Walking through the Brownian zoo

Paris, 3-7 June 2019

Brownian excursions and lookdown spaces

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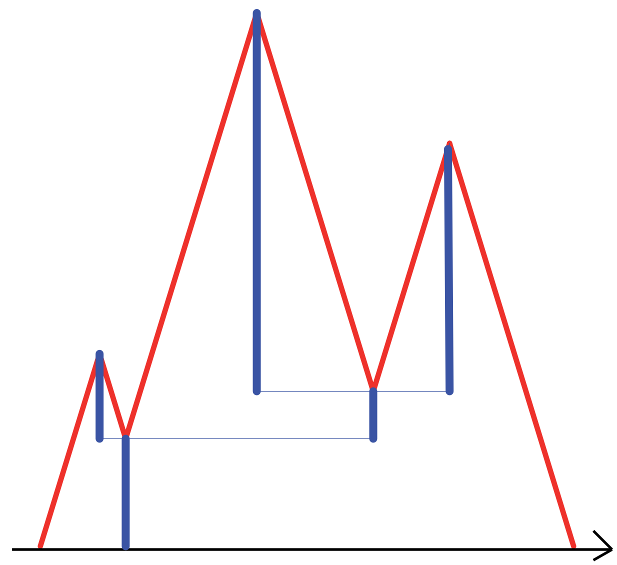
joint with Stephan Gufler (Technion) and Götz Kersting (GU FfM)

Dedicated to Jean-François Le Gall

“**Walks and trees** are abstractly identical objects ... ”

(Ted Harris (1952))



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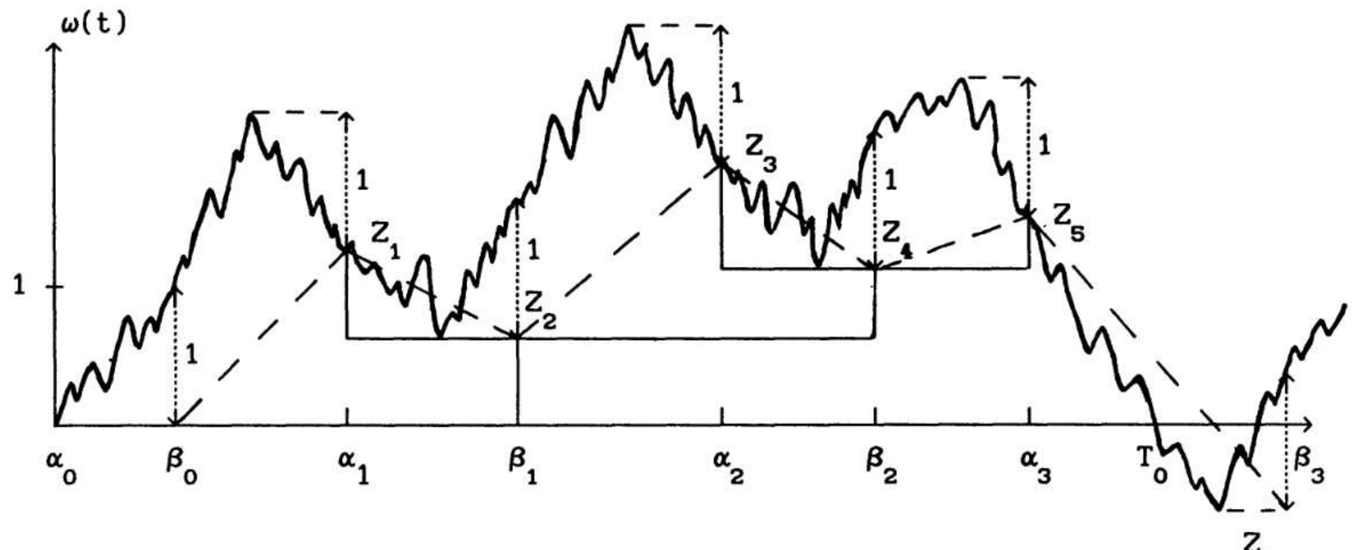
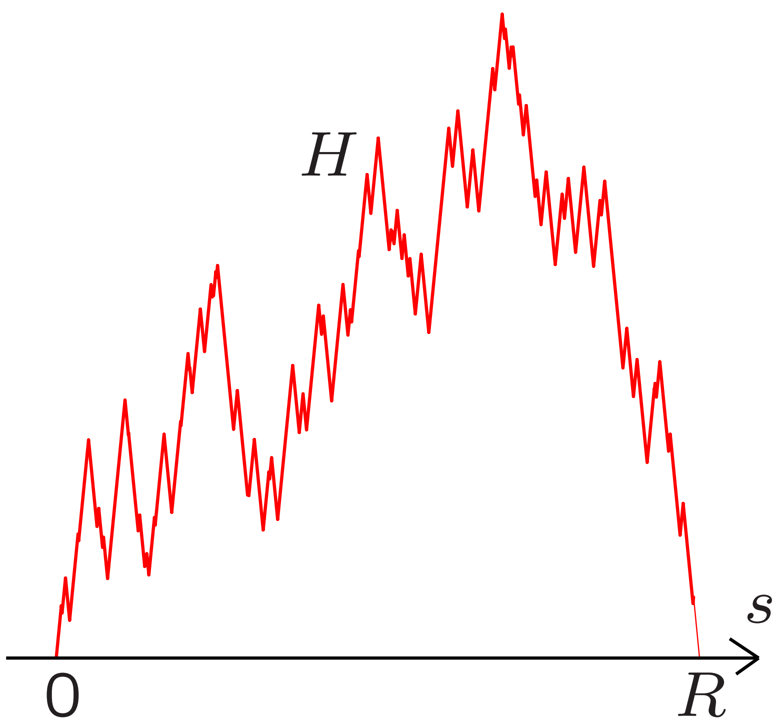


Figure 3 from: J.-F. Le Gall, Marches aléatoires, mouvement brownien et processus de branchement, Séminaire de Probabilités, XXIII, 1989

The *rooted, ordered* $\mathbb{R}$-tree $(T^H, d, \prec)$:

For $0 \leq s_1 \leq s_2 \leq R$:

$s_1 \sim s_2 :\Longleftrightarrow H(s_1) = H(s_2) = \min\{H(s) : s \in [s_1, s_2]\}$



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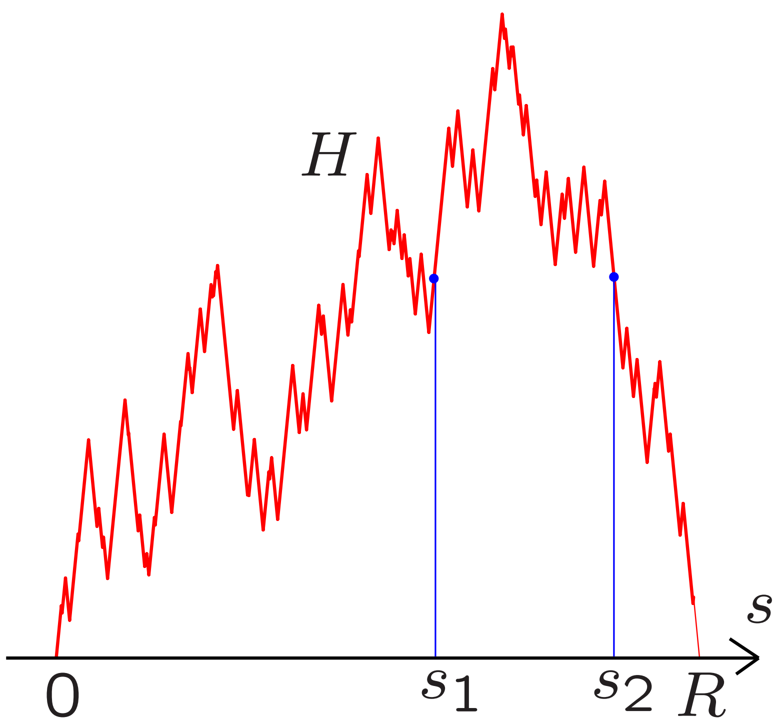

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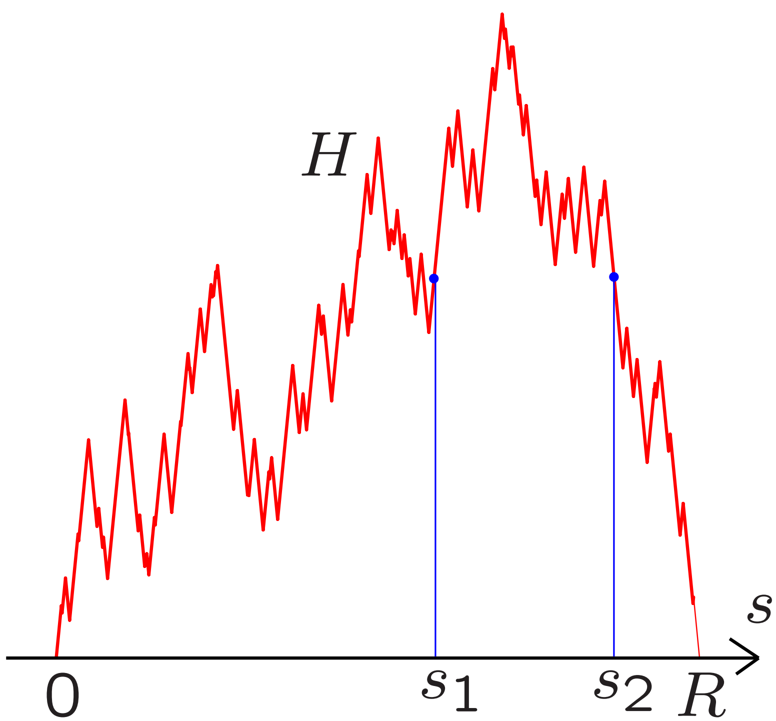

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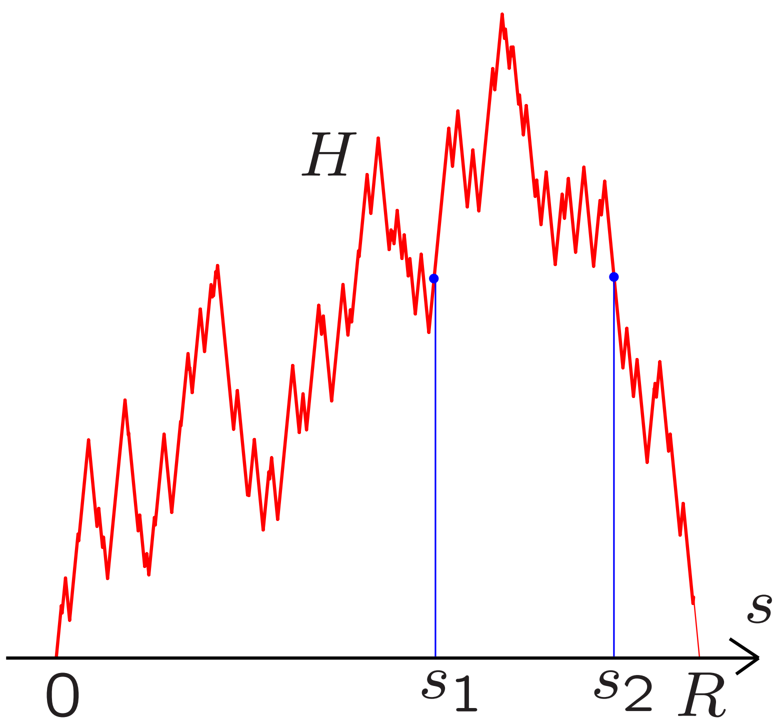

