BRANCHING PROCESSES AND THEIR APPLICATIONS:
LECTURE 11: Branching processes with immigration at zero; transient phenomena; continuous time Markov branching processes

June 3, 2005

1 The Galton-Watson process with immigration at zero:

\[ f(s) = E s^\xi, \quad g(s) = E s^\eta = \sum_{k=1}^{\infty} P(\eta = k) s^k. \]

We have

\[ Y(n + 1) = \xi^{(n)}_1 + \ldots + \xi^{(n)}_{Y(n)} + \eta^{(n)} I \{ Y(n) = 0 \}. \]

\[ \xi^{(n)} \overset{d}{=} \xi, \quad \eta^{(n)} \overset{d}{=} \eta \text{ and iid.} \]

If

\[ \Pi(n, s) = E s^{Y(n)} \]

then

\[ \Pi(n + 1, s) = \Pi(n, f(s)) - \Pi(n, 0) + \Pi(n, 0) g(s) \]

\[ = \Pi(n, f(s)) - (1 - g(s)) \Pi(n, 0) \]

\[ = \Pi(0, f_{n+1}(s)) - \sum_{k=0}^{n} (1 - g(f_k(s))) \Pi(n - k, 0). \]

In particular, if \( Y(0) = 0 \) then

\[ \Pi(n + 1, 0) = 1 - \sum_{k=0}^{n} (1 - g(f_k(0))) \Pi(n - k, 0). \]

If \( A < 1 \) and

\[ g'(1) = b, \quad g(0) > 0, \]
then we have a stationary distribution for the process $Y(n)$ as $n \to \infty$.

Indeed, it is known that if a Markov chain is irreducible and nonperiodic then either

1) for any pair of states $p_{ij}^{(n)} \to 0, n \to \infty$, and, therefore, there exists no stationary distribution;

or

2) all the states are ergodic, that is,

$$\lim_{n \to \infty} p_{ij}^{(n)} = \pi_j > 0$$

and in this case $\{\pi_j\}$ is a stationary distribution and no other stationary distributions exists.

In our case take $p_{00}^{(n)} = \Pi(n, 0) = P(Y(n) = 0)$. Assuming that there is NO stationary distribution we get by dominated convergence theorem a contradiction:

$$\lim_{n \to \infty} \Pi(n + 1, 0) = 0 = 1 - \lim_{n \to \infty} \sum_{k=0}^{n} (1 - g(f_k(0))) \Pi(n - k, 0) = 1$$

since the series

$$\sum_{k=0}^{\infty} (1 - g(f_k(0))) \leq b \sum_{k=0}^{\infty} (1 - f_k(0)) \leq b \sum_{k=0}^{\infty} A^k < \infty.$$

Thus, we have a stationary distribution

$$\Pi(s) = E_s^Y = \lim_{n \to \infty} E_s^{Y(n)}$$

where

$$\Pi(s) = \Pi(f(s)) - \pi_0 (1 - g(s))$$

or

$$\Pi(s) = 1 - \pi_0 \sum_{k=0}^{\infty} (1 - g(f_k(s))).$$

From here

$$\pi_0 = 1 - \pi_0 \sum_{k=0}^{\infty} (1 - g(f_k(0)))$$

leading to

$$\pi_0 = \frac{1}{1 + \sum_{k=0}^{\infty} (1 - g(f_k(0)))}.$$

Hence

$$\Pi(s) = 1 - \frac{\sum_{k=0}^{\infty} (1 - g(f_k(s)))}{1 + \sum_{k=0}^{\infty} (1 - g(f_k(0)))} = 1 - \frac{R(s)}{1 + R(0)}$$
with
\[ R(s) = \sum_{k=0}^{\infty} (1 - g(f_k(s))) \]

