theta + c_\eta\, t\, \theta^\gamma, \qquad \theta, t, u \geq 0, \tag{16}$$

where it is convenient for us to normalize the constant $c_\eta > 0$ to

$$c_\eta \;:=\; 1/\Gamma(1+\gamma), \tag{17}$$

where Γ denotes the Gamma function. From this we deduce two other processes, the process

$$k_t \;:=\; \inf\{s > 0 :\ \eta_s > t\}, \qquad t \geq 0, \tag{18}$$

inverse to η, and the (limiting) *residual lifetime process*

$$\vartheta_t \;:=\; \inf\left\{\eta_s - t :\ \eta_s > t,\ s \geq 0\right\} \;=\; \eta_{k_t} - t, \qquad t \geq 0. \tag{19}$$

Note that ϑ is a càdlàg time-homogeneous strong Markov process, and that $\mathrm{d}k_t$ is a continuous and time-homogeneous additive functional of ϑ. (See [Ber96, p.123] for background.) In particular, ϑ can be started from $\vartheta_0 = 0$. Also recall that in the special case $\gamma = \frac{1}{2}$ the process $t \mapsto k_{\eta_0 + t}$ is just the local time at 0 of a Brownian motion started at 0, and ϑ describes the residual lifetimes of the Brownian excursions from 0.

We pass now to the pair $\hat{\xi} := (\vartheta, \xi)$ of independent processes whose laws we denote by $\hat{P}_e\,,\ e \in \mathsf{E}$, thus arriving at a càdlàg time-homogeneous Markov process $(\hat{\xi}, \hat{P}_e\,,\ e \in \mathsf{E})$ in $\mathsf{E} = \mathsf{R}_+ \times \mathsf{R}^d$. This $\hat{\xi}$ will serve as the "motion" process of an intrinsic particle in the superprocess $\hat{X}$ which we now will introduce. The time-homogeneous continuous additive functional $\mathrm{d}k_t$ of $\hat{\xi}$ will serve as the branching functional for $\hat{X}$.

Recall that $\mathcal{D} = \mathcal{D}(\mathsf{R}_+\,, \mathcal{M}_p)$ denotes the set of all càdlàg measure-valued paths $\omega : \mathsf{R}_+ \to \mathcal{M}_p = \mathcal{M}_p(\mathsf{E})$ endowed with the Skorohod topology. Write $\mathcal{P}$ for the set of all probability laws on $\mathcal{D}$ furnished with the topology of weak convergence.

Lemma 4 (Unique existence of the (d,α,β,γ)–superprocess $\hat{X}$) *Fix constants*

$$0<\alpha\le 2,\quad 0<\beta\le 1,\quad 0<\gamma<1,\quad \textit{and}\quad \varrho>0. \tag{20}$$

To each measure $\mu\in\mathcal{M}_p$, there is a unique law $\mathbb{P}_\mu\in\mathcal{P}$ of a time-homogeneous Markov process $\hat{X}$ with log-Laplace transition functional

$$-\log\mathbb{E}_\mu\exp\langle\hat{X}_t\,,-\psi\rangle\;=\;\langle\mu,\hat{V}_t\psi\rangle,\qquad t\ge 0,\quad \psi\in\hat{\mathcal{C}}_p^+, \tag{21}$$

where the function $\hat{v}=\hat{V}\psi=\{\hat{V}_t\psi\,(e):\ t\ge 0,\ e\in\mathsf{E}\}\ge 0$ uniquely solves the integral equation

$$\hat{v}_t(e)\;=\;\hat{E}_e\Big[\psi(\hat{\xi}_t)\;-\;\varrho\int_0^t \mathrm{d}k_s\;\hat{v}_{t-s}^{1+\beta}(\hat{\xi}_s)\Big],\qquad t\ge 0,\quad e\in\mathsf{E}. \tag{22}$$

In particular, the following expectation formula holds:

$$\mathbb{E}_\mu\langle\hat{X}_t\,,\psi\rangle\;=\;\int_{\mathsf{E}}\mu(\mathrm{d}e)\,\hat{E}_e\psi(\hat{\xi}_t)\qquad \mu\in\mathcal{M}_p\,,\quad t\ge 0,\quad \psi\in\hat{\mathcal{C}}_p^+, \tag{23}$$

(criticality).

We call $(\hat{X},\,\mathbb{P}_\mu,\,\mu\in\mathcal{M}_p)$ the (d,α,β,γ)–*superprocess* with branching rate ϱ. The construction of such a process is nowadays standard and we will abstain from this (see [Sch99]).

A good interpretation of this measure-valued process $\hat{X}$ is provided by the particle system approximation worked out in the next subsection.

Remark 5 (Decoupling) Note that the integral in (22) can be rewritten as

$$\int_0^t \mathrm{d}k_s\;\hat{v}_{t-s}^{1+\beta}(0,\xi_s) \tag{24}$$

since

$$\int_0^\infty \mathrm{d}k_s\;1\{\vartheta_s>0\}\;=\;0,\quad \hat{P}_e\text{–a.s.},\ e\in\mathsf{E}, \tag{25}$$

which follows just from the definitions of the processes k and ϑ. But k and ξ are independent, hence from

$$\hat{E}_e k_s\;=\;E_u k_s\;=\;1_{[u,\infty)}(s)\,(s-u)^\gamma,\qquad e=(u,a),\quad s\ge 0, \tag{26}$$

we conclude that (22) can be rewritten as [3)]

$$\hat{v}_t(e)\;=\;\big[\hat{E}_e\psi(\hat{\xi}_t)\big]\;-\;\varrho\int_{u\wedge t}^t \mathrm{d}_s(s-u)^\gamma\;E_a\,\hat{v}_{t-s}^{1+\beta}(0,\xi_s), \tag{27}$$

$t\ge 0,\quad e=(u,a)\in\mathsf{E}.$ ◇

[3)] Here d_s indicates that the Stieltjes integral is formed with respect to the variable s [of the monotone function $s\mapsto(s-u)^\gamma$].

1.6 Approximation of $\hat{X}$ by particle systems

As already indicated, $\hat{X}$ arises from the $(d, \alpha, \beta, \gamma)$–branching particle system $\hat{Z}$ via a diffusion type approximation, which we now want to make precise. The proof of the next proposition follows the lines of [KS98], Section 4, see in particular also Corollary 1 there.

Proposition 6 (Approximation of $\hat{X}$ by particle systems) *For $n \geq 1$, let $\hat{Z}^{(n)}$ denote a $(d, \alpha, \beta, \gamma)$–branching particle system but with lifetime distribution function G of Hypothesis 1(b) replaced by $G^{(n)}$:*

$$G^{(n)}(u) := G\big(n^{\beta/\gamma}\, u\big), \qquad u \geq 0. \tag{28}$$

Moreover, assume that the $\hat{Z}^{(n)}$ start with (deterministic) initial populations $\hat{Z}_0^{(n)} \in \mathcal{N}_p^\circ$ satisfying

$$\frac{1}{n}\, \hat{Z}_0^{(n)} \underset{n\uparrow\infty}{\longrightarrow} \mu \quad \text{in } \mathcal{M}_p\,. \tag{29}$$

Then

$$\frac{1}{n}\, \hat{Z}^{(n)} \underset{n\uparrow\infty}{\Longrightarrow} \hat{X} \quad \text{in law,} \tag{30}$$

where $\hat{X}$ is the $(d, \alpha, \beta, \gamma)$–superprocess from Lemma 4 starting from μ, and with special branching rate

$$\varrho \;=\; \frac{\sin \pi\gamma}{\pi}\, \frac{c_f}{c_G} \tag{31}$$

with c_f from (3) and c_G from (2).

In simple terms, if in the $(d, \alpha, \beta, \gamma)$–branching particle system $\hat{Z}$ the particles' lifetimes τ are replaced by the rescaled (and smaller) lifetimes $n^{-\beta/\gamma}\tau$, if their unit masses are replaced by the small masses $\frac{1}{n}$, and if the initial populations $\hat{Z}_0^{(n)}$ are chosen such that the rescaled measures $\frac{1}{n}\, \hat{Z}_0^{(n)}$ converge as in (29), then the whole rescaled branching particle systems $\frac{1}{n}\, \hat{Z}^{(n)}$ tend to the $(d, \alpha, \beta, \gamma)$–superprocess $\hat{X}$ of Lemma 4 with special branching rate ϱ as defined in (31).

1.7 The marginal measure process X

The residual lifetime process ϑ, starting from a non-zero state, is *deterministic* until it reaches the state 0. Thus, the "motion" process $\hat{\xi}$ does not have transition densities, hence the superprocess $\hat{X}$ will also not have "smooth" states. But integrating out the residual lifetimes, that is, passing to the spatial marginal process

$$X_t(\cdot) := \hat{X}_t\big(\mathsf{R}_+ \times (\cdot)\big), \qquad t \geq 0, \tag{32}$$

the situation changes, and this will be enough for our description for the long-term behavior of $\hat{Z}$. In fact, in dimensions $d < \frac{\alpha}{\beta}$, the measure states X_t are absolutely continuous. This property we want to expose as a proposition, which we will verify in Subsection 2.2 below by a modification of methods borrowed from [Kle00b]. For this purpose, let $\chi \geq 0$ denote a bounded continuous function on R^d with $\langle \ell, \chi \rangle = 1$, and consider the regularization

$$\chi_\varepsilon(a) \;:=\; \varepsilon^{-d} \chi(\varepsilon^{-1} a), \qquad a \in \mathsf{R}^d, \quad \varepsilon > 0, \quad \varepsilon \downarrow 0, \tag{33}$$

of the δ–function δ_0 on R^d.