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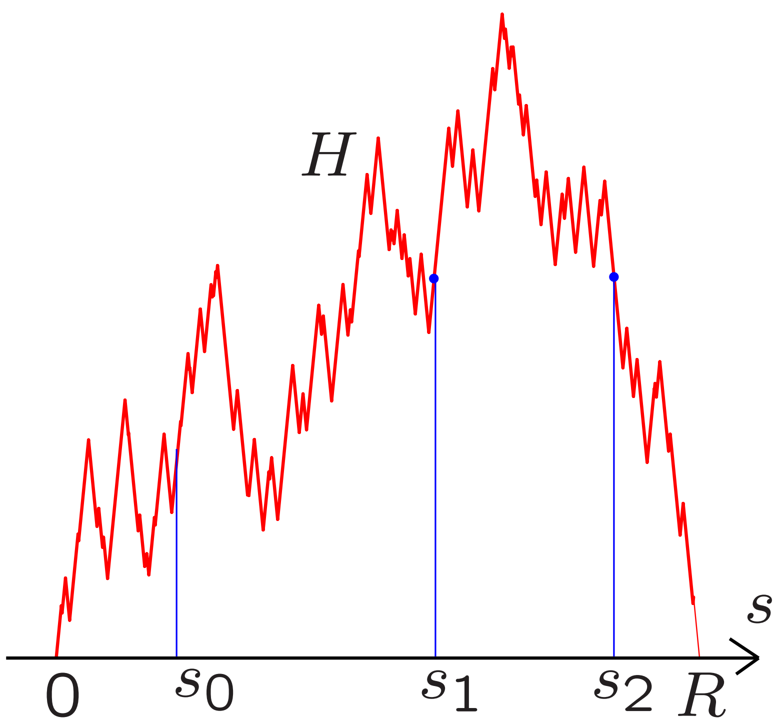

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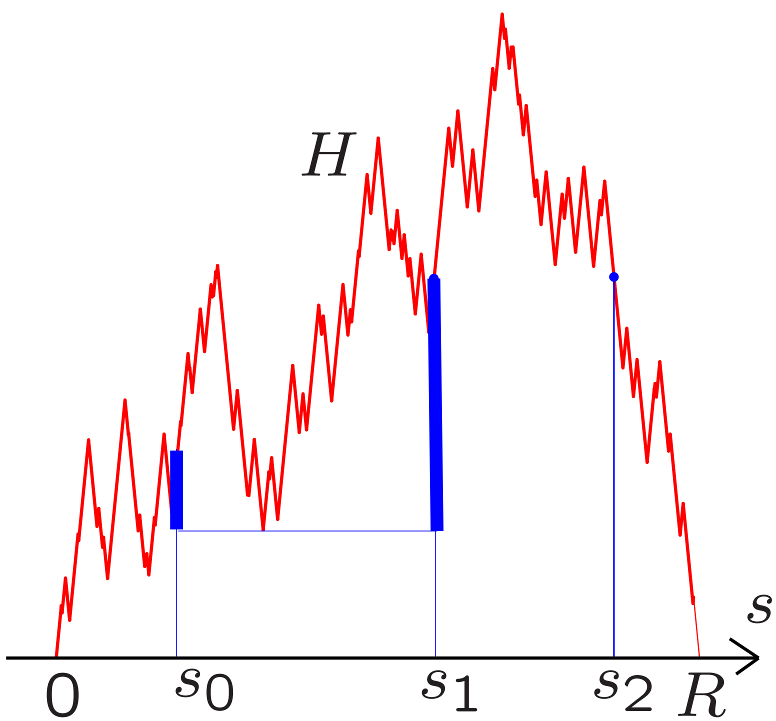

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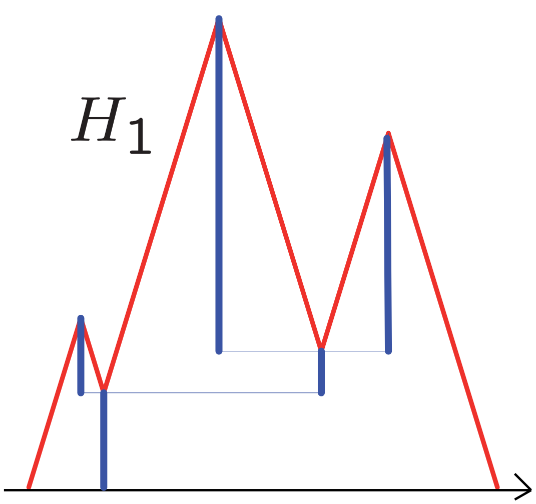
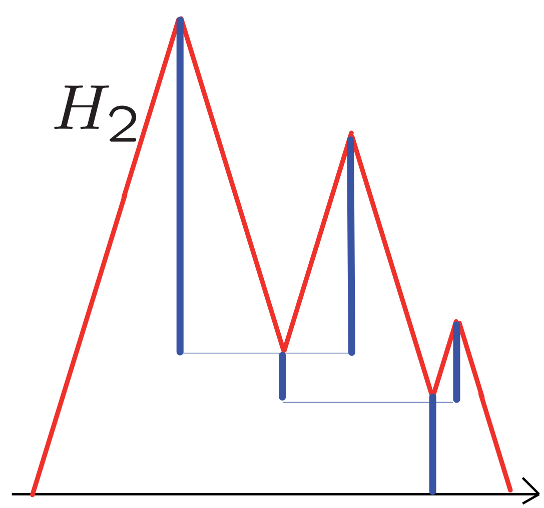
$$d(\langle s_0 \rangle, \langle s_1 \rangle) := H(s_0) + H(s_1) - 2\min\{H(s) : s \in [s_0, s_1]\}$$

The isomorphy class of $(T^H, d, \prec)$ will be denoted by $\mathbb{T}^H_\prec$.

The root-preserving isometry class of (T^H, d) will be denoted by $\mathbb{T}^H$.

Example:





$$\mathbb{T}^{H_1}_\prec \neq \mathbb{T}^{H_2}_\prec$$

but

$$\mathbb{T}^{H_1} = \mathbb{T}^{H_2}.$$

For H a normalized Itô excursion

(i.e. conditioned to $R = 1$) ,

$\mathbb{T}^H$ is the “classical” CRT of Aldous.

The uniform random tree in a Brownian excursion

Jean-François Le Gall

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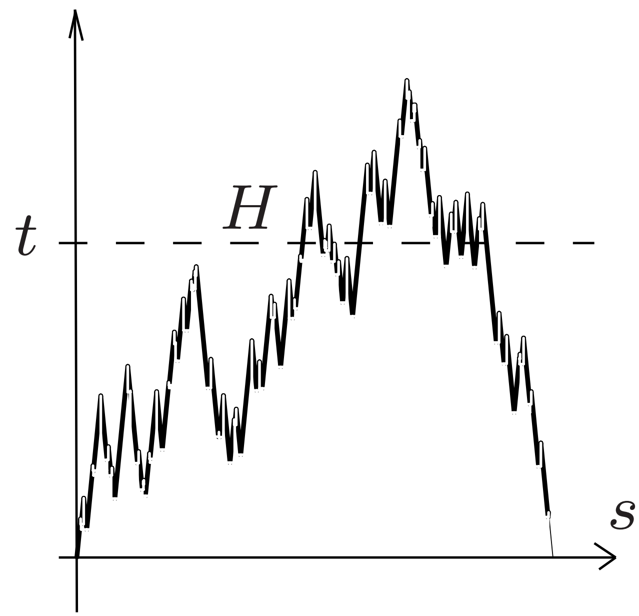
Received October 12, 1992

Summary. To any Brownian excursion e with duration $\sigma(e)$ and any $t_1, \ldots, t_p \in [0, \sigma(e)]$, we associate a branching tree with p branches denoted by $T_p(e, t_1, \ldots, t_p)$, which is closely related to the structure of the minima of e. Our main theorem states that, if e is chosen according to the Itô measure and $(t_1, \ldots, t_p)$ according to Lebesgue measure on $[0, \sigma(e)]^p$, the tree $T_p(e, t_1, \ldots, t_p)$ is distributed according to the uniform measure on the set of trees with p branches. The proof of this result yields additional information about the "subexcursions" of e corresponding to the different branches of the tree, thus generalizing a well-known representation theorem of Bismut. If we replace the Itô measure by the law of the normalized excursion, a simple conditioning argument leads to another remarkable result originally proved by Aldous with a very different method.

“Counting” the number of subexcursions above height t:



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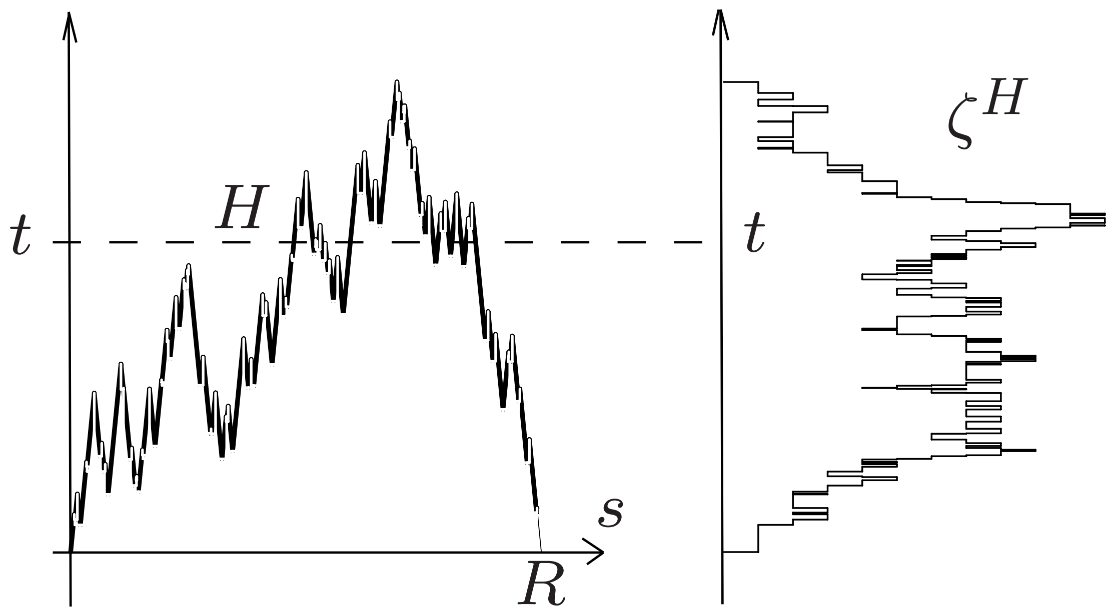

