1 Crump-Mode-Jagers process counted by random characteristics

We give here only an informal description of the Crump-Mode-Jagers process counted by random characteristics or, what is the same, of the general branching process counted by random characteristics. A particle, say, $x$, of this process is characterised by three random processes 

$$(\lambda_x, \xi_x(\cdot), \chi_x(\cdot))$$

which are iid copies of a triple $(\lambda, \xi(\cdot), \chi(\cdot))$ and whose components have the following sense:

- if a particle was born at moment $\sigma_x$ then 
  $\lambda_x$ is the life-length of the particle;
- $\xi_x(t - \sigma_x)$ is the number of children produced by the particle within the time-interval $[\sigma_x, t]$; $\xi_x(t - \sigma_x) = 0$ if $t - \sigma_x < 0$;
- $\chi_x(t - \sigma_x) \geq 0$ is a stochastic process subject to changes ONLY within the time-interval $[\sigma_x, \sigma_x + \lambda_x)$ while outside the interval it has the form 

$$\chi_x(t - \sigma_x) = \begin{cases} 
0 & \text{if } t - \sigma_x < 0 \\
\chi_x(\lambda_x) & \text{if } t - \sigma_x \geq \lambda_x 
\end{cases}$$

(it is NOT assumed that $\chi_x(t)$ is a nondecreasing function in $t \geq 0$).

The stochastic process 

$$Z^x(t) = \sum_x \chi_x(t - \sigma_x)$$

where summation is taken over all particles $x$ born in the process up to moment $t$ is called the general branching process counted by random characteristics.
Examples:
1) \( \chi(t) = I \{ t \in [0, \lambda) \} \) in this case \( Z^\chi(t) = Z(t) \) is the number of particles existing in the process up to moment \( t \);
2) \( \chi(t) = tI \{ t \in [0, \lambda) \} + \lambda I \{ \lambda < t \} \) then
\[
Z^\chi(t) = \int_0^t Z(u)du;
\]
3) \( \chi(t) = I \{ t \geq 0 \} \) then \( Z^\chi(t) \) is the total number of particles born up to moment \( t \).

Classification. \( E \xi(\infty) <,=,> 1 \) - subcritical, critical and supercritical, respectively.

Let
\[
0 \leq v(1) \leq v(2) \leq \ldots \leq v(n) \leq \ldots
\]
be the birth moments of the children of the initial particle. Then
\[
\xi_0(t) = \# \{ n : v(n) \leq t \}
\]
is the number of children born by the initial particle up to moment \( t \). We have
\[
Z^\chi(t) = \chi_0(t) + \sum_{x \neq 0} \chi_x(t - \sigma_x) = \chi_0(t) + \sum_{v(n) \leq t} Z^\chi_n(t - v(n))
\]
where \( Z^\chi_n(\cdot), n = 1, 2, \ldots \) are iid copies of \( Z^\chi(\cdot) \). Hence it follows that
\[
E Z^\chi(t) = E \chi(t) + E \left[ \sum_{v(n) \leq t} Z^\chi_n(t - v(n)) \right]
\]
\[
= E \chi(t) + E \left[ \sum_{v(n) \leq t} E[Z^\chi_n(t - v(n)) | v(1), v(2), \ldots, v(n), \ldots] \right]
\]
\[
= E \chi(t) + E \left[ \sum_{v(n) \leq t} E[Z^\chi_n(t - v(n)) | v(n)] \right]
\]
\[
= E \chi(t) + E \left[ \sum_{u \leq t} E[Z^\chi(t - u)] (\xi_0(u) - \xi_0(u-)) \right]
\]
\[
= E \chi(t) + \int_0^t E Z^\chi(t - u) E \xi(du).
\]
Thus, we get the following renewal-type equation for \( A^\chi(t) = E Z^\chi(t) \) and \( \mu(t) = E \xi(t) \) :
\[
A^\chi(t) = E \chi(t) + \int_0^t A^\chi(t - u) \mu(du).
\] (1)
**Malthusian parameter:** a number \( \alpha \) is called the Malthusian parameter of the process if
\[
\int_0^\infty e^{-\alpha t} \mu(dt) = 1
\]
(such a solution not always exists). For the critical processes \( \alpha = 0 \), for the supercritical processes \( \alpha > 0 \), and for the subcritical processes \( \alpha < 0 \) (if exists).

If the Malthusian parameter exists we can rewrite (1) as
\[
C(t) = e^{-\alpha t} \int_0^t C(t-u) d\left( \int_0^u e^{-\alpha y} \mu(dy) \right)
\]
where \( C(t) = e^{-\alpha t} A(t) \). In view of (2) and given that, say, \( e^{-\alpha t} E(t) \) is directly Riemann integrable and
\[
\int_0^\infty e^{-\alpha t} E(t) dt < \infty, \quad \int_0^\infty te^{-\alpha t} \mu(dt) < \infty
\]
we can apply the key renewal theorem to conclude that if the measure
\[
M(t) = \int_0^t e^{-\alpha y} \mu(dy)
\]
is non-lattice then
\[
\lim_{t \to \infty} C(t) = \lim_{t \to \infty} e^{-\alpha t} A(t) = \int_0^\infty e^{-\alpha t} E(t) dt \left( \int_0^\infty te^{-\alpha t} \mu(dt) \right)^{-1}.
\]
In particular, if \( G(t) \) is the life-length distribution of particles and \( \chi(t) = I \{ t \in [0, \lambda] \} \) we get
\[
E(t) = P(\lambda > t) = 1 - G(t)
\]
and
\[
\lim_{t \to \infty} e^{-\alpha t} EZ(t) = \frac{\int_0^\infty e^{-\alpha t} (1 - G(t)) dt}{\int_0^\infty te^{-\alpha t} \mu(dt)}
\]
if the respective integrals converge.