Introduce the following classes of functions: \( K_1 = K(b_1, b_2, \gamma_1(y)) = \{ g \} \) of probability generating functions (PGF) specified by \( b_1, b_2, \gamma_1(y) \):
\[ g(1 - y) = 1 - (b + \alpha_1(y))y \]
where
\[ 0 < b_1 \leq b \leq b_2, \quad \sup_{g \in K_1} |\alpha_1(y)| \leq \gamma_1(y) = o(1), y \to 0, \]
and \( K_2 = K_2(B_3, B_4, \gamma_2(y)) = \{ f \} \) of PGF specified by \( B_3, B_4, \gamma_2(y) \):
\[ f(1 - y) = 1 - Ay + (B + \alpha_3(y))y^2 + \frac{Ay}{1 + (BA^{-1} + \alpha_4(y))y} \]
where
\[ 0 < B_3 \leq B \leq B_4, \quad \sup_{f \in K_2} |\alpha_i(y)| \leq \gamma_2(y) = o(1), y \to 0, \quad i = 3, 4. \]

Let \( H \) be the class of immigration processes such that \( g \in K_1, f \in K_2. \)

**Theorem 1** If \( \{ Y(n) \} \in H \) then
\[ \lim_{A \to 1} P \left( \frac{\ln Y}{\ln \frac{1}{1-x}} \leq x \right) = x, \quad x \in (0, 1]. \]

**Proof.** It follows from the conditions of the theorem that for any \( \varepsilon \in (0, B_2) \) there exists \( \delta = \delta(\varepsilon) > 0 \) such that for all \( 0 < y < \delta \) and all \( \{ Y(n) \} \in H \)
\[ \frac{A(1-s)}{1 + B(1 + \varepsilon)(1-s)} \leq 1 - f(s) \leq \frac{A(1-s)}{1 + B(1 - \varepsilon)(1-s)} \]
and
\[ b(1 - \varepsilon)y \leq 1 - g(1 - y) \leq b(1 + \varepsilon)y. \]

Let for \( \varepsilon \in (0, B_2) \)
\[ f^\pm(s) = 1 - \frac{A(1-s)}{1 + B(1 \pm \varepsilon)(1-s)} \]
and
\[ f_n^+(s) = f^+(f_{n-1}^+(s)), f_n^-(s) = f^-(f_{n-1}^-(s)). \]
Since the functions are fractional-linear and the derivative of \( f^+ \) and \( f^- \) at point \( s = 1 \) are less than 1 it is not difficult to show that
\[ f_n^\pm(s) = 1 - \frac{A^n(1-s)}{1 + B(1 \pm \varepsilon)(1-s)^{1-A^n}} \]
and that the inequalities are preserved. Thus, for \( s \) sufficiently close to 1 we have
\[
\sum_{k=0}^{\infty} (1 - g(f_k^+(s))) \leq R(s) \leq \sum_{k=0}^{\infty} (1 - g(f_k^-(s))).
\]

In particular, if \( M \) is such that \( f_M(0) > 1 - \varepsilon \) then
\[
\sum_{k=M}^{\infty} (1 - g(f_k^+(f_M(0)))) \leq R(0) = \sum_{k=0}^{\infty} (1 - g(f_k(0)))
\]
\[
= \sum_{k=0}^{M-1} (1 - g(f_k(0))) + \sum_{k=0}^{\infty} (1 - g(f_k(f_M(0))))
\]
\[
\leq \sum_{k=0}^{M-1} (1 - g(f_k(0))) + \sum_{k=0}^{\infty} (1 - g(f_k^-(f_M(0))))
\]
\[
\leq \sum_{k=0}^{M-1} (1 - g(f_k(0))) + \sum_{k=0}^{\infty} (1 - g(f_k(0)))
\]

and using the arguments to calculate integral in the previous section one can show that
\[
R(0) \sim -\frac{b}{B} \ln (1 - A), \ A \uparrow 1. \tag{1}
\]

Now for \( x \in (0, 1) \) let \( s = \exp \{-\lambda(1 - A)^x\} \). Clearly,
\[
1 - s \sim \lambda(1 - A)^x, \ A \uparrow 1.
\]