Proposition 7 (Marginal measure process X) *Let $d < \frac{\alpha}{\beta}$, and fix $t > 0$. Consider the marginal process $X = \hat{X}\big(\mathsf{R}_+ \times (\cdot)\big)$ of the $(d, \alpha, \beta, \gamma)$–superprocess $\hat{X}$ with law $\mathbb{P}_\mu$, $\mu \in \mathcal{M}_p(\mathsf{E})$.*

(a) (Random density at a point) *For all $a \in \mathsf{R}^d$, the limit in law*

$$\lim_{\varepsilon \downarrow 0} \langle X_t \,, \delta_a * \chi_\varepsilon \rangle \;=:\; X_t(a) = \langle X_t \,, \delta_a \rangle \tag{34}$$

exists. Moreover, the random density $X_t(a)$ of X_t at site a has the log-Laplace transform

$$-\log \mathbb{E}_\mu \exp\big[-\theta X_t(a)\big] \;=\; \big\langle \mu, \hat{v}_t^{\theta,a} \big\rangle, \qquad \theta \geq 0, \tag{35}$$

where (for $\theta \geq 0$ and $a \in \mathsf{R}^d$ fixed)

$$\hat{v}^{\theta,a} \;=\; \hat{v} \;=\; \big\{ \hat{v}_t(e') : \;\; t > 0, \;\; e' = (u', a') \in \mathsf{E} \big\} \geq 0 \tag{36}$$

uniquely solves the equation

$$\hat{v}_t(e') \;=\; \theta \, \mathrm{p}_t(a - a') \;-\; \varrho \, \hat{E}_{e'} \int_0^t \mathrm{d}k_s \; \hat{v}_{t-s}^{1+\beta}(0, \xi_s), \tag{37}$$

$t > 0$, $e' = (u', a') \in \mathsf{E}$. In particular, the following expectation formula holds:

$$\mathbb{E}_\mu X_t(a) \;=\; \int_{\mathsf{E}} \mu\big(\mathrm{d}(u', a')\big) \, \mathrm{p}_t(a - a'). \tag{38}$$

(b) (Absolutely continuous measure states) *$\mathbb{P}_\mu$–almost surely, X_t is absolutely continuous.*

The occurrence of the first term at the right hand side of (37) is not surprising. In fact, proceeding formally,

$$X_t(a) \;=\; \langle X_t \,, \delta_a \rangle \;=\; \big\langle \hat{X}_t \,, 1 \otimes \delta_a \big\rangle \tag{39}$$

with δ_a the delta function at a and

$$1 \otimes \delta_a \;=:\; \overline{\delta_a}\,, \qquad a \in \mathsf{R}^d, \tag{40}$$

its constant extension to E. Moreover, replacing ψ in the log-Laplace equation (22) by the generalized function $\overline{\delta_a}$ on E, we get for $e' = (u', a') \in \mathsf{E}$,

$$\hat{E}_{e'}\,\psi(\hat{\xi}_t) \;=\; \hat{E}_{(u',a')} 1(\vartheta_t)\,\delta_a(\xi_t) \;=\; E_{a'}\,\delta_a(\xi_t) \;=\; \mathrm{p}_t(a - a'), \tag{41}$$

since ξ has continuous transition density function p. Thus, (37) can be considered as a special case of (22), and

$$\hat{v}^{\theta,a} \;=\; \hat{V}\big(\theta\overline{\delta_a}\big). \tag{42}$$

Of course, part of the statement in (a) is that (37) is a well-posed equation. Note also that $\overline{\delta_a}$ is the generalized density function of the measure $\ell_+ \times \delta_a$ on E, where ℓ_+ denotes the Lebesgue measure on R restricted to R_+, and we will not make a difference in notation between this measure and its generalized density function, that is, we write $\overline{\delta_a}$ for both.

Note also that the index γ does not enter into the dimension restriction $d < \frac{\alpha}{\beta}$ of this proposition. Roughly speaking, the enlargement of the lifetime of an "intrinsic particle" in the superprocess does not have an effect to the local issue of smoothness of the marginal measure states X_t. In particular, concerning absolute continuity of the states, there is no difference between the "classical" case corresponding to $\gamma = 1$ and our cases $\gamma < 1$. (Of course, one expects singularity of the marginal measure states if $d \geq \frac{\alpha}{\beta}$.)

We introduced the random densities $X_t(a)$, since they will show up as random intensities of certain mixed homogeneous Poissonian particle systems occurring in our main result (Theorem 11 below). But before returning to particle systems, we present some other interesting properties of the superprocess $\hat{X}$.

1.8 Scaling properties of $\hat{X}$ at the critical dimension

The next property follows from the log-Laplace representation in Lemma 4 by standard arguments (we skip the details).

Proposition 8 (Scaling properties of $\hat{X}$) *Consider $d = \frac{\alpha\gamma}{\beta}$. Then the $(d, \alpha, \beta, \gamma)$–superprocess has the following properties.*

(a) (Scaling for finite initial masses) *For each $a \in \mathsf{R}^d$ and $K, i_0 > 0$, the process*

$$\left\{ \Big(K^{-d/\alpha} \hat{X}_{Kt}\big(K(\,\cdot\,) \times K^{1/\alpha}(\,\cdot\,)\big) \Big)_{t \geq 0} \;\Big|\; \hat{X}_0 = i_0\,\delta_{(0,a)} \right\} \tag{43}$$

coincides in law with the process

$$\left\{ \hat{X} \;\Big|\; \hat{X}_0 = K^{-d/\alpha}\, i_0\, \delta_{\left(0, K^{-1/\alpha} a\right)} \right\}. \tag{44}$$

(b) (Self-similarity) *If instead* $\hat{X}_0 = i_0\,\delta_0 \times \ell$, *then* $\hat{X}$ *is self-similar:*

$$\Big\{K^{-d/\alpha}\hat{X}_{Kt}\big(K(\,\cdot\,)\times K^{1/\alpha}(\,\cdot\,)\big)\colon\ t\geq 0\Big\} \stackrel{\mathcal{L}}{=} \hat{X}, \qquad K>0. \tag{45}$$

Scaling properties are often a very useful tool. In the present case, we may combine them with Proposition 7, in order to get immediately the following result. Define $\hat{X}^{(K)}$ just as we introduced $\hat{Z}^{(K)}$ in (14). Recall the limiting residual lifetime distribution G^t_∞ introduced in (15).

Corollary 9 (Limiting scaling behavior of $\hat{X}$) *Let* $d = \frac{\alpha\gamma}{\beta}$, *and fix* $t, i_0 > 0$. *Then the* (d,α,β,γ)*–superprocess* $\hat{X}$ *has the following limiting scaling behavior.*

(a) (Asymptotics of $\hat{X}^{(K)}$ for finite initial masses) *Fix* $a \in \mathsf{R}^d$. *Assume that* $\hat{X}_0 = K^{d/\alpha}\,i_0\,\delta_{(0,K^{1/\alpha}a)}$. *Then*

$$\hat{X}^{(K)}_t \underset{K\uparrow\infty}{\Longrightarrow} X'_t(0)\,G^t_\infty \times \ell \quad \textit{in law,} \tag{46}$$

where $\hat{X}'$ *is the* (d,α,β,γ)*–superprocess starting from* $\hat{X}'_0 = i_0\,\delta_{(0,a)}$, *and* $X'_t(0)$ *is the random density at time* t *at the origin from the marginal process* $X' := \hat{X}'\big(\mathsf{R}_+ \times (\,\cdot\,)\big)$ *according to Proposition* 7(a).

(b) (Persistent convergence of $\hat{X}^{(K)}$) *If instead* $\hat{X}_0 = i_0\,\delta_0 \times \ell$, *then*

$$\hat{X}^{(K)}_t \underset{K\uparrow\infty}{\Longrightarrow} X_t(0)\,G^t_\infty \times \ell \quad \textit{in law,} \tag{47}$$

where $X_t(0)$ *is the random density at time* t *at site* 0 *of the marginal process* $X = \hat{X}\big(\mathsf{R}_+ \times (\cdot)\big)$, *having full expectation* i_0.

Specifying first to $t = 1$ and then to $K = t$ in the previous corollary and paying attention only to the spatial marginal measures leads to the following result.

Corollary 10 (Long-term behavior of X) *Let* $d = \frac{\alpha\gamma}{\beta}$. *The marginal process* X *of the* (d,α,β,γ)*–superprocess* $\hat{X}$ *has the following long-term behavior. Fix* $i_0 > 0$.

(a) (Asymptotics of X_t for t–dependent finite initial measures) *Consider a whole family* $\{{}^t\hat{X} : t \geq 0\}$ *of* (d,α,β,γ)*–superprocesses starting from* ${}^t\hat{X}_0 = t^{d/\alpha}\,i_0\,\delta_{(0,t^{1/\alpha}a)}$, $a \in \mathsf{R}^d$. *Then the spatial marginal processes* ${}^tX := {}^t\hat{X}\big(\mathsf{R}_+ \times (\,\cdot\,)\big)$ *obey*

$${}^tX_t \underset{t\uparrow\infty}{\Longrightarrow} X'_1(0)\,\ell \quad \textit{in law,} \tag{48}$$

where $X'_1(0)$ *is the random density at time* 1 *at the origin of the marginal process* $X' := \hat{X}'\big(\mathsf{R}_+ \times (\,\cdot\,)\big)$ *(see Proposition* 7(a)*) of the* (d,α,β,γ)*–superprocess* $\hat{X}'$ *starting from* $\hat{X}'_0 = i_0\,\delta_{(0,a)}$.