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$$\zeta_t^H := L^H(t,R)$$

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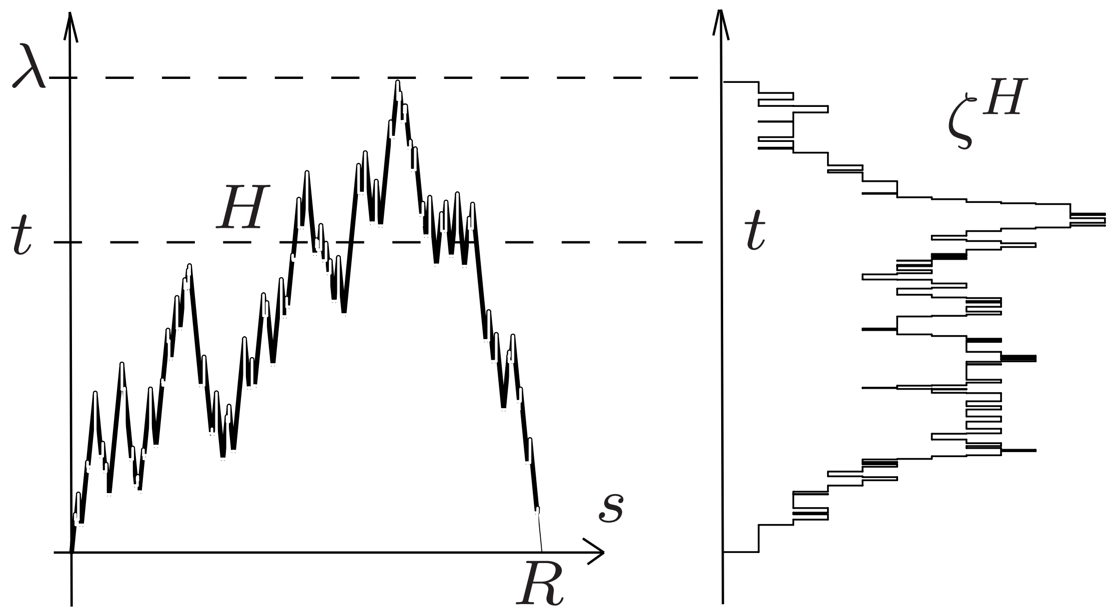

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$$\zeta^H_t := L^H(t,R)$$

$$\zeta^H := \left(\zeta^H_t\right)_{0\le t\le\lambda}$$ ... the local time profile of H

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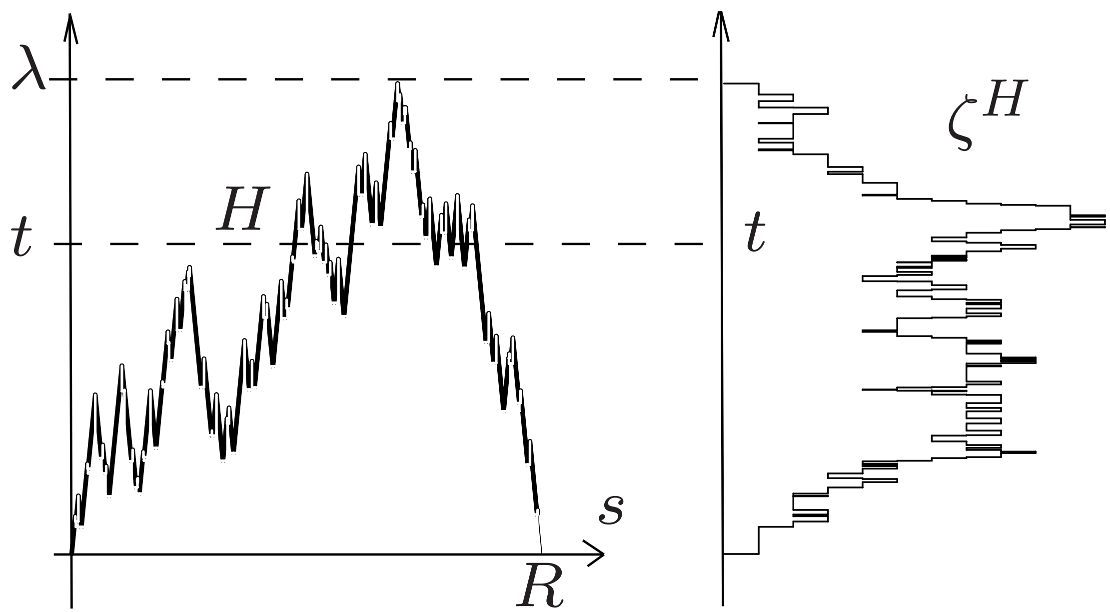

By the Ray-Knight theorem, $H \mapsto \zeta^H$ transports the Itô excursion measure into the excursion measure of Feller’s branching diffusion $d\zeta_t = \sqrt{4\zeta_t}\, dW_t$.

We will condition on $\left\{ \sup_{0 \le s \le R} H_s > 1 \right\}$

This turns the excursion measure into a probability measure and allows to speak of H and ζ^H as random variables.

How to go back from ζ^H

to H, $\mathbb{T}^H_{\prec}$ and $\mathbb{T}^H$?

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Quote from D. Aldous (1998), *Brownian excursion conditioned on its local time*:

"Given a local time profile ζ, can we define a process H^ζ whose law is, in some sense, the conditional law of H given $L = \zeta$?"

We will see that H is made up of

three independent ingredients $\zeta^H, \Lambda^H, \gamma^H$,

with

the triple $\left(\zeta^H, \Lambda^H, \gamma^H\right)$ coding for $\mathbb{T}^H_{\prec}$,

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Let us now turn to the second ingredient Λ^H.

Λ^H is a point measure whose points (i, j, τ) are in 1-1 correspondence with the local minima of H.

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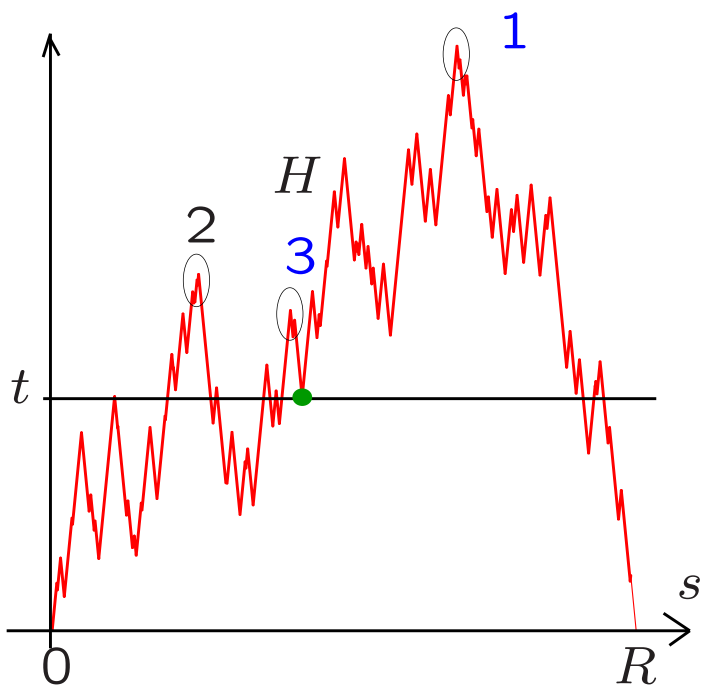
Let $t = H_s$ be the height at which a local minimum occurs. $i < j$ are the height ranks of the two subexcursions in H above t that are attached to this local minimum among all subexcursions in H above t.



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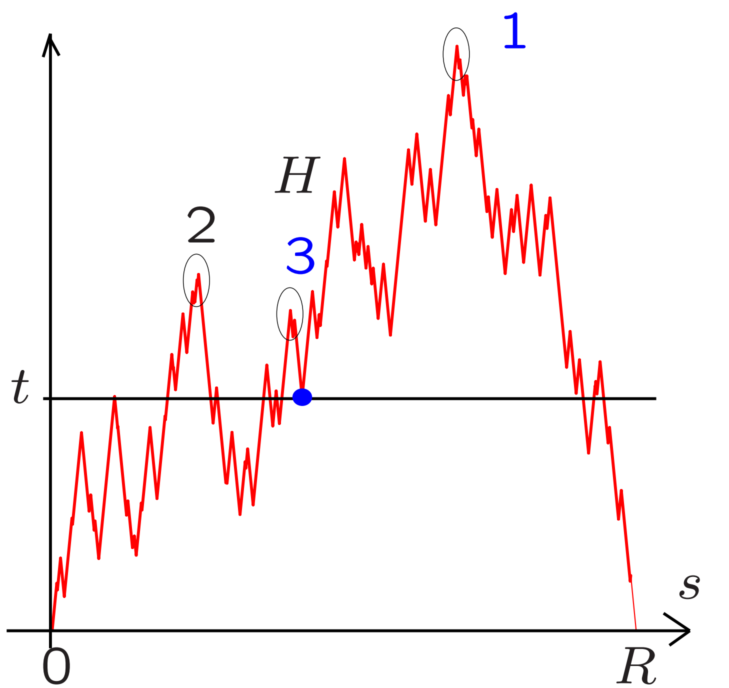

