BRANCHING PROCESSES AND THEIR APPLICATIONS

LECTURE 2: Elementary properties of generating functions; branching processes and simple random walk

V.A.Vatutin
Department of Discrete Mathematics
Steklov Mathematical Institute
Gubkin street, 8
119991 Moscow
RUSSIA
e-mail: vatutin@mi.ras.ru

April 21, 2005

1 Elementary properties of generating functions.

Let
\[ f(s) = E_s^\xi = \sum_{k=0}^{\infty} P(\xi = k) s^k = \sum_{k=0}^{\infty} p_k s^k \]
be the offspring probability generating function. Assume \( p_0 + p_1 < 1 \). Then

1) \( f(s) \) is strictly convex and increasing in \([0,1]\); \( f'(s) > 0, f''(s) > 0; \)
2) \( f(0) = p_0 = P(Z(1) = 0 | Z(0) = 1); \)
3) if \( A \leq 1 \) then \( f(s) > s, s \in [0,1] \), since \( f'(s) - 1 < 0, s \in [0,1]; \)
4) if \( A > 1 \) then \( f(s) = s \) has a unique root \( r \) in \([0,1]\) and \( f(s) > s \) if \( s < r \) and \( f(s) < s \) if \( s > r \) since \( f'(0) - 1 < 0, f'(1) - 1 = A - 1 > 0 \) and \( f''(s) > 0. \)

Extinction probability

\[ f_n(s) = E_s^{Z(n)} = \sum_{k=0}^{\infty} P(Z(n) = k) s^k, \]

\[ f_n(0) = P(Z(n) = 0) \leq P(Z(n + 1) = 0) = f_{n+1}(0). \]
It follows that the sequence

\[ P(n) = P(\text{extinction by generation } n) = P(Z(n) = 0) = f_n(0), n = 1, 2 \ldots \]

must increase to the extinction probability, which we denote by \( P \),

\[ \lim_{n \to \infty} P(n) = P. \]

Since \( f(0) < r = f(r) \)

\[ P(n) = f_n(0) = f(f_{n-1}(0)) = f(P(n-1)) < f(r) = r \]

and the function \( f \) is continuous, it follows that \( P = f(P) \). Hence \( P = r \).

Thus, the subcritical and critical processes die with probability 1 while supercritical with probability \( P < 1 \) being the smallest root of \( f(s) = s, s \in [0, 1) \).

**Example.** For binary splitting, we must solve \( q + px^2 = x \) with the result

\[ P = \frac{q}{p} \]

provided \( p > 1/2 \) and one otherwise. For geometrically distributed offspring, the equation

\[ q/(1 - xp) = x \]

yields the same result, \( P = q/p \) for \( p > 1/2 \). In this case \( A = p/q \) so that also \( P = 1/A \). A process with a mean reproduction of, say 1.2, therefore has a population doubling time of four generations (if \( Z(0) = N, \mathbb{E}[Z(4)] = N(1.2)^4 = N \times 2.07 \)), and an extinction probability for each particular family higher than 80% \( (P = 1/A = 1/1.2 = 0.83 \text{ if } Z(0) = 1) \).

## 2 Branching processes and simple random walk

**Branching process:** Consider a branching process with geometric probability generating function for the offspring number:

\[ f(s) = \frac{q}{1 - ps} = Es^\xi, \ p + q = 1, \ pq > 0. \]  

(1)

It follows from the consideration above that the probability of extinction of this process, being a solution of \( f(P) = P \), is

\[ P = \min \left\{ \frac{q}{p}, 1 \right\} \]

and, besides the standard recurrence relation

\[ Z(n + 1) = \xi^{(n)}_1 + \cdots + \xi^{(n)}_{Z(n)} \]  

(2)

is valid, where \( \xi^{(n)}_i \) are iid, \( \xi^{(n)}_i \overset{d}{=} \xi \) with \( P(\xi = j) = qp^j, \ j = 0, 1, \ldots \).

**Random walk:** Consider a random walk

\[ S_0 = 1, S_k = X_1 + \cdots + X_k \]
with 
\[ P(X_i = 1) = p, \quad P(X_i = -1) = 1 - p = q. \]

Let \( S_k^* \) be the random walk stopped at zero at moment \( \tau = \min \{ k : S_k = 0 \} \).