(b) (Persistent convergence of X_t) *If, instead,* $\hat{X}_0 = i_0\,\delta_0 \times \ell$, *then*

$$X_t := \hat{X}_t\,(\mathsf{R}_+ \times (\,\cdot\,)) \underset{t\uparrow\infty}{\Longrightarrow} X_1'(0)\,\ell \quad \textit{in law,} \tag{49}$$

where $X_1'(0)$ *is as in* (a) *but with* $\hat{X}_0' = i_0\,\delta_0 \times \ell$, *hence the limiting state has full expectation* $i_0\ell$.

Note that in the superprocess setting with $\hat{X}_0 = i_0\,\delta_0 \times \ell$ as in (b) it is easy to understand why in the critical dimension mixed Lebesgue measures occur in the spatial component in the limit as $K \uparrow \infty$, instead of the zero measure in the "classical" case of a (d, α, β)–superprocess. This behavior is caused by the coincidence that one has nice scaling properties *and* the existence of absolutely continuous states. While (d, α, β)–superprocesses have singular states at their critical dimension $d_{\mathrm{c}} = \frac{\alpha}{\beta}$, in contrast the spatial projections of $(d, \alpha, \beta, \gamma)$–superprocesses have absolutely continuous states at their critical dimension $d_{\mathrm{c}} = \frac{\alpha\gamma}{\beta}$ $(< \frac{\alpha}{\beta})$.

Recall also that the statement in Corollary 10(b) is quite analogous to a result on the super-Brownian reactant with a super-Brownian catalyst at the critical dimension $d = 2$, see [FK99]. (Note that also there an analog of part (a) of Corollary 10 can be established.)

1.9 Persistent convergence of Z at the critical dimension

Now we return to our marginal branching particle system $Z = \hat{Z}\big(\mathsf{R}_+ \times (\cdot)\big)$. First we state the analog of Corollary 10(b). Note that the following theorem deepens the statement in [VW99] that non-trivial limit points exist.

Theorem 11 (Persistent convergence to a mixed Poisson system) *Consider the* $(d, \alpha, \beta, \gamma)$–*branching particle system* $\hat{Z}$ *with* $\hat{Z}_0 = \pi_{i_0\,G\times\ell}$. *Assume that* $d = \frac{\alpha\gamma}{\beta}$. *Then*

$$Z_t \underset{t\uparrow\infty}{\Longrightarrow} \pi_{X_1(0)\,\ell} \quad \textit{in law,} \tag{50}$$

where $X_1(0)$ *is the random density at time* 1 *at site* 0 *of the marginal process* $X = \hat{X}\big(\mathsf{R}_+ \times (\cdot)\big)$ *of the* $(d, \alpha, \beta, \gamma)$–*superprocess* $\hat{X}$ *with* $\hat{X}_0 = i_0\,\delta_0 \times \ell$. *In particular, the limiting state has full expectation* $i_0\ell$ *and is not (spatially) ergodic.*

In other words, the limit population is mixed homogeneous Poisson, and its random intensity is just $X_1(0)$, the random density at time 1 at the origin of the marginal process X of the $(d, \alpha, \beta, \gamma)$–superprocess $\hat{X}$ starting from $i_0\,\delta_0 \times \ell$.

Theorem 11 will actually be derived from our next theorem, and this derivation will be provided at the end of the next subsection.

1.10 Refined asymptotics

The progress relative to [VW99], exhibited in Theorem 11, was possible by revealing a refined asymptotics as $K \uparrow \infty$ for the marginal process $Z^{(K)} := \hat{Z}^{(K)}\big(\mathsf{R}_+ \times (\cdot)\big)$ of the rescaled $(d, \alpha, \beta, \gamma)$–branching particle system $\hat{Z}^{(K)}$ defined in (14) and starting from *finite* and asymptotically large initial populations [the particle analog of Corollary 9(a)], which we now want to deal with. It can be counted as a spatial generalization of a result on age-dependent branching processes in [Vat86].

For this purpose, we need to introduce further notation. Let $\mathcal{C}_p = \mathcal{C}_p(\mathsf{R}^d)$ denote the set of all continuous functions $\varphi : \mathsf{R}^d \to \mathsf{R}$ such that $|\varphi| \le c_\varphi\, \phi_p$ for some constant c_φ and such that φ/ϕ_p can continuously be extended to a function on the one-point compactification $\dot{\mathsf{R}}^d$. Note that the

$$\text{constant extension } \bar{\varphi} := 1 \otimes \varphi \text{ of } \varphi \in \mathcal{C}_p \tag{51}$$

belongs to $\hat{\mathcal{C}}_p$.

The following notation, though potentially ambiguous at first glance, will be convenient.

Notation 12 (Mixed laws) We denote by $\mathbf{P}_{\delta_{(\tau,a)}}$ the ("annealed", deterministic) law $\int_0^\infty \mathrm{d}G(u)\, \mathbf{P}_{\delta_{(u,a)}}$ of $\bar{Z}$ started from a single particle with position a and residual life time τ distributed according to G. Also in other cases of mixed laws we will use a similar notation. For instance, we write $\hat{P}_{(\tau,a)}$ for $\int_0^\infty \mathrm{d}G(u)\, \hat{P}_{(u,a)}$. ◇

For $\varphi \in \mathcal{C}_p^+$, set

$$Q_t\varphi\,(a) \;:=\; 1 - \mathbf{E}_{\delta_{(\tau,a)}} \exp\big\langle \hat{Z}_t\,, -\bar{\varphi}\big\rangle \;=\; 1 - \mathbf{E}_{\delta_{(\tau,a)}} \exp\langle Z_t\,, -\varphi\rangle, \tag{52}$$

$t \ge 0,\ a \in \mathsf{R}^d$. Note that $Q_t\varphi\,(a)$ occurs in the log-Laplace functional of the state $\hat{Z}_t$ of the $(d, \alpha, \beta, \gamma)$–branching particle system starting from the homogeneous Poissonian particle system $\hat{Z}_0 = \pi_{i_0\, G\times\ell}$:

$$-\log \mathbf{E}_{\pi_{i_0\, G\times\ell}} \exp\big\langle \hat{Z}_t\,, -\bar{\varphi}\big\rangle \;=\; \langle i_0\,\ell, Q_t\varphi\rangle. \tag{53}$$

Applied to the rescaled process $\hat{Z}^{(K)}$ introduced in (14) we get

$$\begin{aligned} -\log \mathbf{E}_{\pi_{i_0\, G\times\ell}} \exp\big\langle \hat{Z}_t^{(K)}, -\bar{\varphi}\big\rangle \;&=\; -\log \mathbf{E}_{\pi_{i_0\, G\times\ell}} \exp\big\langle Z_t^{(K)}, -\varphi\big\rangle \\ &=\; i_0 \int_{\mathsf{R}^d} \mathrm{d}a\; K^{d/\alpha}\, Q_{Kt}\varphi\,(K^{1/\alpha} a). \end{aligned} \tag{54}$$

The asymptotics we are interested in concerns the quantity

$$V_t^{(K)}\varphi\,(a) \;:=\; K^{d/\alpha}\, Q_{Kt}\varphi\,(K^{1/\alpha} a) \tag{55}$$

occurring in the integrand in (54). Here is our *key result*:

Theorem 13 (Refined asymptotics) *Assume that* $d = \frac{\alpha\gamma}{\beta}$. *Then, for each* $\varphi \in \mathcal{C}_p^+$, $t > 0$, *and* $a \in \mathsf{R}^d$,

$$V_t^{(K)}\varphi\,(a) \underset{K\uparrow\infty}{\longrightarrow} v_t^{\theta(\varphi)}(a), \tag{56}$$

where

$$\theta(\varphi) := \int_{\mathsf{R}^d} \mathrm{d}b\,\left[1 - \mathrm{e}^{-\varphi(b)}\right] \tag{57}$$

and

$$v_t^{\theta(\varphi)}(a) := \hat{v}_t^{\theta(\varphi),0}(0,a) \tag{58}$$

with $\hat{v}^{\theta(\varphi),0}$ *the unique solution to equation* (37) *in the case* $\theta = \theta(\varphi)$ *and* $a = 0$ *there, and with special* ϱ *as in* (31). *Consequently,* $v^{\theta(\varphi)} = v = \{v_t(a) : t > 0,\ a \in \mathsf{R}^d\} \geq 0$ *solves the equation*

$$v_t(a) = \theta(\varphi)\,\mathrm{p}_t(a) - \varrho \int_0^t \mathrm{d}(s^\gamma)\; E_a v_{t-s}^{1+\beta}(\xi_s), \tag{59}$$

$t > 0$, $a \in \mathsf{R}^d$.

The proof of this theorem will be given in Subsection 2.6 below, after preparations in the Subsections 2.3–2.5. The proof will essentially use the assumption $\gamma < 1$ imposed in Hypothesis 1(b) for all of this paper. On the other hand, to get a refined asymptotics for the cases $\gamma = 1$ or $E\tau < \infty$, another approach is needed; these cases will be investigated elsewhere.

Let us also mention that for super-Brownian motions and branching Brownian motions with exponential lifetimes refined asymptotic results in the critical dimension (in the sense of Theorem 13) have been established in [Kle98].

Now we want to explain how Theorem 13 is related to the mentioned asymptotics for the case of large but finite initial populations.

Corollary 14 (Asymptotics starting from large finite populations) *Let* $d = \frac{\alpha\gamma}{\beta}$. *Assume*[4] $\hat{Z}_0 = [K^{d/\alpha} i_0]\,\delta_{(\tau, K^{1/\alpha} a)}$ *with* $i_0 > 0$ *and* $a \in \mathsf{R}^d$. *Then for the marginal rescaled processes* $Z^{(K)} := \hat{Z}^{(K)}\big(\mathsf{R}_+ \times (\cdot)\big)$ *and* $t > 0$ *we have*

$$Z_t^{(K)} \underset{K\uparrow\infty}{\Longrightarrow} \pi_{X_t(0)\,\ell} \quad \textit{in law}, \tag{60}$$

where $X_t(0)$ *is the random density at time* t *at site* 0 *of the marginal process* X *of the* (d,α,β,γ)*–superprocess* $\hat{X}$ *with* $\hat{X}_0 = i_0\,\delta_{(0,a)}$.