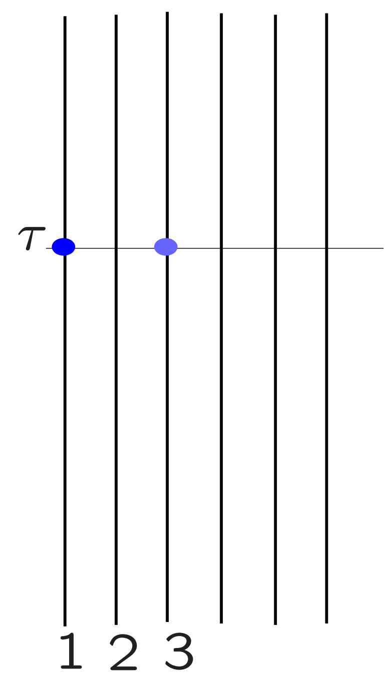

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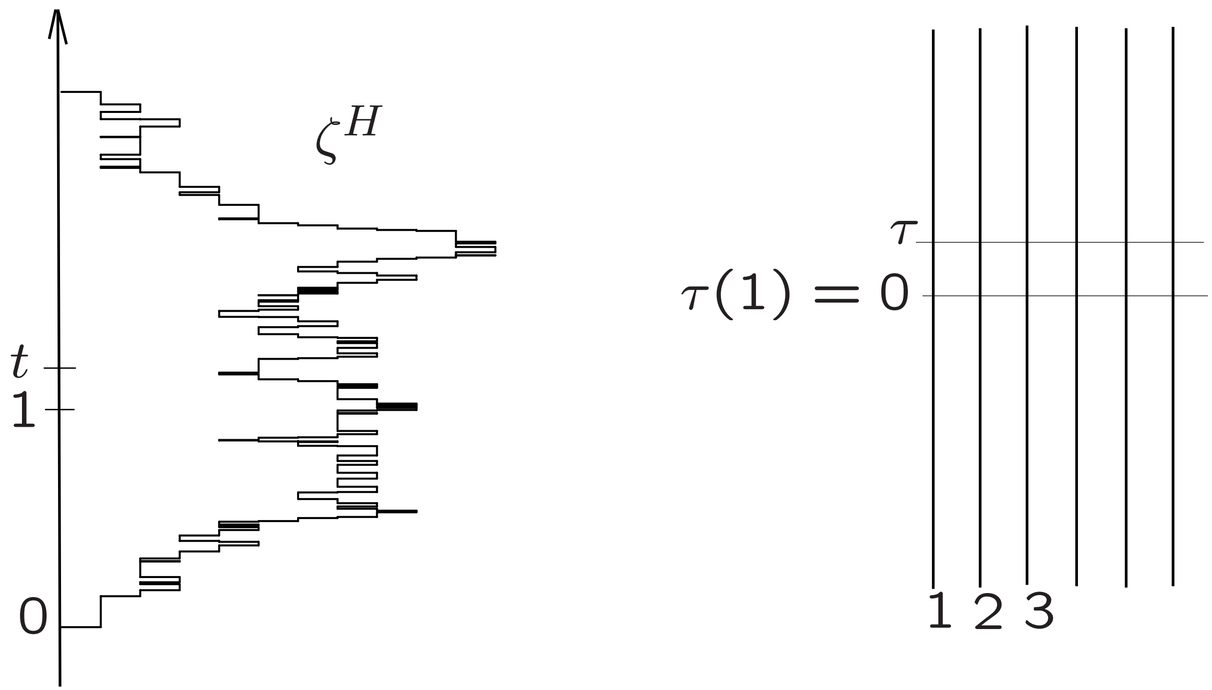
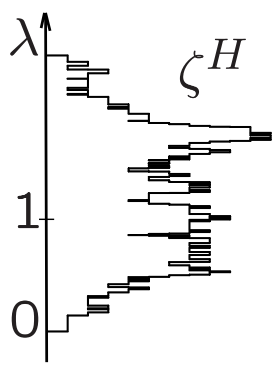
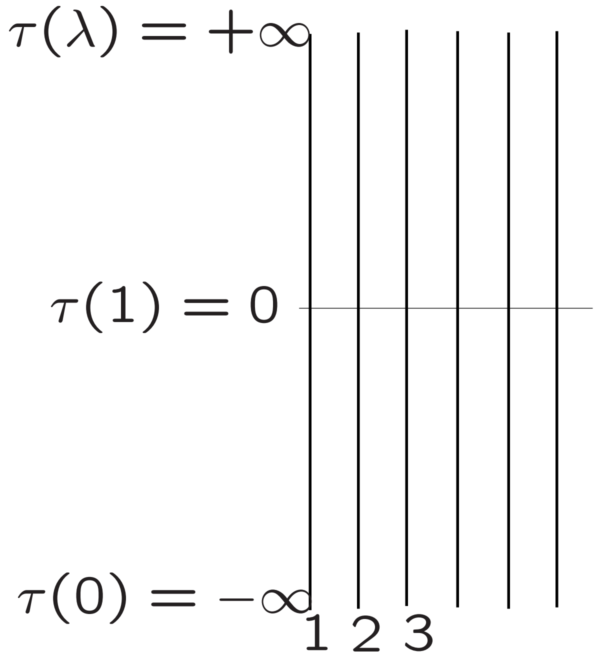
$$\tau := \tau(t) := \int_0^t \frac{4}{\zeta_u^H} du.$$



Almost surely,

$$t \mapsto \tau(t) := \int_1^t \frac{4}{\zeta_u^H}\, du \quad \text{maps } (0, \lambda) \text{ bijectively to } \mathbb{R}.$$





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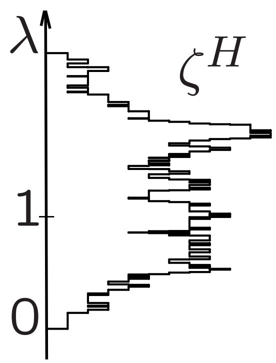
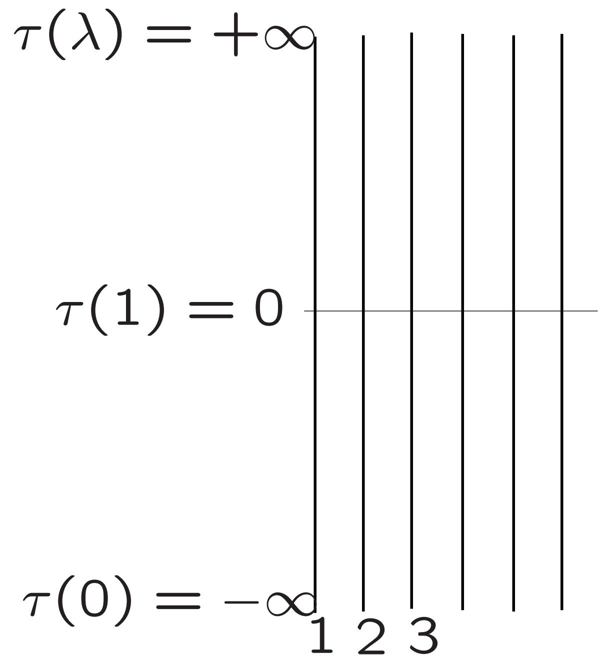
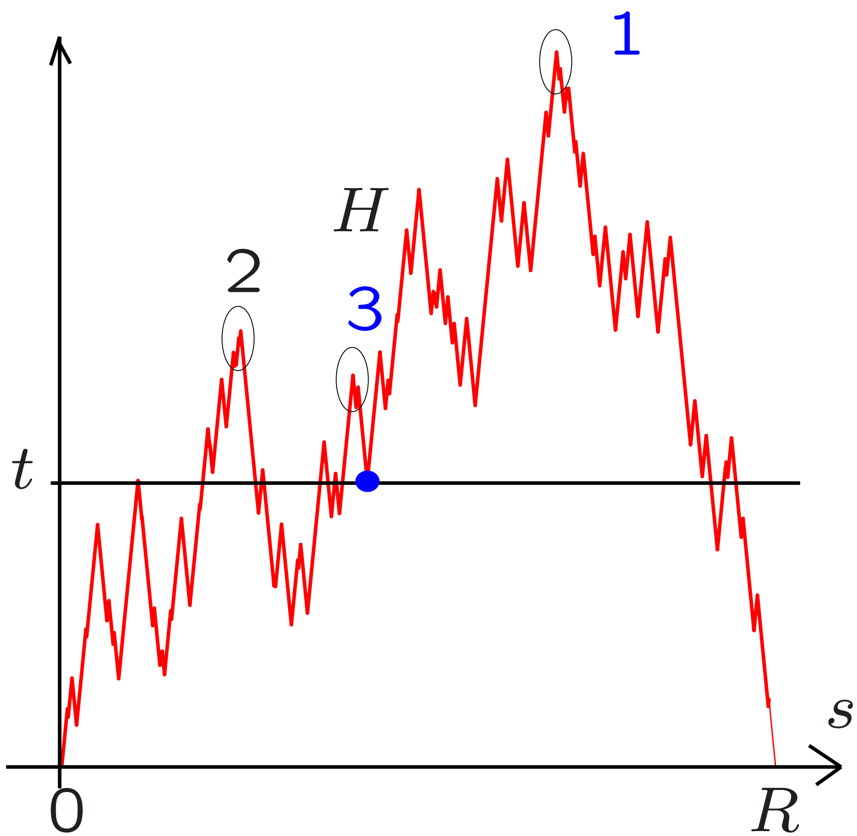


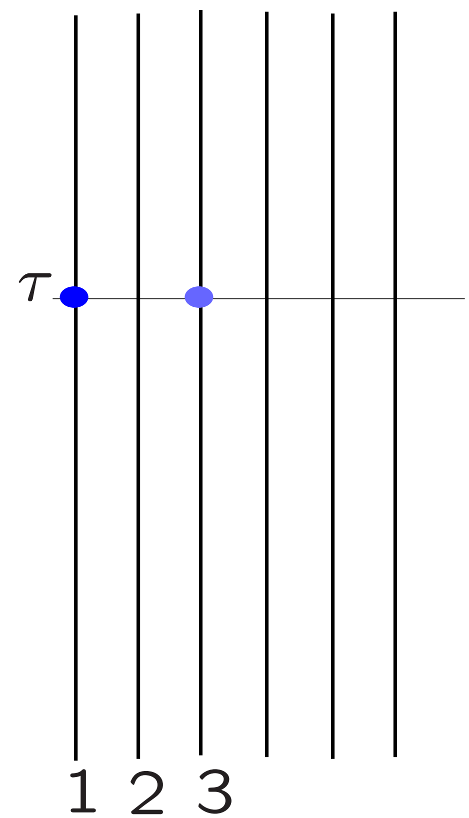


$t \mapsto \tau(t)$ is the time change from Perkin's desintegration theorem relating superbrownian motion to Fleming-Viot processes.

Λ^H is a random point measure on

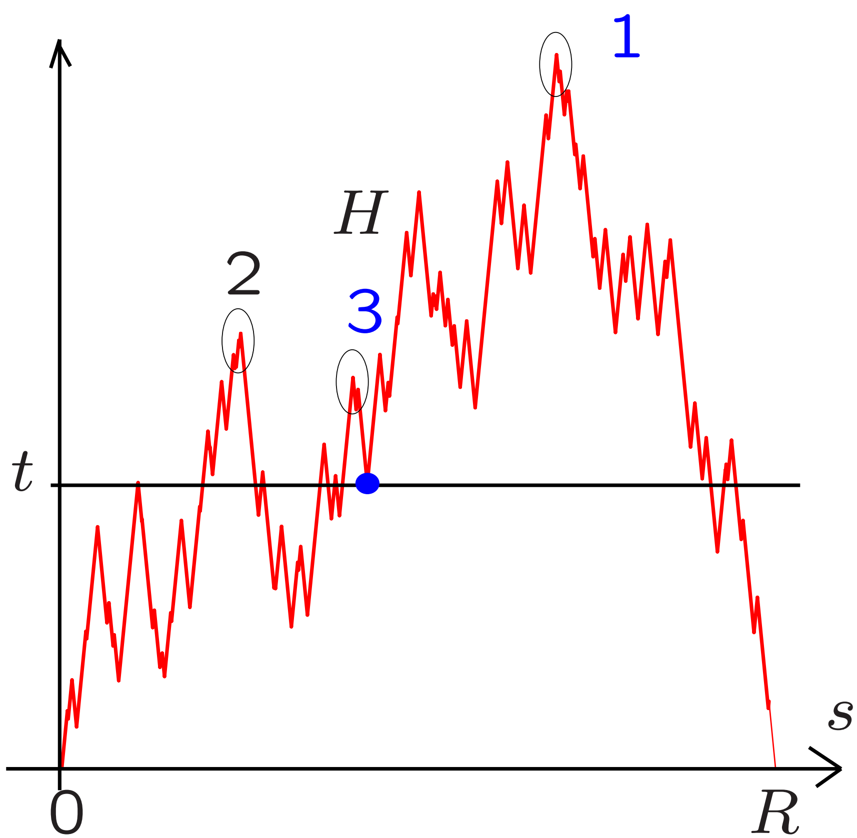
$$\{(i,j) : 1 \le i < j \in \mathbb{N}\} \times \mathbb{R}$$