## 2 M|G|1 system with processor sharing discipline

**The model:** a Poisson flow of customers with intensity \( \Lambda \) comes to a system with one server which has unit service intensity. The service time distribution of a particular customer is (if there are no other customers in the queue) \( B(u) \). If there are \( M \) customers in the system at some moment \( T \) they are served simultaneously with intensity \( M^{-1} \) each.

Let
\[
W_{l_1, \ldots, l_{N-1}}(l_N)
\]
be the waiting time for the end of service of a customer which arrived to the queue at the moment when the queue had $N-1$ customers with remaining service times $l_1, \ldots, l_{N-1}$.

The question is to study the properties of the random variable $W_{l_1, \ldots, l_{N-1}, l_N}$ when $l_N \to \infty$.

To solve this problem we construct an auxiliary general branching process.

**Construction of the branching process.**

Consider a general branching process in which initially at time $t = 0$ there are $N$ particles with remaining life-lengths $l_1, \ldots, l_{N-1}, l_N$ and which constitute the zero generation of this process. The life-length distribution of any newborn particle $\lambda_x$ is $P(\lambda_x \leq u) = B(u)$, the reproduction process $\xi_x(t)$ of the number of children produced by a particle up to moment $t$ has the probability generating function

$$E_{t}^{\xi_x(t)} = \int_{0}^{t} e^{A(s-1)u} dB(u) + e^{A(s-1)t} (1 - B(t))$$

that is, this is an ordinary Poisson flow with intensity $\Lambda$ stopped when the particle dies:

$$E_{t}^{\xi_x(t)} = E_{\Delta}^{\text{Poi}_\Lambda(t^\wedge \lambda_x)}.$$

Let $Z(t; l_1, \ldots, l_{N-1}, l_N)$ denote the number of particles in the process at moment $t$ with the mentioned initial conditions. We use a simplified notation $Z(t)$ if at moment $t = 0$ there is only one particle of zero age in the process.

We will consider also the process with immigration $X(t; l_1, \ldots, l_{N-1}, l_N)$ which has the same initial conditions and development as $Z(t; l_1, \ldots, l_{N-1}, l_N)$ but, in addition, given $X(t; l_1, \ldots, l_{N-1}, l_N) = 0$ it starts again by one individual of zero age after a random time $r_i$ having distribution $P(r_i \leq u) = 1 - e^{-\lambda u}$ (if the process dies out for the $i$-th time). $X(t)$ is used if we initially start by the process $Z(t)$.

Now let $\sigma_x \leq \sigma_x \leq \ldots$ be the sequential moments of jumps of the process $X(t; l_1, \ldots, l_{N-1}, l_N)$. We construct by the general branching process the following queueing system with $S(T)$ being the number of customers in the queue at moment $T$:

1) the queue has $N$ customers at $T = 0$ with remaining service times $l_1, \ldots, l_{N-1}, l_N$;

2) the moment $T_i$ of the $i$-th jump of the queue size $S(\cdot)$ is specified as

$$T_i = \int_{0}^{\sigma_x} X(y; l_1, \ldots, l_{N-1}, l_N) dy + \int_{0}^{\sigma_x} I \{X(y; l_1, \ldots, l_{N-1}, l_N) = 0\} dy.$$

3) the service discipline is such that at each moment $T$ the number of customers in the queue and their remaining service times coincide with the number of individuals and the remaining life-lengths of individuals in the branching process at moment $t(T)$ where

$$T = \int_{0}^{t(T)} X(y; l_1, \ldots, l_N) dy + \int_{0}^{t(T)} I \{X(y; l_1, \ldots, l_N) = 0\} dy.$$
Thus, $t \mapsto T$ is a random change of time.

**Theorem.** The described queueing system is a processor-sharing system with service time of customers $B(u)$ and a Poisson flow of customers with intensity of arrivals $\Lambda$.

**Proof.** Let $S(T)$ be the number of customers in the queue at time $T$ and let $\Theta_1, \Theta_2, \ldots$ be the moments of changes the size of the queue. Let us show that the evolution of the constructed queue coincides with the evolution of a queueing system with processor sharing discipline. It is enough to show that this is true for $T \in [0, \Theta_1]$ and then, using the memoryless property of the Poisson flow to show in a similar way that this is true for $T \in [\Theta_1, \Theta_2]$ and so on.