Clearly, the previous corollary can be restated in terms of a long-term behavior analogously to Corollary 10(a).

[4] $[z]$ denotes the integer part of z.

Proof of Corollary 14 Fix $t, i_0 > 0$, $a \in \mathsf{R}^d$, and $\varphi \in \mathcal{C}_p^+$. By the branching property and the definition (14) of rescaling, for $K > 0$,

$$\log \mathbf{E}_{[K^{d/\alpha}\, i_0]\delta_{(\tau, K^{1/\alpha} a)}} \mathrm{e}^{-\langle Z_t^{(K)}, \varphi\rangle} \;=\; [K^{d/\alpha}\, i_0] \log \mathbf{E}_{\delta_{(\tau, K^{1/\alpha} a)}} \mathrm{e}^{-\langle Z_{Kt}, \varphi\rangle}. \quad (61)$$

Using the definition (52) of $Q\varphi$, the limit as $K \uparrow \infty$ of the right hand side of the latter equation equals

$$\lim_{K\uparrow\infty} [K^{d/\alpha}\, i_0] \log\bigl(1 - Q_{Kt}\varphi\,(K^{1/\alpha} a)\bigr)$$
$$= \; -i_0 \lim_{K\uparrow\infty} V_t^{(K)} \varphi\,(a) \;=\; -i_0\, v_t^{\theta(\varphi)}(a),$$

where we used notation (55) and Theorem 13. But by the definition (58) of $v_t^{\theta(\varphi)}(a)$, and the log-Laplace formula (35) in Proposition 7,

$$-i_0\, v_t^{\theta(\varphi)}(a) \;=\; \log \mathbb{E}_{i_0\, \delta_{(0,a)}} \exp\bigl[-\theta(\varphi)\, X_t(0)\bigr]. \quad (62)$$

However by formula (57), this is the log-Laplace transform of the mixed homogeneous Poissonian particle system $\pi_{X_t(0)\ell}$ as desired, where $\hat{X}$ starts from $\hat{X}_0 = i_0\, \delta_{(0,a)}$. This finishes the proof (based on Theorem 13). ∎

Proof of Theorem 11 First of all, note that by definition (14),

$$Z_t \;=\; \hat{Z}_1^{(t)}\bigl(\mathsf{R}_+ \times (\,\cdot\,)\bigr)\,, \qquad t > 0. \quad (63)$$

We need to combine (54) and (56) [by using notation (55)] to conclude for

$$-\log \mathbf{E}_{\pi_{i_0\, G\times\ell}} \exp\langle Z_t\,, -\varphi\rangle \;\underset{t\uparrow\infty}{\longrightarrow}\; i_0 \int_{\mathsf{R}^d} \mathrm{d}a\; v_1^{\theta(\varphi)}(a) \quad (64)$$

with $\varphi \in \mathcal{C}_p^+$ and $\theta(\varphi)$ as in (57), since the right hand side in (64) equals the log-Laplace transform of the mixed homogeneous Poissonian particle system $\pi_{X_1(0)\ell}$ with $\hat{X}$ starting from $\hat{X}_0 = i_0\delta_0 \times \ell$. For this we only have to justify that in

$$\int_{\mathsf{R}^d} \mathrm{d}a\; t^{d/\alpha}\, Q_t\varphi\,(t^{1/\alpha} a) \quad (65)$$

from (54) the limit as $t \uparrow \infty$ can be provided under the integral. In fact, by the definition (52) of Q,

$$Q_t\varphi\,(a) \;\le\; \mathbf{E}_{\delta_{(\tau,a)}}\,\langle Z_t\,, \varphi\rangle \;=\; E_a\varphi(\xi_t), \quad (66)$$

where we used the criticality of the process Z [see (93) below]. Hence

$$t^{d/\alpha}\, Q_t\varphi\,(t^{1/\alpha} a) \;\le\; t^{d/\alpha} \int_{\mathsf{R}^d} \mathrm{d}b\; \mathrm{p}_t(b - t^{1/\alpha} a)\, \varphi(b). \quad (67)$$

But by the *self-similarity*

$$K^{d/\alpha}\, \mathrm{p}_{Ks}(K^{1/\alpha} b) \;=\; \mathrm{p}_s(b), \qquad K, s > 0, \quad b \in \mathsf{R}^d, \quad (68)$$

of the stable transition densities of Hypothesis 1(a), the right hand side in the estimate (67) can be rewritten as

$$\int_{\mathsf{R}^d} \mathrm{d}b \; \mathrm{p}_1(t^{-1/\alpha}b - a)\, \varphi(b) \tag{69}$$

which converges to

$$\mathrm{p}_1(a) \int_{\mathsf{R}^d} \mathrm{d}b \; \varphi(b) \;=:\; \mathrm{p}_1(a)\, \|\varphi\|_1 \tag{70}$$

as $t \uparrow \infty$ (where we also used the symmetry of p_s). Finally, integrating the expressions in (69) and (70) with respect to $\mathrm{d}a$ we get identically $\|\varphi\|_1$. Thus, we can apply the extended dominated convergence theorem to justify that the limit in (65) can be provided under the integral. This finishes the proof (based on Theorem 13). ■

2 Remaining proofs

After some preparation, we will demonstrate in Subsection 2.2 that X has random densities in each point. The main purpose however is to prove the refined asymptotics Theorem 13, which is done in the Subsections 2.3 – 2.6.

2.1 Preliminaries: Some notation

With c we will always denote a positive constant which might vary from place to place. A c with some additional mark (as c_f or c_G) will however denote a specific constant. A constant of the form $c_{(\#)}$ means, this constant first occurred related to formula line (#).

Recall that p denotes the stable transition density function from Hypothesis 1(a). From the self-similarity (68) we immediately get the following simple estimate

$$\mathrm{p}_r(a) \;\le\; c_{(71)}\, r^{-d/\alpha}, \qquad r > 0, \quad a \in \mathsf{R}^d. \tag{71}$$

On the other hand, we also have

$$\mathrm{p}_r(a) \;\le\; c_{(72)}\, r\, |a|^{-d-\alpha}, \qquad r > 0, \quad a \in \mathsf{R}^d, \tag{72}$$

see, for instance, [FG86, formula (A.8)].

We denote by $\hat{S} = \{\hat{S}_t : t \ge 0\}$ the semigroup of the Markov process $\hat{\xi}$.

Recall also that the boldface letter $\mathbf{P}$ is related to a law of a particle system, the blackboard letter $\mathbb{P}$ to superprocesses, and the italics P to a law of a "basic" random object as ξ, τ, η. Corresponding expectations are expressed as $\mathbf{E}, \mathbb{E}, E$, respectively.

2.2 Absolutely continuous measure states of X (proof of Proposition 7)

Here we will *prove Proposition* 7. It will be based on some modification of arguments in [Kle00b]. The main result there is a general criterion for the absolute continuity of the measure states of some catalytic superprocesses. For this it is assumed in [Kle00b] that the underlying motion process has transition densities, which our motion process $\hat{\xi}$ for $\hat{X}$ does not have. However, we only want to show that the projected process X has absolutely continuous states. To prove this, we will heavily exploit the methods of [Kle00b], and in this sense we also meet the invitation in [Kle00b] "to think about interesting new examples" for the main theorem there.

Recall that $d < \frac{\alpha}{\beta}$. Fix $\mu \in \mathcal{M}_p(\mathsf{E})$ and $t > 0$.

1° (*Absolutely continuous expectation*) The expectation measure $\mathbb{E}_\mu X_t$ on R^d is absolutely continuous with continuous density function

$$a \;\mapsto\; \int_{\mathsf{E}} \mu\big(\mathrm{d}(u,b)\big)\; \mathrm{p}_t(b-a) \;=:\; \mu * \mathrm{p}_t\,(a) \tag{73}$$

[see the expectation formula (23)].

2° (*Absolutely continuous states*) Assume for the moment, that

$$\lim_{\varepsilon \downarrow 0} \Big(-\log \mathbb{E}_\mu \exp\big\langle X_t\,,\, -\theta\, \delta_a * \chi_\varepsilon \big\rangle \Big) \;=:\; w(a,\theta) \tag{74}$$

exists for each $a \in \mathsf{R}^d$ and $\theta > 0$, and that

$$\frac{\partial}{\partial \theta} w(a,\theta)\big|_{\theta=0+} \;=\; \mu * \mathrm{p}_t\,(a), \qquad a \in \mathsf{R}^d. \tag{75}$$

Then, for each $a \in \mathsf{R}^d$, the random density $X_t(a) = \langle X_t\,, \delta_a \rangle$ exists as claimed in (34), has log-Laplace transform

$$-\log \mathbb{E}_\mu \exp\big[-\theta X_t(a)\big] \;=\; w(a,\theta), \qquad \theta > 0, \tag{76}$$

and "full" expectation

$$\mathbb{E}_\mu X_t(a) \;=\; \mu * \mathrm{p}_t\,(a). \tag{77}$$

Hence, combined with step 1°, we recognize that the singular measure component of the random measure X_t must disappear a.s., that is, we get the absolute continuity claim (b) (see, for instance, [Kle00b, Lemma 2.2] for a more careful formulation).