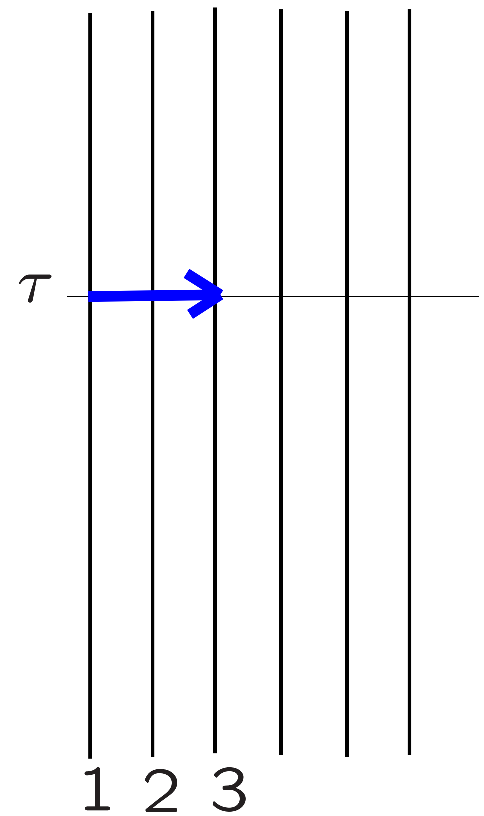




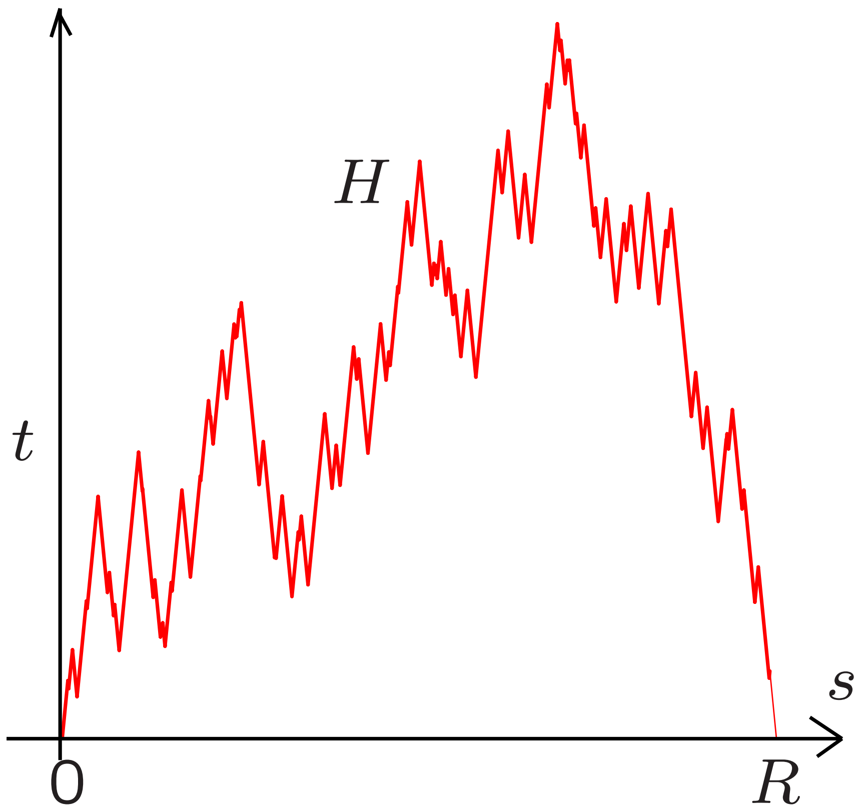
Visualize a point (i, j, τ)

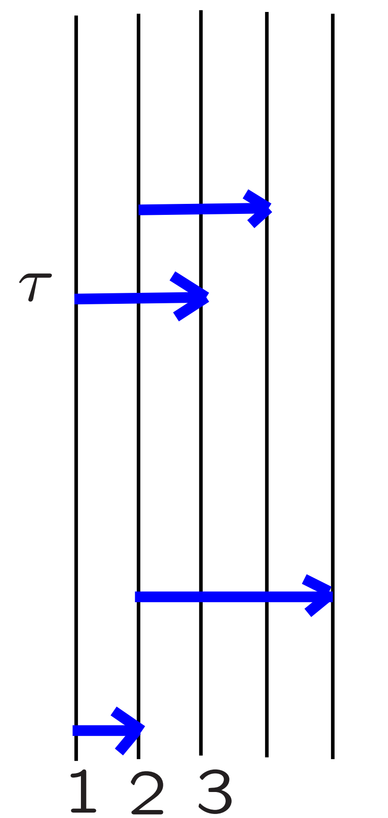
by an arrow from i to j at time τ.





Then Λ^H becomes a random configuration of horizontal arrows on $\mathbb{N} \times \mathbb{R}$.





Theorem 1 (S. Gufler, PhD, 2017)

$$\Lambda^H_{ij} := \Lambda^H(\{(i,j)\} \times (\cdot))$$

are independent rate 1 Poisson point processes,

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A precursor of this result is
J. & N. Berestycki (2009), *Kingmans coalescent and Brownian motion*.
Among others, they cite Perkins (1991), Le Gall (1989, 1993), Aldous (1991,93,98), Warren and Yor (1998), Warren (1999).
Gufler (2017) relates the Brownian excursion to the full lookdown picture (between times $-\infty$ and $+\infty$) of Donnelly and Kurtz (1999).

The third ingredient $\gamma^H = \left(\gamma^H_{ijk}\right)$:

For $i < j$, let $(\tau_{ijk})_{k\in\mathbb{Z}\setminus\{0\}}$ be the time coordinates of the points in Λ_{ij}, with the convention

$$\cdots < \tau_{i,j,-2} < \tau_{i,j,-1} < 0 < \tau_{i,j,1} < \tau_{i,j,2} < \cdots .$$

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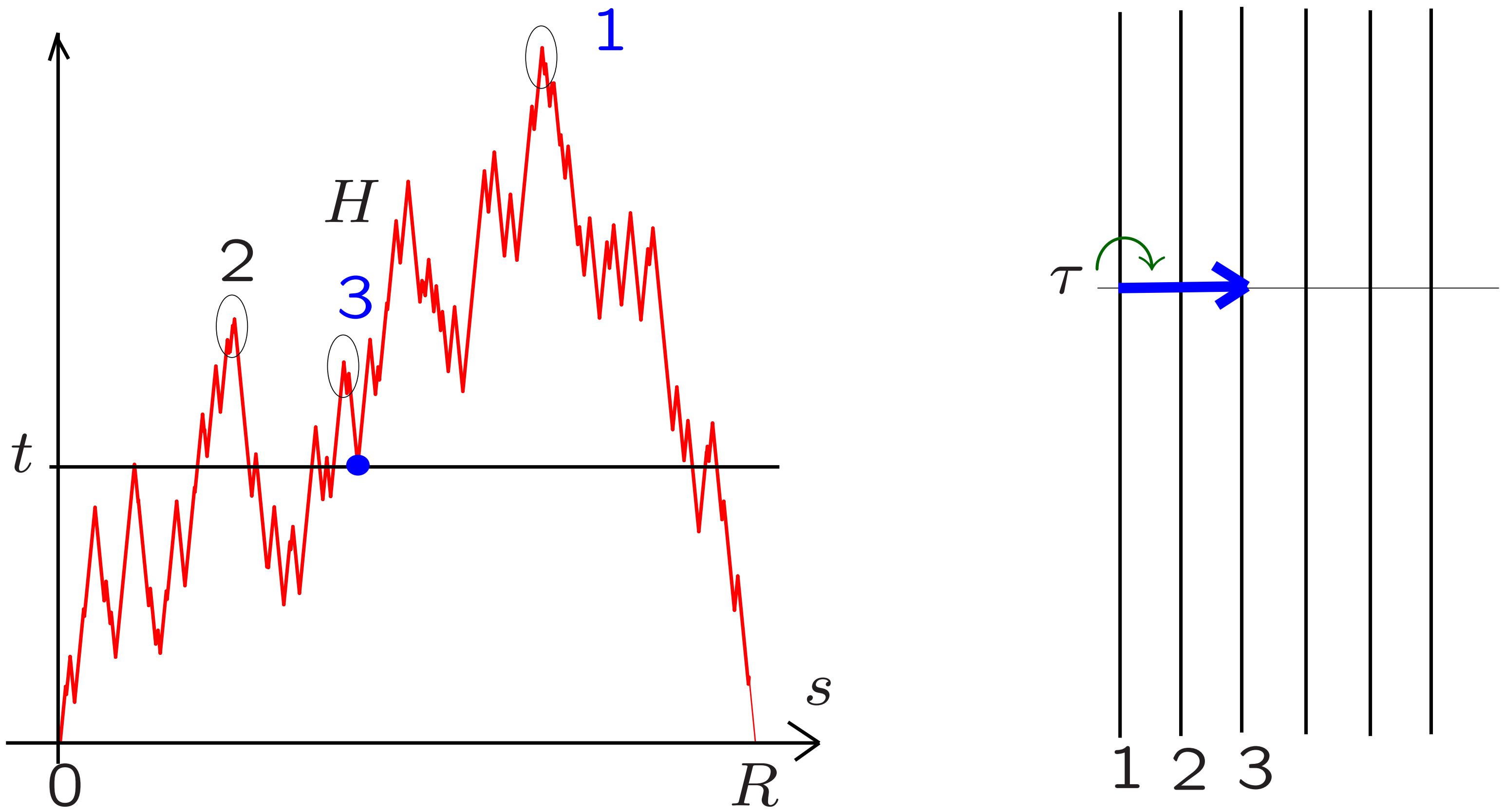
Put $\gamma^H_{ijk} := \curvearrowleft$

if the bigger of the two excursions attached to the corresponding local minimum is to the left