It is known that
\[ P(\tau < \infty) = \min \left\{ \frac{q}{p}, 1 \right\}. \]

Set
\[ Y(n) = \text{the number of } k \text{ such that } S_k^* = n + 1, S_{k+1}^* = n. \]

Then the random variable
\[ Y(1) = \text{the number of } k \text{ such that } S_k^* = 2, S_{k+1}^* = 1 \]

has the following probability law:
\[ P(Y(1) = 0) = q; \quad P(Y(1) = 1) = pq \]

and, in general, the Geometric distribution with
\[ P(Y(1) = j) = P(\eta = j) = qp^j. \]

Besides,
\[ Y(n + 1) = \eta_1^{(n)} + \ldots + \eta_Y^{(n)} \]

where \( \eta_i^{(n)} \overset{d}{=} \eta. \)

Thus, we get the same stochastic process as in (2).

If \( p \leq 1/2 \) then the branching process dies out and if \( T \) is the moment of extinction then
\[ \sigma = Z(0) + Z(1) + \ldots + Z(T - 1) \]

is the total number of particles in the process and
\[ \sigma = 2\tau - 1. \]

We know that for the probability generating function \( f(s) \) of the form (1) with \( p = 1/2 \)
\[ 1 - f_n(0) = \frac{1}{n + 1} \]

while for \( p < 1/2 \)
\[ 1 - f_n(0) = \frac{(\frac{1}{q})^n (1 - \frac{1}{q})}{1 - \left(\frac{p}{q}\right)^{n+1}} \]

In terms of our random walk interpretation this fact is easy to explain.

Indeed, assume that our random walk \( S_n \) starts from a point \( S_0 = m \in [0, n + 1] \) and stops when it hits for the first time either 0 or \( n + 1 \). What is the probability to hit \( n + 1 \) earlier than 0 (this corresponds to \( P(Z(n) > 0) \))?
Let
\[ P_m(N) = P( \text{random walk starts at } m \text{ and hits } n + 1 \text{ within the time interval } [0, N] \text{ and earlier than 0}). \]

Clearly, for \( 1 \leq m \leq n \) we have
\[ P_m(N) = pP_{m+1}(N - 1) + qP_{m-1}(N - 1) \]
while
\[ P_0(N) = 0, P_{n+1}(N) = 1. \]

Letting \( N \to \infty \) we get
\[ P_m = pP_{m+1} + qP_{m-1}, 1 \leq m \leq n, \]
with
\[ P_0 = 0, P_{n+1} = 1. \]

It is necessary to search for a solution of this difference system of equations by solving the characteristic equation
\[ \lambda^m = p\lambda^{m+1} + q\lambda^{m-1} \]
or
\[ \lambda = p\lambda^2 + q \]
giving
\[ \lambda_1 = 1, \lambda_2 = \frac{q}{p}. \]

Thus, for \( p \neq q \) the general solution is
\[ P_m = a\lambda_1^m + b\lambda_2^m = a + b \left( \frac{q}{p} \right)^m. \]

Hence, on account of our boundary conditions
\[ a + b = 0 \text{ and } a + b \left( \frac{q}{p} \right)^{n+1} = 1 \]
and, therefore,
\[ a = \frac{1}{1 - \left( \frac{q}{p} \right)^{n+1}} = -b. \]

Consequently,
\[ P_m = \frac{1 - \left( \frac{q}{p} \right)^m}{1 - \left( \frac{q}{p} \right)^{n+1}} = \frac{\left( \frac{p}{q} \right)^n \left( \frac{p}{q} \right)^{1-m} - \frac{p}{q}}{1 - \left( \frac{q}{p} \right)^{n+1}} \]
with

\[ P_1 = \frac{(\frac{p}{q})^n (1 - \frac{p}{q})}{1 - (\frac{p}{q})^{n+1}} = P(Z(n) > 0). \]

If \( p = q \) then the general solution is

\[ P_m = a + bm \]

and, on account of our boundary conditions,

\[ a = 0, \quad b = \frac{1}{n+1} \]

leading to

\[ P_m = \frac{m}{n+1}. \]

Hence, in particular,

\[ P_1 = \frac{1}{n+1} = P(Z(n) > 0). \]