3° (*Uniform regularity*) To attack the proof of (74) and (75), we go back from the marginal measure $X_t = \hat{X}_t\big(\mathsf{R}_+ \times (\cdot)\big)$ to $\hat{X}_t$, and use the log-Laplace representation (21). This means we have to study the log-Laplace equation (22) with the function ψ replaced by

$$\theta\, 1 \otimes \delta_a^\varepsilon \;=:\; \theta\, \overline{\delta_a^\varepsilon}, \qquad a \in \mathsf{R}^d, \tag{78}$$

where

$$\delta_a^\varepsilon(b) := \delta_a * \chi_\varepsilon(b) = \chi_\varepsilon(b-a), \qquad b \in \mathsf{R}^d, \tag{79}$$

with χ_ε from (33), and we have to let $\varepsilon \downarrow 0$. Note that $\overline{\delta_a^\varepsilon}$ approaches the constantly extended δ–function $\overline{\delta_a} = 1 \otimes \delta_a$ on E from (40), which of course does not belong to the set $\hat{\mathcal{C}}_p^+$ of test functions occurring in the log-Laplace equation (22), so we have to justify its usage in order to come to (37).

In line with (41) we define

$$\hat{S}_t \overline{\delta_a} := 1 \otimes (\delta_a * \mathrm{p}_t), \qquad t > 0, \quad a \in \mathsf{R}^d. \tag{80}$$

We mention that $\hat{S}_t \overline{\delta_a}$ could also be introduced by using the collision measure ([EP91, Myt98]) between the transition measure related to $\hat{S}_t$ which – by the independence of $(\vartheta, \xi) = \hat{\xi}$ – is a product measure with continuous density function p_t of the second component, and the measure $\overline{\delta_a} = \ell_+ \times \delta_a$.

First we show that the family $\{\overline{\delta_a} : a \in \mathsf{R}^d\}$ of measures on E has the following *uniform regularity* property: For all $(u,b) \in \mathsf{E}$ and $0 \le r \le t$,

$$\sup_{a \in \mathsf{R}^d} \hat{E}_{(u,b)} \int_r^t \mathrm{d}k_s \, \left[\hat{S}_{t-s} \overline{\delta_a}(\hat{\xi}_s)\right]^{1+\beta} \tag{81}$$

$$\le c_{(82)}\, t^{-d/\alpha} \int_{\{u \vee r \le s \le t\}} \mathrm{d}_s (s-u)^\gamma \, (t-s)^{-\beta d/\alpha} < \infty. \tag{82}$$

In fact, by (80), and, since k and ξ are independent, by using (26), the expectation expression in (81) is bounded from above by

$$\varrho \int_{\{u \vee r \le s \le t\}} \mathrm{d}_s (s-u)^\gamma \, E_b \, \mathrm{p}_{t-s}^{1+\beta}(\xi_s - a)$$

$$\le c\, t^{-d/\alpha} \int_{\{u \vee r \le s \le t\}} \mathrm{d}_s (s-u)^\gamma \, (t-s)^{-\beta d/\alpha}, \tag{83}$$

where we first applied the simple estimate (71) to p^β, then Chapman-Kolmogorov to the expectation on p, and finally again (71). But the integral in (83) is finite since γ and $\beta d/\alpha$ belong to $(0,1)$. Consequently, the estimate (81-82) is true.

4° (*Uniform integrability in the regularization*) Next we verify that for $u \ge 0$ and $a, b \in \mathsf{R}^d$,

$$\lim_{r \uparrow t} \limsup_{\varepsilon \downarrow 0} \hat{E}_{(u,b)} \int_r^t \mathrm{d}k_s \left[\hat{S}_{t-s} \overline{\delta_a^\varepsilon}(\hat{\xi}_s)\right]^{1+\beta} = 0 \tag{84}$$

[recall notation (78)]. Indeed, from $\overline{\delta_a^\varepsilon} = 1 \otimes \delta_a^\varepsilon$ we get

$$\hat{S}_t \overline{\delta_a^\varepsilon}(\hat{\xi}_s) = S_t \delta_a^\varepsilon(\hat{\xi}_s) = \int_{\mathsf{R}^d} \mathrm{d}a' \, \chi_\varepsilon(a') \, \mathrm{p}_t(a + a' - \xi_s)$$

(with S denoting the semigroup related to p). Hence, by Jensen's inequality,

$$\big[\hat{S}_{t-s}\overline{\delta^{\varepsilon}_{a}}\,(\hat{\xi}_s)\big]^{1+\beta} \;\le\; \int_{\mathsf{R}^d} \mathrm{d}a'\; \chi_{\varepsilon}(a')\big[\hat{S}_{t-s}\overline{\delta_{a+a'}}\,(\hat{\xi}_s)\big]^{1+\beta} \tag{85}$$

[recall (80)]. Therefore, the expectation expression in (84) can be bounded from above by

$$\sup_{a\,\in\,\mathsf{R}^d} \hat{E}_{(u,b)} \int_r^t \mathrm{d}k_s\, \big[\hat{S}_{t-s}\overline{\delta_a}\,(\hat{\xi}_s)\big]^{1+\beta} \tag{86}$$

which does not dependent on ε and, moreover, tends to 0 as $r \uparrow t$ by the uniform regularity property (81-82). This gives (84).

5° (*Conclusions*) According to step 3°, for a in R^d fixed, $\overline{\delta_a}$ is a regular measure on E in the sense of [Kle00b, Definition 1.1]. Here our $\hat{S}_t\overline{\delta_a}$ takes the role of the density function of the motion process applied to the measure ν there, which corresponds to our $\overline{\delta_a}$. Now by Proposition 1.2 of [Kle00b] (or rather carrying over its proof line by line with the obvious changes since we modified his Definition 1.1), equation (22) with ψ replaced by $\theta\overline{\delta_a}$ where $0 < \theta \le 1$, that is equation (37), has exactly one solution $\hat{v} = \hat{v}^{\theta,a} = \hat{V}(\theta\overline{\delta_a})$, and

$$\frac{\partial}{\partial\theta}\hat{v}^{\theta,a}_t(e')\big|_{\theta=0+} \;=\; \mathrm{p}_t\,(a-a'), \qquad e'=(u',a') \in \mathsf{E}. \tag{87}$$

Moreover, as in the proof of formula line (2.12) in [Kle00b], by using (84),

$$\hat{V}_t\big(\theta\,\overline{\delta^{\varepsilon}_{a}}\big)(e') \underset{\varepsilon\downarrow 0}{\longrightarrow} \hat{V}_t(\theta\overline{\delta_a})(e') \;=\; \hat{v}^{\theta,a}_t(e'), \qquad e' \in \mathsf{E}. \tag{88}$$

Hence, from the general *domination* formula

$$0 \;\le\; \hat{V}_t\psi\,(e) \;\le\; \hat{E}_e\psi(\hat{\xi}_t), \qquad t \ge 0, \quad e \in \mathsf{E}, \quad \psi \in \hat{\mathcal{C}}^{+}_{p}, \tag{89}$$

and dominated convergence, we obtain

$$\big\langle \mu, \hat{V}_t(\theta\,\overline{\delta^{\varepsilon}_{a}})\big\rangle \underset{\varepsilon\downarrow 0}{\longrightarrow} \big\langle \mu, \hat{V}_t(\theta\overline{\delta_a})\big\rangle \;=\; \big\langle \mu, \hat{v}^{\theta,a}_t\big\rangle.$$

But this gives the needed statements (74) and (75) in step 2° with $w(a,\theta) = \big\langle \mu, \hat{v}^{\theta,a}_t\big\rangle$, and at the same time claims (35-37), finishing the proof of Proposition 7. ∎

2.3 Renewal type equation and its scaling

Our starting point for the asymptotic properties of $V^{(K)}\varphi$ as needed for Theorem 13 is the fact, that it satisfies an integral equation (Lemma 16 below). For this we need first some further notation.

Recall that τ with law G denotes the lifetime of a newly born particle in

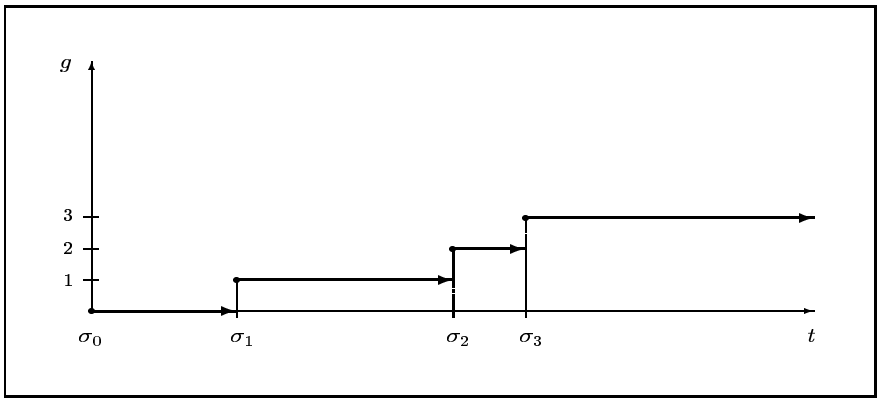

Figure 1: Generation number process g

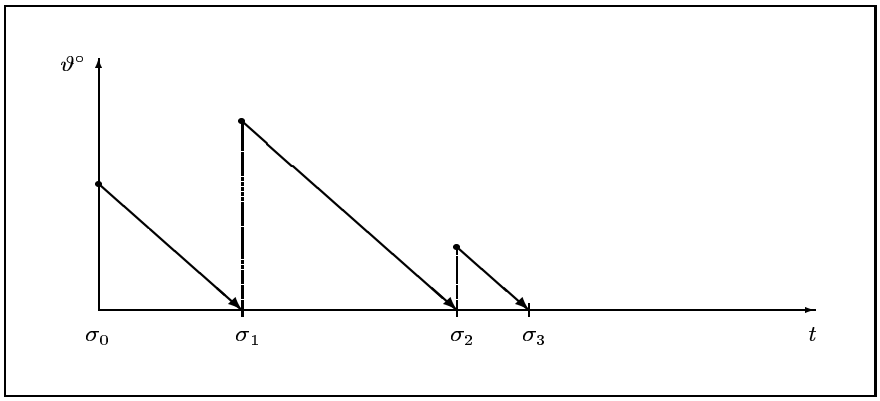

Figure 2: Residual lifetime process ϑ°

our branching particle system $\hat{Z}$. Let $\tau_1, \tau_2, \ldots$ denote independent copies of τ, and set $\sigma_n := \sum_{1 \le i \le n} \tau_i$, $n \ge 0$. With