and $\gamma^H_{ijk} := \curvearrowright$

if the bigger of these two excursions is to the right.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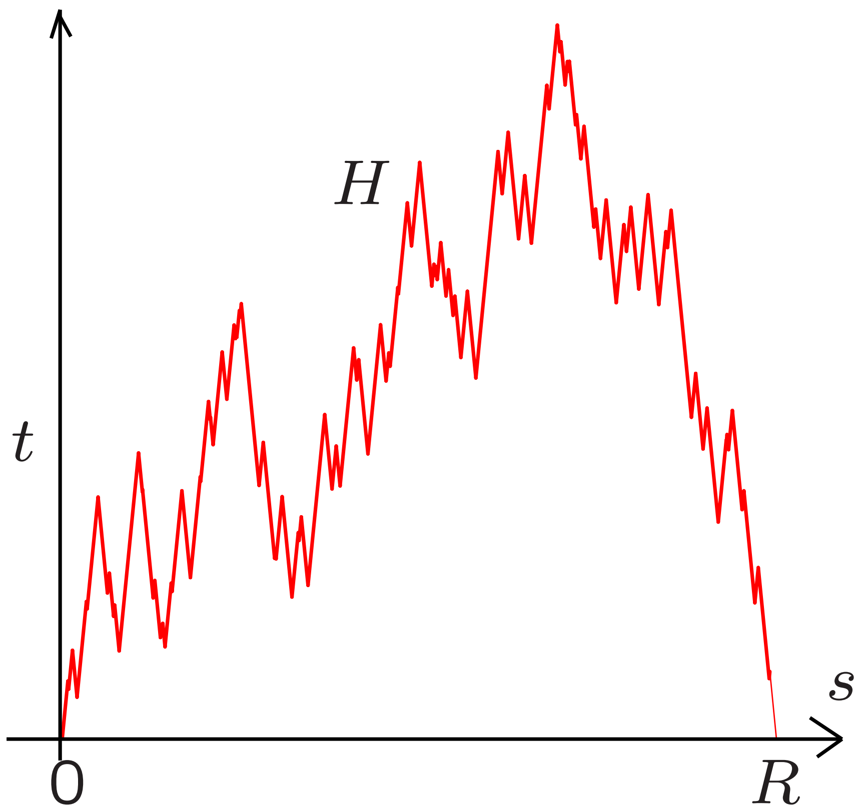
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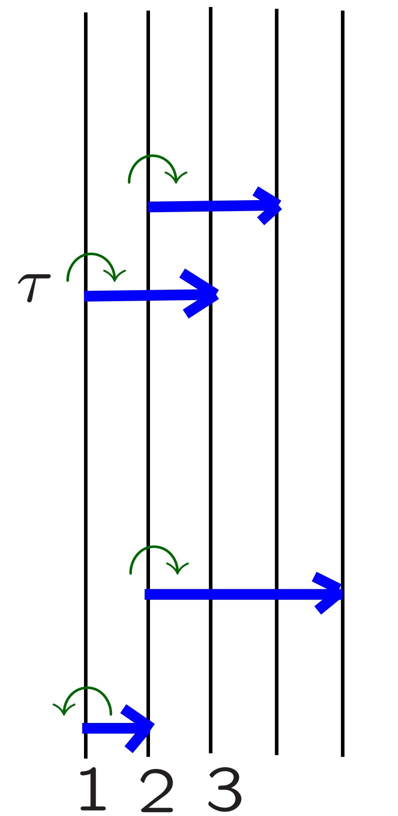
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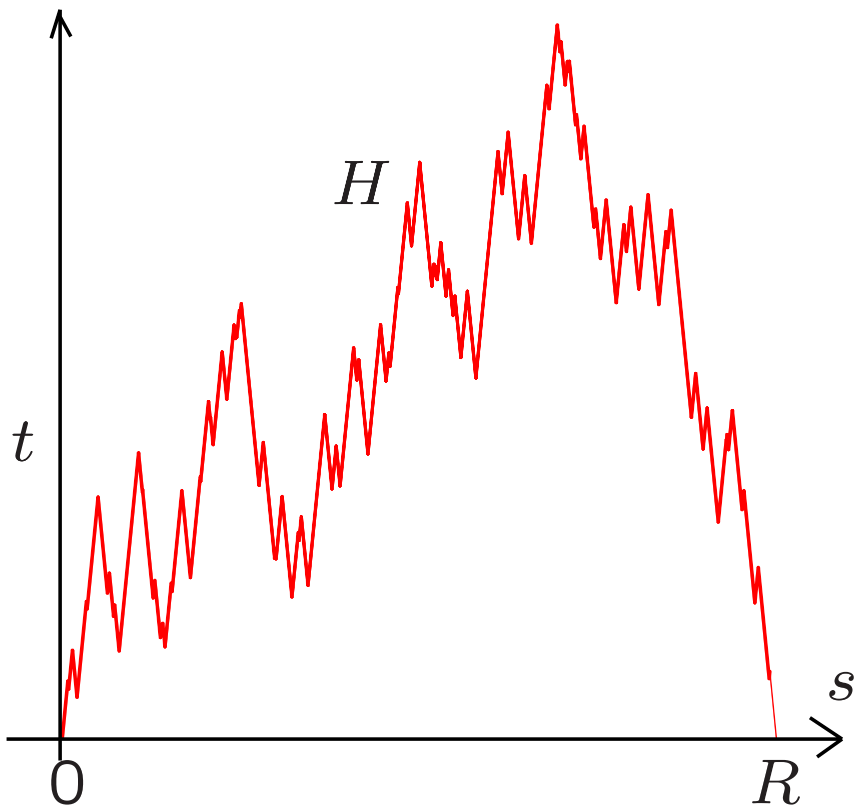
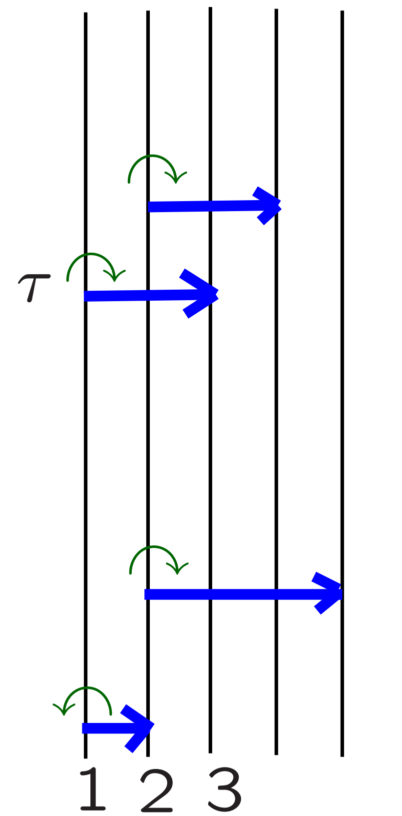




Then $\gamma^H = \left(\gamma^H_{ijk}\right)$ is a fair coin tossing array.

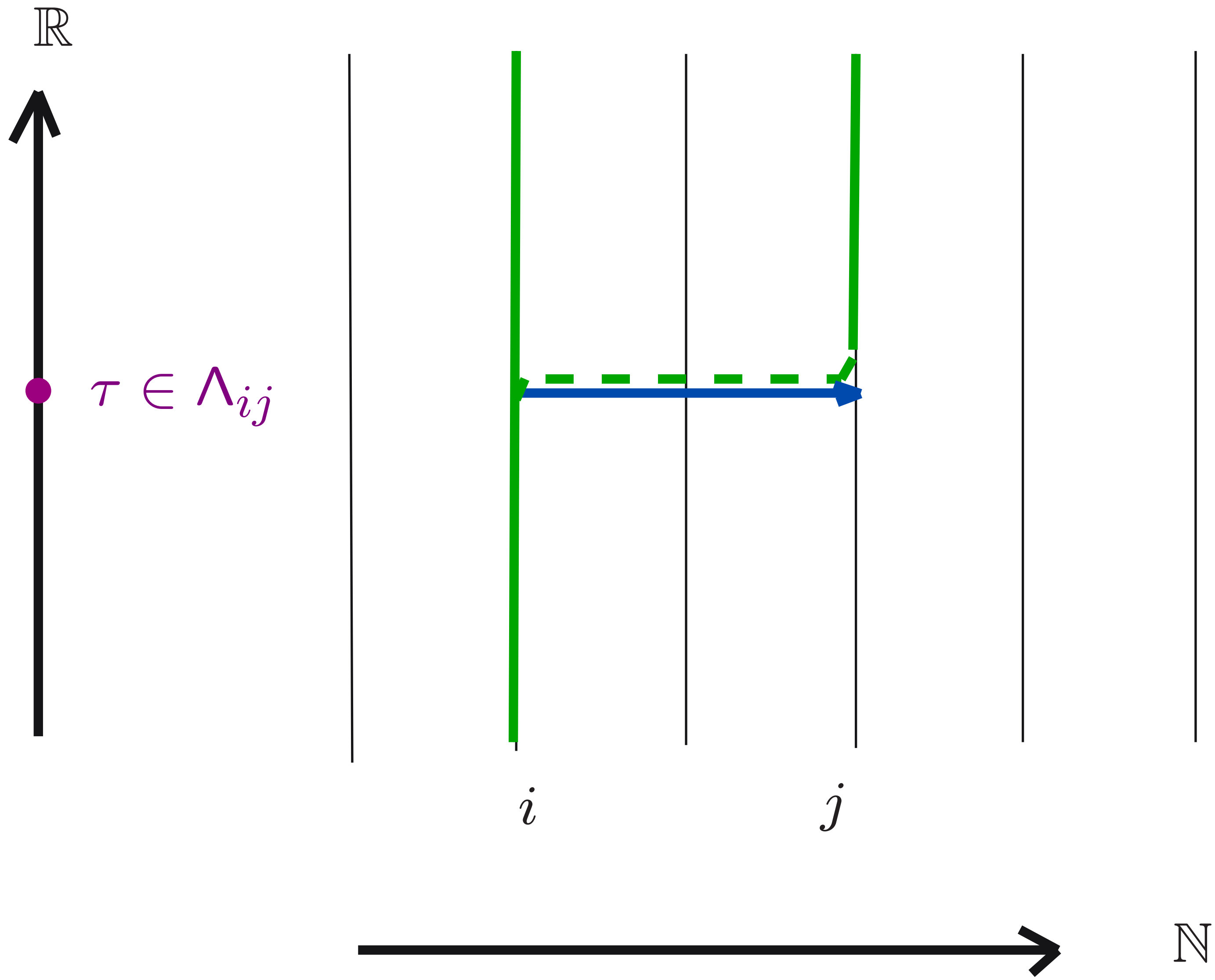
The lookdown space obtained from Λ:

Let Λ_{ij}, $1 \leq i < j$,

be independent rate 1 Poisson point processes.