To demonstrate this it is enough to check that:

1) $\Theta_1 = Nl_1 \land \ldots \land Nl_N \land d$ where $P(d \leq u) = 1 - e^{-\Lambda u}$;
2) If $\Theta_1 = Nl_i$ then at this moment the $i$-th customer comes out of the queue; if $\Theta_1 = d$ then one new customer arrives;
3) at any moment $T \in [0, \Theta_1]$ the remaining service times of the initial $N$ customers are $l_1 - N^{-1}T, \ldots, l_N - N^{-1}T$.

Let $\theta_1$ be the first moment of change of $X(t; l_1, \ldots, l_N)$. Clearly,

$$\theta_1 = l_1 \land \ldots \land l_N \land d_1 \land \ldots \land d_N$$

where $P(d_i \leq u) = 1 - e^{-\Lambda u}$ and where the sense of $d_i$ is the birth of an individual by the initial particle labelled $i$. On the interval $u \in [0, \theta_1]$ the processing time of the queueing system $T$ and the time $t$ passed from the start of the evolution of the general branching process are related by $T = Nt$. Hence 3) is valid.

Further, $\Theta_1 = N(l_1 \land \ldots \land l_N \land d_1 \land \ldots \land d_N) = Nl_1 \land \ldots \land Nl_N \land (N(d_1 \land \ldots \land d_N))$ and

$$P(N(d_1 \land \ldots \land d_N) \geq y) = \left(e^{-y/N}\right)^N = e^{-y}.$$

This proves 1). Point 2) is evident.

**Corollary 1.**

$$S(T) = X(t(T); l_1, \ldots, l_N).$$

**Corollary 2.**

$$W_{l_1, \ldots, l_{N-1}}(l_N) = \int_0^{l_N} Z(y; l_1, \ldots, l_N)dy.$$

**More detailed construction:**

Let $L$ be the life-length of a particle and let $0 \leq \delta(1) \leq \delta(2) \leq \ldots$ be the birth moments of her children. Denote

$$\xi(t, L) = \#\{n: \delta(n) \leq t\}.$$

Then the process generated by this particle can be treated as a process with immigration stopped at moment $L$ where

$$E_S^{\xi(t, L)} = e^{\Lambda(s-1)\min(t, L)}.$$
and, since each newborn particle generates an *ordinary* process without immigration, we see that the offspring size of new particles at moment $t$ in the process is

$$
\int_0^t Z_{(u,L)}(t-u)\xi(du, L)
$$

where $Z_i(y)$ are independent branching processes initiated by one individual of zero age. Thus,

$$
Z(y; l_1, \ldots, l_N) = I \{ l_1 \geq y \} + \int_0^y Z_{(u,l_1)}(y-u)\xi(du,l_1)
$$

$$
+ \ldots + I \{ l_N \geq y \} + \int_0^y Z_{(u,l_N)}(y-u)\xi(du,l_N)
$$

and, in particular, we have

$$
W_{l_1, \ldots, l_{N-1}}(l_N) = \int_0^{l_N} \int_0^y Z(y; l_2, \ldots, l_N) dy
dy
$$

Since the birth moments of new particles constitute a Poisson flow with intensity $\Lambda$ we have $E[\xi(u,l)|l] = \min(u,l)$. Hence

$$
E \left[ \int_0^{l_k} dy \int_0^y Z_{(u,l_k)}(y-u)\xi(du,l_k) \right]
$$

$$
= E \left[ \int_0^{l_k} dy \int_0^y Z_{(u,l_k)}(y-u)\xi(du,l_k) | \xi(u,l_k), 0 \leq u \leq l_k \right]
$$

$$
= E \left[ \int_0^{l_k} dy \int_0^y E \left[ Z_{(u,l_k)}(y-u) | \xi(u,l_k), 0 \leq u \leq l \right] \xi(du,l_k) \right]
$$

$$
= E \left[ \int_0^{l_k} dy \int_0^y E \left[ Z(y-u) \right] \xi(du,l_k) \right]
$$

$$
= E \left[ \int_0^{l_k} dy \int_0^y E \left[ Z(y-u) \right] E \left[ \xi(du,l_k) | l_k \right] \right]
$$

$$
= E \left[ \int_0^{l_k} dy \int_0^y E \left[ Z(y-u) \right] \Lambda du \right] = \Lambda E \left[ \int_0^{l_k} dy \int_0^y E \left[ Z(u) \right] du \right].
$$

Hence

$$
E W_{l_1, \ldots, l_{N-1}}(l_N) = E \left[ \sum_{k=1}^N \min(l_N, l_k) + \Lambda \sum_{k=1}^N \int_0^{l_k} dy \int_0^y E \left[ Z(u) \right] du \right].
$$
One can prove also that if

$$\beta_1 = \mathbf{E}l_N = \int_0^\infty udB(u) < \infty$$

and $\Lambda \beta_1 < 1$ then for fixed $l_1, \ldots, l_{N-1}$

$$\lim_{l_N \to \infty} W(l_1, \ldots, l_{N-1}, l_N) = \frac{1}{1 - \Lambda \beta_1}$$

almost surely (in particular, if it comes to an empty system).