$$g_t := \sum_{n \ge 1} \mathbf{1}_{\{\sigma_n \le t\}}, \qquad t \ge 0, \tag{90}$$

we get the "*generation number process*" (see Figure 1), and with

$$\vartheta^\circ_t := \sigma_{g_t+1} - t, \qquad t \ge 0, \tag{91}$$

the *residual lifetime process* related to a single particle in our branching particle system (see Figure 2). Recalling that ξ denotes the particles' motion process, we pass now to the pair $\hat{\xi}^\circ := (\vartheta^\circ, \xi)$ of independent processes whose laws we denote by $\hat{P}^\circ_e$, $e \in \mathsf{E}$. Note that $(\hat{\xi}^\circ, \hat{P}^\circ_e, e \in \mathsf{E})$ is a càdlàg

time-homogeneous Markov process in $\mathsf{E} = \mathsf{R}_+ \times \mathsf{R}^d$, and that the laws $\hat{P}^\circ_e$, $e \in \mathsf{E}$, describe the renewal process σ and the generation number process g as well. Recall the notation $Q\varphi$ introduced in (52).

Lemma 15 (Renewal type equation) *Fix $\varphi \in \mathcal{C}_p^+$. Then*

$$Q_t\varphi\,(a) \;=\; E_a\big[1 - \mathrm{e}^{-\varphi(\xi_t)}\big] - c_f\,\hat{E}^\circ_{(\tau,a)} \int_{(0,t]} \mathrm{d}g_s\;\big[Q_{t-s}\varphi\,(\xi_s)\big]^{1+\beta}, \qquad (92)$$

$t \geq 0,\;\; a \in \mathsf{R}^d$.

Proof Follow the proof of [KS98, Lemma 3] with the obvious changes to adapt to our setting with the residual lifetimes in the description. In particular, first distinguish between $\tau \geq t$ and $\tau < t$; then, in the latter case further iterate. ■

Consequently, although the lifetime distribution is not exponential, one can exploit the renewal in the time points of death of a particle, and some invariance properties related to these points.

Note that (92) yields the *expectation formula*

$$\mathbf{E}_{\delta_{(\tau,a)}}\big\langle \hat{Z}_t\,, \bar{\varphi}\big\rangle \;=\; E_a\varphi(\xi_t), \qquad a \in \mathsf{R}^d, \quad t \geq 0, \quad \varphi \in \mathcal{C}_p^+, \qquad (93)$$

(replace, for instance, φ by $\theta\varphi$ and differentiate with respect to $\theta > 0$ at $\theta = 0+$).

Recall that we are interested in the rescaled quantity $V^{(K)}\varphi$ (where $K > 0$ and $\varphi \in \mathcal{C}_p^+$) introduced in (55). For $t \geq 0$ and $a \in \mathsf{R}^d$, put

$$W_t^{(K)}\varphi\,(a) \;:=\; K^{d/\alpha}\,E_{K^{1/\alpha}a}\Big(1 - \exp\big[-\varphi(\xi_{Kt})\big]\Big), \qquad (94)$$

and introduce the (right-continuous) *renewal function* N corresponding to the lifetime distribution function G:

$$N_s \;:=\; Eg_s \;=\; \sum_{k=1}^{\infty} G^{*k}(s), \qquad s \geq 0, \qquad (95)$$

[recall notation (90)]. Note that by assumption (2) in Hypothesis 1(b),

$$N_K \;\sim\; D\,K^\gamma \quad \text{as } K \uparrow \infty \qquad (96)$$

with the constant

$$D \;:=\; \frac{\sin\pi\gamma}{\pi}\,\frac{1}{c_G}\,. \qquad (97)$$

Lemma 16 (Scaled equation) *Fix $K > 0$ and $\varphi \in \mathcal{C}_p^+$. The functional $V^{(K)}\varphi$ defined in (55) satisfies the integral equation*

$$V_t^{(K)}\varphi\,(a) \;=\; W_t^{(K)}\varphi\,(a) - c_f \int_{(0,t]} \frac{\mathrm{d}_s N_{Ks}}{K^\gamma}\;E_a\big[V_{t-s}^{(K)}\varphi\,(\xi_s)\big]^{1+\beta}, \qquad (98)$$

$t \geq 0,\;\; a \in \mathsf{R}^d$.

Proof Scaling (92) as in (55), the left hand side of (92) changes to $V_t^{(K)}\varphi\,(a)$, and the first part of the right hand side of (92) changes to the $W^{(K)}$–term as in (98). For the remaining part, exploit the independence of the generation number process g and the motion process ξ, and apply the definition (95) of N to get

$$-\,c_f\,K^{d/\alpha}\,E_{K^{1/\alpha}a}\int_{(0,Kt]}\mathrm{d}N_s\,\left[Q_{Kt-s}\varphi\,(\xi_s)\right]^{1+\beta} \tag{99}$$

$$=\,-c_f\,K^{d/\alpha}\int_{(0,t]}\mathrm{d}_sN_{Ks}\,E_{K^{1/\alpha}a}\left[Q_{K(t-s)}\varphi\,(\xi_{Ks})\right]^{1+\beta}. \tag{100}$$

Then by the *self-similarity* of the stable process ξ, that is,

$$E_{K^{1/\alpha}a}\big(K^{-1/\alpha}\xi_{Ks}\in\,\cdot\,\big)\;=\;E_a\,(\xi_s\in\,\cdot\,), \tag{101}$$

$a\in\mathsf{R}^d$, $s\geq 0$, $K>0$, and the assumed critical parameter constellation $d\beta/\alpha=\gamma$, using once more (55), also the remaining term of (98) occurs, finishing the proof. ■

2.4 Convergence to the limiting equation

Our next task is to let $K\uparrow\infty$ in the integral equation (98), provided that $t>0$. For this purpose, we first note that by the self-similarity as formulated in (101), $W^{(K)}$ defined in (94) can be written as

$$W_t^{(K)}\varphi\,(a)\;=\;\int_{\mathsf{R}^d}\mathrm{d}b\;\mathrm{p}_t\big(K^{-1/\alpha}b-a\big)\big[1-\mathrm{e}^{-\varphi(b)}\big]. \tag{102}$$

By dominated convergence, this implies

$$W_t^{(K)}\varphi\,(a)\;\underset{K\uparrow\infty}{\longrightarrow}\;\theta(\varphi)\,\mathrm{p}_t(a),\qquad t>0,\quad a\in\mathsf{R}^d, \tag{103}$$

with $\theta(\varphi)$ from (57). On the other hand, by (96), for $T>0$ fixed,

$$\frac{N_{Ks}}{N_K}\;\underset{K\uparrow\infty}{\longrightarrow}\;s^\gamma,\qquad\text{uniformly in }\;s\in[0,T]. \tag{104}$$

In fact, take $\varepsilon\in(0,T)$ and deal with $s\leq\varepsilon$ and $\varepsilon<s\leq T$ separately. Then from (96) and (104),

$$\frac{N_{Ks}}{K^\gamma}\;=\;\frac{N_K}{K^\gamma}\,\frac{N_{Ks}}{N_K}\;\underset{K\uparrow\infty}{\longrightarrow}\;D\,s^\gamma\qquad\text{uniformly in }\;s\in[0,T]. \tag{105}$$

Inserting (103) and (105) into (98) by using the fact that

$$c_fD\;=\;\varrho \tag{106}$$

[recall (97) and (31)], we expect that

$$V_t^{(K)}\varphi\,(a) \underset{K\uparrow\infty}{\longrightarrow} v_t(a) \tag{107}$$

where v solves equation (59).

We start with showing (107) in some uniform $L^1 = \big(L^1(\mathrm{d}a), \|\cdot\|_1\big)$ sense. For later use in an analytic extension argument, we introduce an additional factor $\lambda \geq 0$ in front of φ, and write

$$v^{(\lambda)} := v^{\theta(\lambda\varphi)}, \tag{108}$$

instead of v. Then (59) changes to

$$v_t^{(\lambda)}(a) = \theta(\lambda\varphi)\,\mathrm{p}_t(a) - \varrho\int_0^t \mathrm{d}(s^\gamma)\,E_a\big[v_{t-s}^{(\lambda)}(\xi_s)\big]^{1+\beta}, \tag{109}$$

$t>0,\ a\in\mathsf{R}^d$.

Lemma 17 (Uniform L^1–convergence for small λ) *Fix $0<L<T$ and $\varphi\in\mathcal{C}_p^+$. There exists a constant $\Lambda=\Lambda(T,\varphi)>0$ such that for $V^{(K)}(\lambda\varphi)$ defined in (55) we have*

$$\lim_{K\uparrow\infty}\sup_{L\leq t\leq T}\sup_{\lambda\in[0,\Lambda]}\Big\|V_t^{(K)}(\lambda\varphi)-v_t^{(\lambda)}\Big\|_1 = 0, \tag{110}$$

where $v^{(\lambda)}\geq 0$ solves (109) with the constant $\varrho=c_f D$.