$\Lambda = (\Lambda_{ij})$ induces a semi-metric $\rho = \rho^{\Lambda}$ on $\mathbb{N} \times \mathbb{R}$

via coalescent ancestral lineages



ℝ
$\tau \in \Lambda_{ij}$
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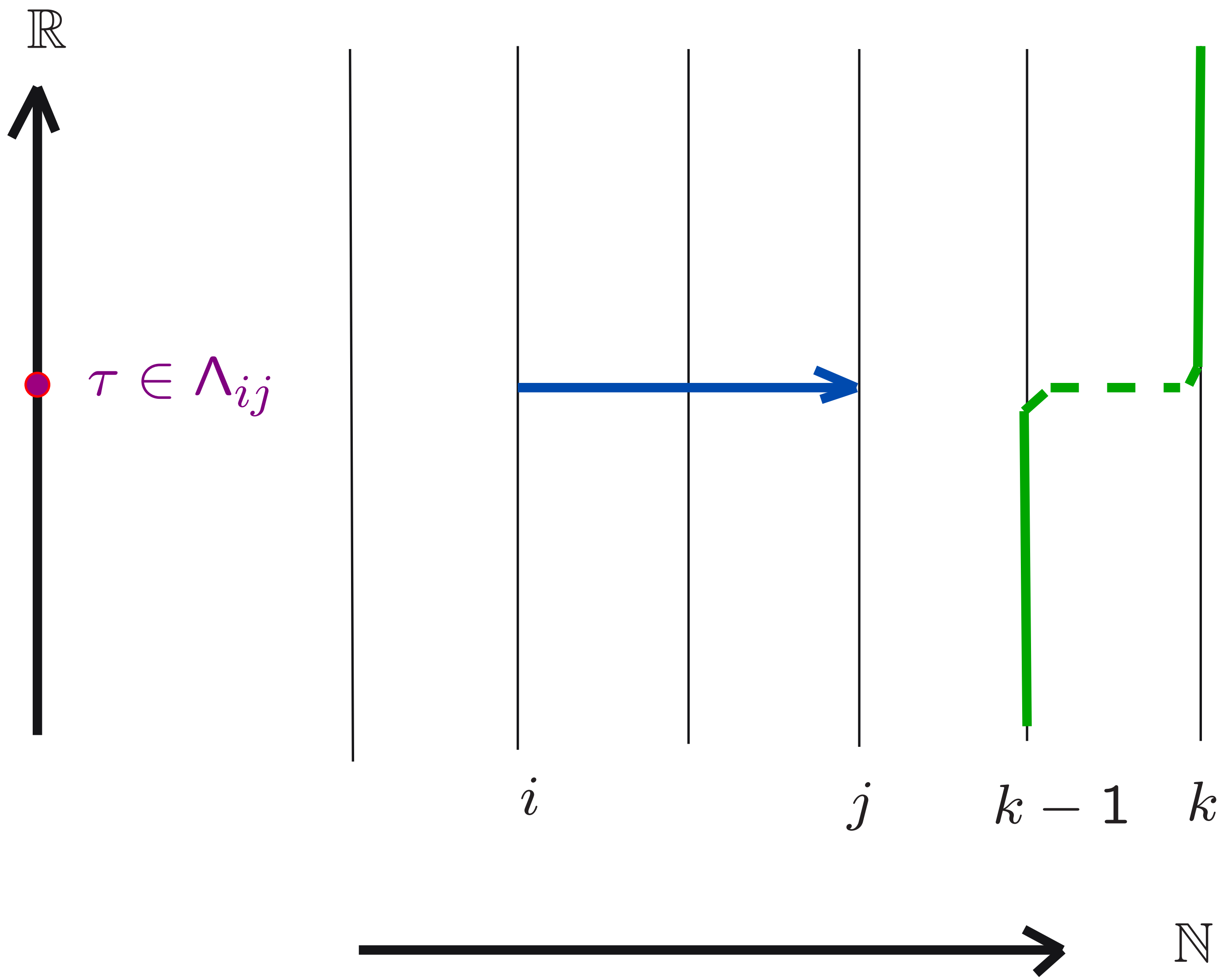
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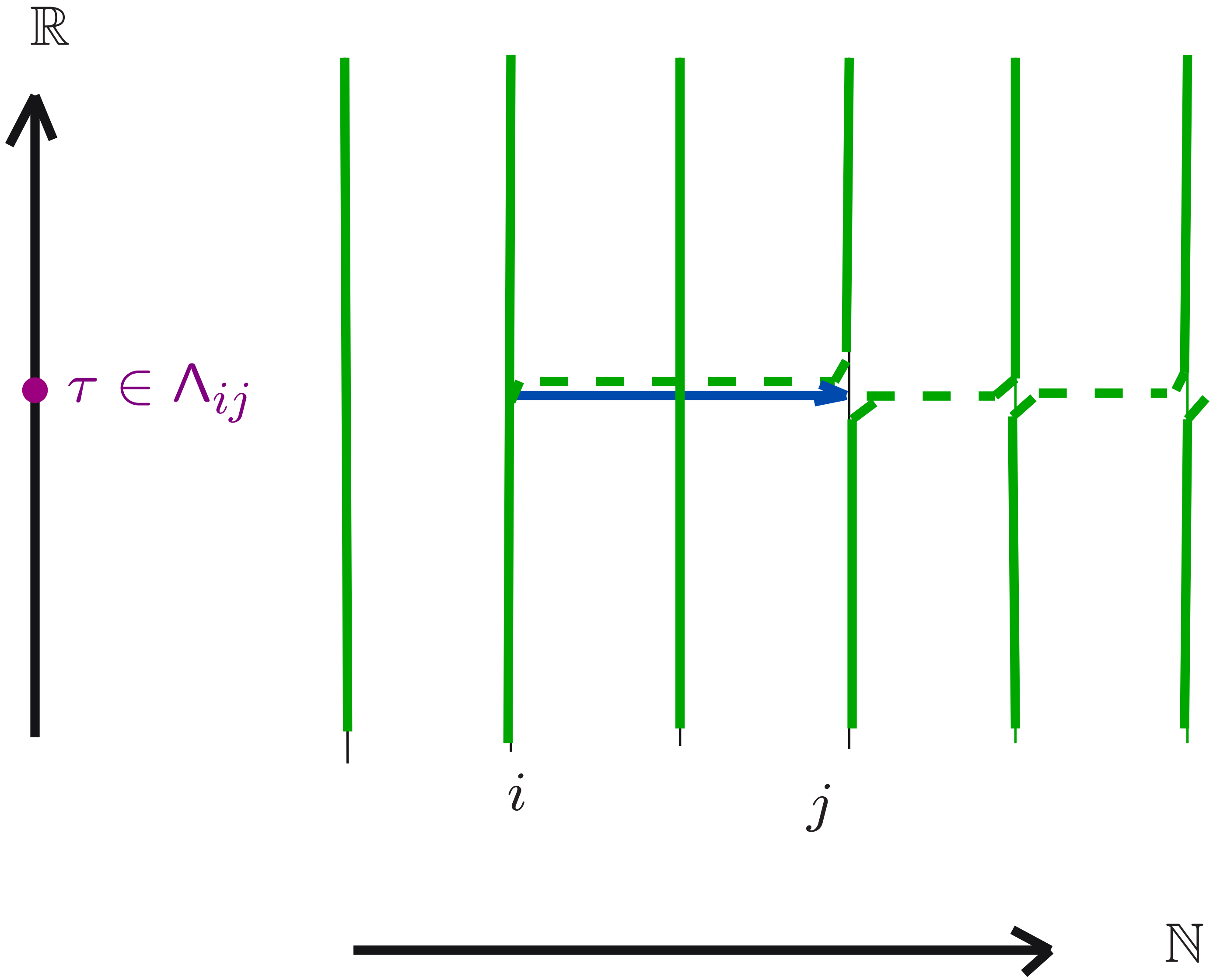

$\mathbb{R}$
$\tau \in \Lambda_{ij}$
i
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$\mathbb{N}$

The closure of $(\mathbb{N} \times \mathbb{R}, \rho^{\wedge})$

is denoted by $(Z^{\wedge}, \rho^{\wedge}) =: (Z, \rho)$,

and called the (random) lookdown space.

(Z, ρ) is a random non-compact $\mathbb{R}$-tree,

and can be compactified to $\bar{Z} := Z \cup \{z_{\mathrm{root}}, z_{\mathrm{top}}\}$,

where we say that

$$z_n \to z_{\mathrm{root}} \quad \text{if } \tau(z_n) \to -\infty =: \tau(z_{\mathrm{root}}),$$

$$z_n \to z_{\mathrm{top}} \quad \text{if } \tau(z_n) \to +\infty =: \tau(z_{\mathrm{top}}).$$

If ζ is a Feller excursion (independent of Λ), we put

$$\rho_\zeta((i,\tau),(i,\tau+d\tau)) := \tfrac{1}{4}\zeta(t(\tau))\,d\tau,$$

$$\rho_\zeta((i,\tau_{ijk}),(j,\tau_{ijk})) := 0,$$

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Now think of Λ and ζ both arising from a Brownian excursion H.

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Now think of Λ and ζ both arising from a Brownian excursion H.

Proposition 1:

For $(Z,\rho) := \left(Z^{\Lambda^H}, \rho^{\Lambda^H}\right)$ and $\zeta := \zeta^H$,

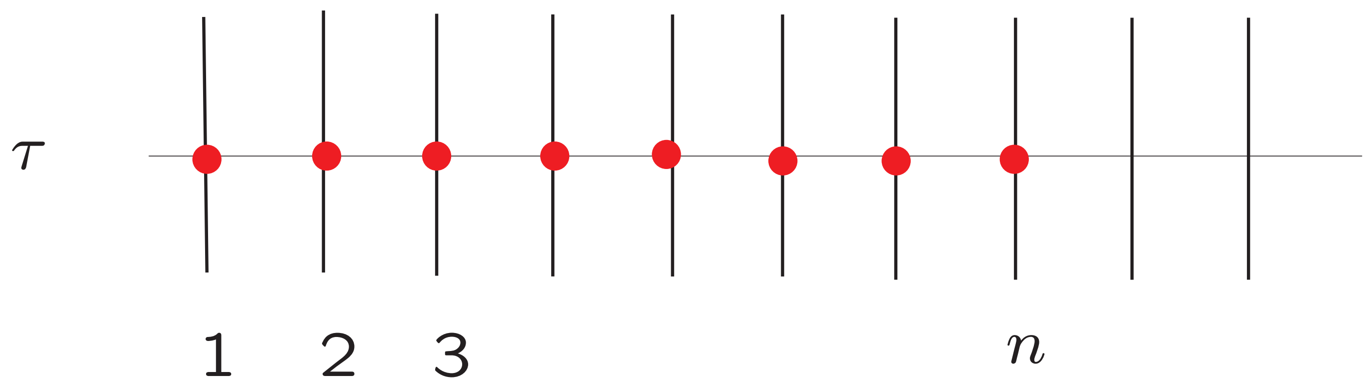
(T^H, d) and $\left(\bar{Z}, \rho_\zeta\right)$ are a.s. root-preserving isometric.

Theorem 2 (S. Gufler, EJP, 2018)

The lookdown space (Z, ρ) carries a family $(\mu_\tau)_{\tau \in \mathbb{R}}$ of probability measures such that a.s. for all $\tau \in \mathbb{R}$,

$$\mu_\tau = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{(i,\tau)},$$

in the weak topology on the probability measures on (Z, ρ).



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The EJP paper contains a direct proof of Thm 2.
An alternative (elegant) proof (see Gufler's PhD thesis) works
by embedding the lookdown space into a Brownian excursion H,
i.e. by appealing to Theorem 1 and proving the assertion for $(Z^{\lambda^H}, \rho^{\lambda^H})$.
This is achieved via the *uniform downcrossing representation*
for local times due to Chacon, Le Jan, Perkins and Taylor (1981).

Marrying Λ and γ leads to an ordering of Z^{Λ}:

Let $(Z, \rho) = (Z^{\Lambda}, \rho^{\Lambda})$ be a lookdown space, and

$\gamma = (\gamma_{ijk})$ be an $\{\curvearrowleft, \curvearrowright\}$-valued array, independent of Z.