To prepare for the proof of this lemma, by using equations (98) (with φ replaced by $\lambda\varphi$) and (109), we can estimate the L^1–norm expression in the claim (110):

$$\Big\|V_t^{(K)}(\lambda\varphi)-v_t^{(\lambda)}\Big\|_1 \leq A_1 + c_f\,A_2 + c_f\,A_3 + c_f\,A_4 + c_f D\,A_5\,. \tag{111}$$

Here, by using the identity

$$\int_{\mathsf{R}^d}\mathrm{d}a\;E_a F(\xi_s) = \|F\|_1\,,\qquad \text{measurable } F\geq 0,\quad s\geq 0, \tag{112}$$

for fixed $t>0$ and $\lambda\geq 0$,

$$A_1 := \Big\|W_t^{(K)}(\lambda\varphi)-\theta(\lambda\varphi)\,\mathrm{p}_t\Big\|_1, \tag{113a}$$

$$A_2 := \int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{K^\gamma}\,\Big\|\big[V_{t-s}^{(K)}(\lambda\varphi)\big]^{1+\beta}-\big[v_{t-s}^{(\lambda)}\big]^{1+\beta}\Big\|_1, \tag{113b}$$

$$A_3 := \frac{(N_{Kt}-N_{Kt-})}{K^\gamma}\,\Big\|\big[V_0^{(K)}(\lambda\varphi)\big]^{1+\beta}\Big\|_1, \tag{113c}$$

$$A_4 := \Big|\frac{N_K}{K^\gamma}-D\Big|\,\Big\|\int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{N_K}\,\big[v_{t-s}^{(\lambda)}\big]^{1+\beta}\Big\|_1, \tag{113d}$$

$$A_5 := \Big|\int_{(0,t)}\Big(\frac{\mathrm{d}_s N_{Ks}}{N_K}-\mathrm{d}(s^\gamma)\Big)\Big\|\big[v_{t-s}^{(\lambda)}\big]^{1+\beta}\Big\|_1\Big|. \tag{113e}$$

2.5 Uniform L^1–convergence (proof of Lemma 17)

Here we prove Lemma 17. We proceed in several steps.

1° (*Error term* A_1) First of all, we consider any constant $\Lambda > 0$ and all $\lambda \in [0, \Lambda]$. By (102),

$$A_1 \;\le\; \int_{\mathsf{R}^d} \mathrm{d}a \int_{\mathsf{R}^d} \mathrm{d}b \;\Big|\mathrm{p}_t\big(K^{-1/\alpha}b - a\big) - \mathrm{p}_t(a)\Big| \left[1 - \mathrm{e}^{-\lambda\varphi(b)}\right] \tag{114}$$

$$\le\; c_\varphi\, \Lambda \int_{\mathsf{R}^d} \mathrm{d}b\; \phi_p(b) \int_{\mathsf{R}^d} \mathrm{d}a\;\Big|\mathrm{p}_t\big(K^{-1/\alpha}b - a\big) - \mathrm{p}_t(a)\Big|. \tag{115}$$

For $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that

$$\int_{|b|\ge C} \mathrm{d}b\; \phi_p(b) \;\le\; \varepsilon, \tag{116}$$

whereas the internal integral in (115) is bounded by 2 since the p_s are probability density functions. On the other hand, by the uniform continuity of the stable transition density functions p on $[L, T] \times \mathsf{R}^d$, for $|b| \le C$ the internal integral in (115) is smaller then ε for K sufficiently large (depending on C). This shows that

$$\lim_{K\uparrow\infty} \sup_{\substack{L\le t\le T,\\ \lambda\in[0,\Lambda]}} A_1 \;=\; 0. \tag{117}$$

2° (*Error term* A_3) Use the simply bound

$$\Big\| \big[V_0^{(K)}(\lambda\varphi)\big]^{1+\beta} \Big\|_1 \;\le\; K^{\beta d/\alpha}\, \lambda \|\varphi\|_1\,, \tag{118}$$

which follows from

$$V_0^{(K)}(\lambda\varphi)\,(a) \;\le\; \lambda K^{d/\alpha} \varphi(K^{1/\alpha}a) \tag{119}$$

[by (98) and (94)], criticality $\beta d/\alpha = \gamma$, and

$$\lim_{K\uparrow\infty} \sup_{L\le t\le T} \left(N_{Kt} - N_{Kt-}\right) \;=\; 0, \tag{120}$$

(see, e.g., [Fel71, (9.1.9)]), in order to see that

$$\lim_{K\uparrow\infty} \sup_{\substack{L\le t\le T,\\ \lambda\in[0,\Lambda]}} A_3 \;=\; 0, \tag{121}$$

too.

3° (*Error term* A_4) From (109), by domination, for $s > 0$,

$$0 \;\le\; v_s^{(\lambda)}(a) \;\le\; \theta(\lambda\varphi)\, \mathrm{p}_s(a) \;\le\; \Lambda\, \|\varphi\|_1\, \mathrm{p}_s(a). \tag{122}$$

Hence, using the simple estimate (71) and the identity $d\beta/\alpha = \gamma$, we get

$$0 \;\le\; \left[v_s^{(\lambda)}(a)\right]^{1+\beta} \;\le\; c\left(\Lambda\,\|\varphi\|_1\right)^{1+\beta}\;\frac{\mathrm{p}_s(a)}{s^\gamma}\,. \tag{123}$$

However, the p_s are probability density functions. Therefore

$$A_4 \;\le\; c\,\left(\Lambda\,\|\varphi\|_1\right)^{1+\beta}\,\left|\frac{N_K}{K^\gamma} - D\right|\,\int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{N_K}\,\frac{1}{(t-s)^\gamma}\,. \tag{124}$$

But by substitution, for $0 \le r < t$,

$$\int_r^t \mathrm{d}(s^\gamma)\;(t-s)^{-\gamma} \;\le\; \gamma\,t^{1-\gamma}\int_{r/t}^1 \frac{\mathrm{d}s}{s^{1-\gamma}\,(1-s)^\gamma} \;<\; \infty. \tag{125}$$

Thus, from (104), uniformly in $0 \le r < t \le T$,

$$\int_{[r,t)}\frac{\mathrm{d}_s N_{Ks}}{N_K}\,\frac{1}{(t-s)^\gamma} \;\underset{K\uparrow\infty}{\longrightarrow}\; \gamma\int_{r/t}^1 \frac{\mathrm{d}s}{s^{1-\gamma}\,(1-s)^\gamma} \;\le\; \gamma\,B(\gamma,\gamma+1) \tag{126}$$

with B denoting the Beta function. Indeed, approximate the integral by sums, use monotonicity in time and continuity of the limit in (104). Thus, by (105),

$$\lim_{K\uparrow\infty}\;\sup_{\substack{0<t\le T,\\ \lambda\in[0,\Lambda]}}\; A_4 \;=\; 0. \tag{127}$$

4° (*Error term* A_5) Let $\varepsilon \in (0,1)$. We split the integral in A_5 concerning the cases $(1-\varepsilon)t \le s < t$ and $0 \le s \le (1-\varepsilon)t$. In the first case, we pass from the difference to the sum and use (123). Then apply (126) with $r = (1-\varepsilon)t$ to see that we end up with an ε–term, uniformly in t, λ, and K. In the second case, we use that the map

$$s \;\mapsto\; \left\|\left[v_{t-s}^{(\lambda)}\right]^{1+\beta}\right\|_1 \quad \text{on} \quad \left[0,(1-\varepsilon)t\right] \tag{128}$$

is bounded and continuous, and the weak convergence

$$\frac{\mathrm{d}_s N_{Ks}}{N_K} \;\underset{K\uparrow\infty}{\longrightarrow}\; \frac{\gamma\,\mathrm{d}s}{s^{1-\gamma}} \tag{129}$$

on the same interval, uniformly in t. Putting both together, we get

$$\lim_{K\uparrow\infty}\;\sup_{\substack{L\le t\le T,\\ \lambda\in[0,\Lambda]}}\; A_5 \;=\; 0. \tag{130}$$

5° (A *bound for* A_2) Using the elementary inequality

$$\left|x^{1+\beta} - y^{1+\beta}\right| \;\le\; (1+\beta)\;|x-y|\,\left(x^\beta + y^\beta\right), \qquad x, y \ge 0, \tag{131}$$

the domination

$$0 \;\le\; V_t^{(K)}(\lambda\varphi)\,(a) \;\le\; W_t^{(K)}(\lambda\varphi)\,(a) \tag{132}$$

[by (98)] implying

$$0 \;\le\; V_t^{(K)}(\lambda\varphi)\,(a) \;\le\; c\,\Lambda\,\|\varphi\|_1\,t^{-d/\alpha} \tag{133}$$

[recall (102) and (71)], as well as dominations (122) and (71), we see that, for $0 \le s < t$, the L^1–norm in A_2 is bounded by

$$c\left(\Lambda\,\|\varphi\|_1\right)^{\beta}\;(t-s)^{-\gamma}\,\left\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\right\|_1 . \tag{134}$$

Thus,

$$A_2 \;\le\; c\left(\Lambda\,\|\varphi\|_1\right)^{\beta}\int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{K^{\gamma}}\;\frac{\left\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\right\|_1}{(t-s)^{\gamma}} . \tag{135}$$

6° (*A Gronwall type inequality*) Inserting (117), (121), (127), (130), and the bound (135) into the estimate (111) gives the following statement (recall that $\Lambda > 0$ is fixed and not yet specified): For each $\varepsilon > 0$, there is a constant $K_0 = K_0(\varepsilon,\Lambda) > 0$ such that

$$\begin{aligned}\left\|V_t^{(K)}(\lambda\varphi) - v_t^{(\lambda)}\right\|_1 \;&\le\; \varepsilon \\ &+ c_{(136)}\left(\Lambda\,\|\varphi\|_1\right)^{\beta}\int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{K^{\gamma}}\;\frac{\left\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\right\|_1}{(t-s)^{\gamma}}\end{aligned} \tag{136}$$

for all $K \ge K_0$ and $t \in [L,T]$, as well as $\lambda \in [0,\Lambda]$. From the domination formulas (132) and (122) as well as the representation (102) of $W^{(K)}$ we obtain

$$\left\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\right\|_1 \;\le\; 2\Lambda\,\|\varphi\|_1 , \tag{137}$$

uniformly in $K > 0$, $s < t$ and $\lambda \in [0,\Lambda]$. Hence,

$$\check{D}_t^{(K)} \;:=\; \sup_{\substack{s\,\in\,[0,t),\\ \lambda\,\in\,[0,\Lambda]}} \left\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\right\|_1 \tag{138}$$

is bounded in $K > 0$ and $t > 0$. Then from the Gronwall type inequality in (136) we get

$$\check{D}_t^{(K)} \;\le\; \varepsilon + c_{(136)}\left(\Lambda\,\|\varphi\|_1\right)^{\beta}\;\check{D}_t^{(K)}\;\frac{N_K}{K^{\gamma}}\int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{N_K}\;\frac{1}{(t-s)^{\gamma}} , \tag{139}$$

for all $K \ge K_0$ and $t \in [L,T]$. But by the convergence and boundedness in (126),

$$\limsup_{K\uparrow\infty}\;\sup_{0\,<\,t\,\le\,T}\int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{N_K}\;\frac{1}{(t-s)^{\gamma}} \;:=\; c_{(140)} \;<\; \infty. \tag{140}$$

Thus, introducing the finite number

$$\check{D} \;:=\; \limsup_{K\uparrow\infty} \sup_{L\le t\le T} \check{D}_t^{(K)}, \tag{141}$$

from (139) and asymptotics (96) we obtain

$$\check{D} \;\le\; \varepsilon \;+\; c_{(136)} \left(\Lambda \,\|\varphi\|_1\right)^{\beta} \;\check{D}\, D\, c_{(140)}. \tag{142}$$

Since ε is arbitrary, we arrive at

$$\check{D} \;\le\; c_{(143)} \left(\Lambda \,\|\varphi\|_1\right)^{\beta} \;\check{D}. \tag{143}$$

Choosing now $\Lambda > 0$ so small that

$$c_{(143)} \left(\Lambda \,\|\varphi\|_1\right)^{\beta} \;<\; 1, \tag{144}$$

we obtain $\check{D} = 0$. Hence the claim (110) is true. This finishes the proof of Lemma 17. ∎

2.6 Pointwise convergence (proof of Theorem 13)

The L^1–convergence implies actually pointwise convergence:

Corollary 18 (Pointwise convergence for small λ) *Fix $T > 0$, φ in $\mathcal{C}_p^+$, and take $\Lambda = \Lambda(T,\varphi) > 0$ as in Lemma 17. Then for each $\lambda \in [0,\Lambda]$,*

$$V_t^{(K)}(\lambda\varphi)\,(a) \underset{K\uparrow\infty}{\longrightarrow} v_t^{(\lambda)}(a), \qquad t\in(0,T], \quad a\in \mathsf{R}^d. \tag{145}$$

Proof The proof is very similar to the one of Lemma 17 in the previous subsection, so we skip some details. In analogy to (111), for fixed $t\in(0,T]$, $a\in\mathsf{R}^d$, and $\lambda\in\left[0,\Lambda(T,\varphi)\right]$ we have

$$\left|V_t^{(K)}(\lambda\varphi)\,(a) - v_t^{(\lambda)}(a)\right| \;\le\; A_1' + c_f\,[A_2' + A_3' + A_4' + D\,A_5'], \tag{146}$$

where

$$\begin{aligned}
A_1' \;&:=\; \left|W_t^{(K)}(\lambda\varphi)\,(a) - \theta(\lambda\varphi)\,\mathrm{p}_t(a)\right|,\\
A_2' \;&:=\; \int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{K^\gamma} \int_{\mathsf{R}^d} \mathrm{d}b\; \mathrm{p}_s(b-a) \left|\left[V_{t-s}^{(K)}(\lambda\varphi)\,(b)\right]^{1+\beta} - \left[v_{t-s}^{(\lambda)}(b)\right]^{1+\beta}\right|,\\
A_3' \;&:=\; \frac{(N_{Kt} - N_{Kt-})}{K^\gamma} \int_{\mathsf{R}^d} \mathrm{d}b\; \mathrm{p}_t(b-a)\left[V_0^{(K)}(\lambda\varphi)\,(b)\right]^{1+\beta},\\
A_4' \;&:=\; \left|\frac{N_K}{K^\gamma} - D\right| \int_{(0,t)} \frac{\mathrm{d}_s N_{Ks}}{N_K} \int_{\mathsf{R}^d} \mathrm{d}b\; \mathrm{p}_s(b-a)\left[v_{t-s}^{(\lambda)}(b)\right]^{1+\beta},\\
A_5' \;&:=\; \left|\int_{(0,t)} \left(\frac{\mathrm{d}_s N_{Ks}}{N_K} - \mathrm{d}(s^\gamma)\right) \int_{\mathsf{R}^d} \mathrm{d}b\; \mathrm{p}_s(b-a)\left[v_{t-s}^{(\lambda)}(b)\right]^{1+\beta}\right|.
\end{aligned}$$

Clearly, $A_1' \to 0$ and $A_5' \to 0$ as $K \uparrow \infty$, by dominated convergence. The same is true for A_4'. In fact, use (123) and Chapman-Kolmogorov to get

$$A_4' \;\le\; c\,\big(\Lambda\,\|\varphi\|_1\big)^{1+\beta}\,\Big|\frac{N_K}{K^\gamma} - D\Big|\,p_t(a)\int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{N_K}\,\frac{1}{(t-s)^\gamma}\,, \qquad (148)$$

so we may continue as after (124). Also A_3' will disappear as $K \uparrow \infty$. Indeed, use (119), dominated convergence, criticality, and (120).

It remains to deal with the main term A_2'. Similar to the derivation of the estimate (135),

$$A_2' \;\le\; c\,\big(\Lambda\,\|\varphi\|_1\big)^{\beta} \times \int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{K^\gamma\,(t-s)^\gamma}\int_{\mathbb{R}^d}\mathrm{d}b\;\mathrm{p}_s(b-a)\,\big|V_{t-s}^{(K)}(\lambda\varphi)\,(b) - v_{t-s}^{(\lambda)}(b)\big|. \qquad (149)$$

For $0 < \varepsilon < 1$, we split the integral on $(0,t)$ in three parts:

$$0 < s < \varepsilon t, \quad \varepsilon t \le s \le (1-\varepsilon)t, \quad \text{and} \quad (1-\varepsilon)t < s < t. \qquad (150)$$

In the first and last case, we pass from differences to sums in the integrands, use dominations (132) and (122), as well as (102), Chapman-Kolmogorov, and the estimate (71) to see that the internal integral of (149) is bounded in s and K. But by (105),

$$\lim_{\varepsilon\downarrow 0}\,\limsup_{K\uparrow\infty}\int_{(0,\varepsilon t)\,\cup\,\big((1-\varepsilon)t,t\big)}\frac{\mathrm{d}_s N_{Ks}}{K^\gamma\,(t-s)^\gamma} \;=\; 0. \qquad (151)$$

So it remains to deal with the integral in (149) restricted to $\big[\varepsilon t,\,(1-\varepsilon)t\big]$, for fixed $0 < \varepsilon < 1$. Here $\mathrm{p}_s(b-a) \le c$, and we are back to the L^1–norm

$$\big\|V_{t-s}^{(K)}(\lambda\varphi) - v_{t-s}^{(\lambda)}\big\|_1 \;\le\; \sup_{u\,\in\,\big[\varepsilon t,\,(1-\varepsilon)\,t\big]}\big\|V_u^{(K)}(\lambda\varphi) - v_u^{(\lambda)}\big\|_1 \qquad (152)$$

which by Lemma 17 converges to 0 as $K \uparrow \infty$. But by (105),

$$\limsup_{K\uparrow\infty}\int_{(0,t)}\frac{\mathrm{d}_s N_{Ks}}{K^\gamma\,(t-s)^\gamma} \;<\; \infty, \qquad (153)$$

and we verified (145), finishing the proof of Corollary 18. ∎

Completion of the proof of Theorem 13 Fix φ, t, a as in the theorem, and let $\lambda \ge 0$. By (108) and (35),

$$v_t^{(\lambda)}(a) \;=\; -\log\mathbb{E}_{\delta_{(0,a)}}\exp[-\theta(\lambda\varphi)\,X_t(a)] \qquad (154)$$

with $\theta(\lambda\varphi)$ from (57). On the other hand, by (55) and (52),

$$V_t^{(K)}(\lambda\varphi) \;=\; K^{d/\alpha}\Big[1 \,-\, \mathbf{E}_{\delta_{(\tau,K^{1/\alpha}a)}}\exp\langle Z_{Kt}\,,-\lambda\varphi\rangle\Big]. \qquad (155)$$

Note that the expressions in (154) and (155) are *analytic* functions in $\lambda > 0$ (or $\Re\lambda > 0$). Then from the convergence in Corollary 18 for small λ we get the desired convergence for all $\lambda \geq 0$. This finishes the proof. ■

Acknowledgment We are grateful to Achim Klenke and Jan Swart for comments which helped to improve the presentation of the paper.

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