With the help of γ we define a total order $\prec$ on Z:

For $y, z \in Z$ with z descending from y,

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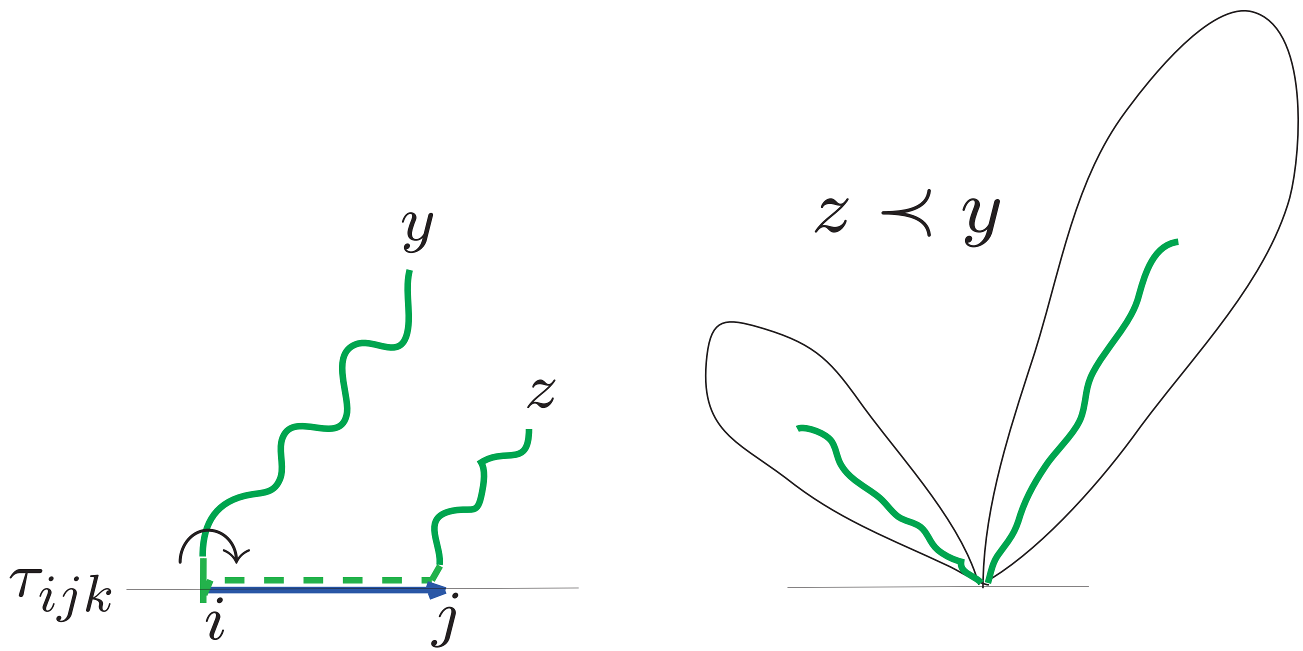
With the help of γ we define a total order $\prec$ on Z:

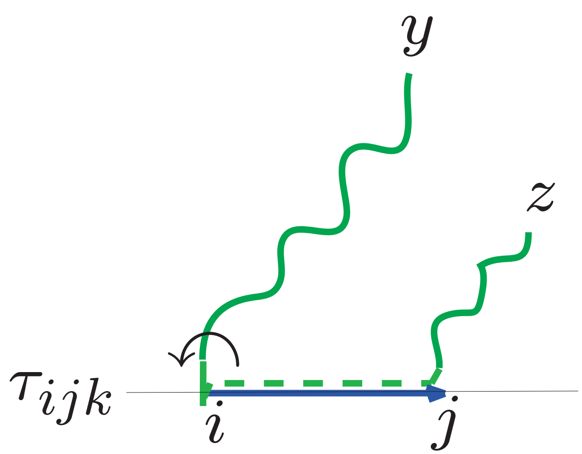
For $y, z \in Z$ not connected by a single line of descent,
their most recent common ancestor is of the form
(i, τ_{ijk}) for some $i < j \in \mathbb{N}$ and some k.

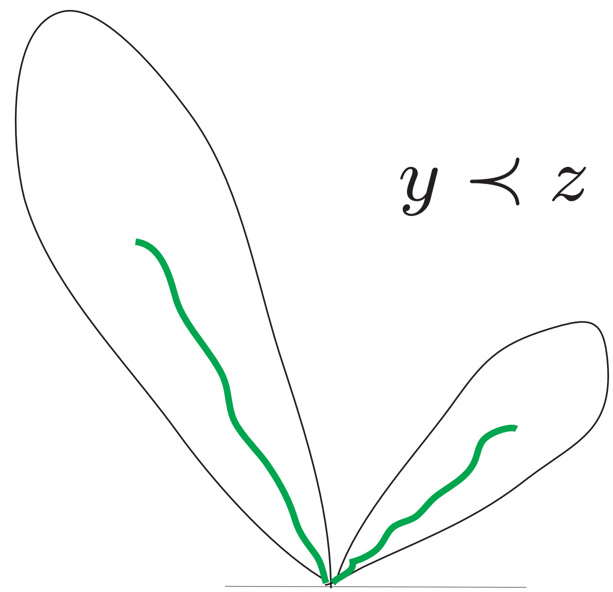
Assume w.l.o.g. that z descends from (j, τ_{ijk}).

We then put $z \prec y$ if $\gamma_{ijk} = \curvearrowright$



$\ldots$ and $y \prec z$ if $\gamma_{ijk} = \curvearrowleft$.





Next we define the *time of the first exploration of* $z \in \bar{Z}$ by

$$s(z) := \int_{-\infty}^{\infty} \mu_{\mathcal{T}}(\{y \prec z\})\,(\zeta_{t(\tau)}^2/4)d\tau\,, \quad z \in Z,$$

$$s(z_{\text{root}}) := 0, \quad s(z_{\text{top}}) := \lim_{z \to z_{\text{top}}} s(z).$$

Theorem 3:

For $(Z, \rho) := \left(Z^{\wedge^H}, \rho^{\wedge^H}\right)$, $\zeta := \zeta^H$ and $\gamma := \gamma^H$, the mapping $z \mapsto \langle s(z) \rangle$ is a root-, order- and measure-preserving isometry from $(\bar{Z}, \rho_\zeta, \prec)$ to $(T^H, d, \prec)$.

The correspondence between the mass measures $\mu_{\mathcal{T}}(dz)$ and $L(t, ds)$ is then given by

$$\mu_{\mathcal{T}}(\{y : y \prec z\}) = L(t(\tau), s(z))/\zeta_{t(\tau)}.$$

Recall:

A Brownian excursion H conditioned to height > 1

corresponds to an independent triple (ζ, Λ, γ)

where

ζ is a Feller branching diffusion excursion conditioned to survive time 1,
Λ is the Poisson process of points (i, j, t) in the lookdown space,
γ is a fair coin-tossing.

Reweighting the law of the excursion

A reweighting of ζ does not affect Λ and γ.

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However, Feller's logistic branching diffusion can also be obtained as the local time profile of H when exposing H to a *local time drift*.

How does a local time drift acting on H affect $\mathbb{T}^H_{\prec}$ and $\mathbb{T}^H$?

For simplicity we focus on

Feller's branching diffusion with competition (quadratic killing)

$$\zeta_t = -c\zeta_t^2 dt + 2\sqrt{\zeta_t} dW_t.$$

With $\zeta := \zeta_H$, consider the two Girsanov reweightings

$$J_c := \exp\left(-\frac{c}{4}\int_0^{\lambda(\zeta)} \zeta_t \, d\zeta_t - \frac{c^2}{8}\int_0^{\lambda(\zeta)} \zeta_t^3 dt\right),$$

$$G_c := \exp\left(-c\int_0^{R} L(H_s, s)dH_s - \frac{c^2}{2}\int_0^{R} L(H_s, s)^2 ds\right).$$

Theorem 4 (Pardoux&W., ECP 2011)

$$\mathscr{L}_{G_c\mathbf{P}}(\zeta^H) = \mathscr{L}_{J_c\mathbf{P}}(\zeta^H)$$

In words: The local time profile of the Itô excursion H under a local time drift is distributed like the Feller excursion ζ under a quadratic killing drift.

Under the local time drift of H the trees are under attack from the left. This induces a skeweness of H.

Therefore we cannot hope that the ordered tree $\mathbb{T}^H_\prec$ has the same distribution under the local time drift of H as it has under a mere reweighting of its local time profile law.

Thus, in general, $\mathscr{L}_{G_c\mathbf{P}}(\mathbb{T}^H_\prec) \neq \mathscr{L}_{J_c\mathbf{P}}(\mathbb{T}^H_\prec)$.

We conjecture that also

$$\mathscr{L}_{G_c\mathbf{P}}(\mathbb{T}^H) \neq \mathscr{L}_{J_c\mathbf{P}}(\mathbb{T}^H).$$

In words the conjecture says that

the root-preserving isometry class of T^H does not have

the same distribution under the local time drift of H

as it has under a mere reweighting of its local time profile law.

Remember that T^H is (ζ^H, Λ^H)-measurable.

Thus the question if (G_c, ζ^H) is independent of Λ

is of relevance for the validity of the conjecture.

We have

$$G_c = \exp\left(-c\int_0^R L(H_s, s)dH_s - \frac{c^2}{2}\int_0^R L(H_s, s)^2 ds\right).$$

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The occupation time formula yields

$$\int_0^R L(H_s, s)^2 ds = \frac{1}{3}\int_0^\lambda \left(\zeta_t^H\right)^3 dt.$$

Thus, the second factor of G_c is independent of Λ^H.

$$G_c = \exp\left(-c\int_0^R L(H_s,s)dH_s - \frac{c^2}{2}\int_0^R L(H_s,s)^2 ds\right)$$

What about the indepencence of $\int_0^R L(H_s,s)\,dH_s$ and Λ^H?

We conjecture they are dependent, but have no proof so far.

Interestingly, given ζ^H,
$\int_0^R L(H_s,s)\,dH_s$ is a Gaussian.

Proposition 2.

Given $\zeta^H = \zeta$, the distribution of $I^H := \int_0^R L(H_s, s)\, dH_s$ is Gaussian with mean $-R/2$ and variance $\frac{1}{12}\int_0^\lambda \zeta_t^3\, dt$.

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Proof: $J_c = \mathbf{E}[G_c|\zeta]$, hence

$$\exp\left(-\frac{c}{4}\int_0^\lambda \zeta_t\, d\zeta_t - \frac{c^2}{8}\int_0^\lambda \zeta_t^3 dt\right)$$

$$= \mathbf{E}\left[\exp\left(-cI^H - \frac{c^2}{2}\int_0^R L(H_s, s)^2 ds\right) \Big| \zeta\right]$$

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$$\exp\left(+\frac{c}{2}\int_0^\lambda \zeta_t\, dt - \frac{c^2}{8}\int_0^\lambda \zeta_t^3 dt\right)$$

$$= \mathbf{E}\left[\exp\left(-cI^H\right)\Big|\,\zeta\right]\exp\left(-\frac{c^2}{6}\int_0^\lambda \zeta_t^3 dt\right)$$

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$$\exp\left(\frac{c}{2}R + \frac{c^2}{24}\int_0^\lambda \zeta_t^3 dt\right) = \mathbf{E}\left[\exp\left(-cI^H\right)\,\Big|\,\zeta\right]$$

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$$\exp\left(-c\left(\frac{-R}{2}\right) + \frac{c^2}{2}\frac{1}{12}\int_0^\lambda \zeta_t^3 dt\right) =$$

$$= \mathbf{E}\left[\exp\left(-cI^H\right)\,\middle|\,\zeta\right] \qquad \square$$