Select \( m = m(\lambda, A, B) : \)
\[
f_{m}(0) \leq \exp \{-\lambda(1 - A)^x\} \leq f_{m+1}(0)
\]
that is
\[
\lambda(1 - A)^x \sim \frac{A^m}{1 + B(1 - \varepsilon)^{\frac{1 - A^m}{1 - A}}}
\]
or
\[
\lambda(1 - A)^x A^{-m} \sim \frac{1}{1 + B(1 - \varepsilon)^{\frac{1 - A^m}{1 - A}}}.
\]

Observe, that under our choice of \( m \) we have \( A^m \sim 1 \) since assuming \( A^m < c < 1 \) we would have as \( A \uparrow 1 \)
\[
\lambda(1 - A)^x \sim \frac{A^m(1 - A)}{B(1 - \varepsilon)(1 - A^m)} \leq \frac{c(1 - A)}{B(1 - \varepsilon)(1 - c)}
\]
which is impossible for \( x < 1 \). This implies
\[
\lambda(1 - A)^x \sim \frac{1}{1 + B(1 - \varepsilon)^{\frac{1 - A^m}{1 - A}}}. \tag{2}
\]
Now
\[ R(e^{-\lambda(1-A)^x}) \leq \sum_{k=0}^{\infty} \left( 1 - g(f_k) \left( e^{-\lambda(1-A)^x} \right) \right) \leq \sum_{k=0}^{\infty} \left( 1 - g(f_k(0)) \right) = \sum_{k=m}^{\infty} \left( 1 - g(f_k(0)) \right) \]
and we have calculated that
\[ \sum_{k=m}^{\infty} \left( 1 - g(f_k(0)) \right) \leq \frac{b(1+\varepsilon)(1-A)}{B(1-\varepsilon) \ln A} \ln \frac{1 + B(1-\varepsilon) \frac{1}{1-A}}{1 + B(1-\varepsilon) \frac{1}{1-A^m}} \tag{3} \]
or, in view of (2)
\[ \sum_{k=m}^{\infty} \left( 1 - g(f_k(0)) \right) \leq \frac{b(1+\varepsilon)(1-A)}{B(1-\varepsilon) \ln A} \ln \left( \frac{\lambda(1-A)^x}{(1-A)^x} \right) \frac{1}{1-A} = \frac{(x-1)b(1+\varepsilon)}{B(1-\varepsilon)} \ln (1-A) \tag{4} \]
Similarly, specifying \( m = m(\lambda, A, B) : \)
\[ f_m^+(0) \leq \exp \{ -\lambda(1-A)^x \} \leq f_{m+1}^+(0) \]
one can show that
\[ \sum_{k=m}^{\infty} \left( 1 - g(f_k^+(0)) \right) \geq -\frac{(x-1)b(1+\varepsilon)}{B(1-\varepsilon)} \ln (1-A) . \tag{5} \]
Since \( \varepsilon > 0 \) can be taken arbitrary small, it follows from (1), (4) and (5) that
\[ \lim_{A \to 1} \Pi \left( e^{-\lambda(1-A)^x} \right) = 1 - \lim_{A \to 1} \frac{R(s)}{1 + R(0)} = 1 - \frac{(x-1)b}{\frac{B}{1-\varepsilon} \ln (1-A)} = 1 + (x-1) = x. \]
Hence
\[ \lim_{A \to 1} E e^{-\lambda Y(1-A)^x} = x. \]
Therefore,
\[ \lim_{A \to 1} P \left( Y(1-A)^x < 1 \right) = \lim_{A \to 1} P \left( \ln Y + x \ln (1-A) < 0 \right) = x \]
or
\[ \lim_{A \to 1} P \left( \frac{\ln Y}{\ln \frac{1}{1-A}} < x \right) = x. \]
1.1 Queueing systems with batch service

$M^{[X]}|G|1$

$\lambda$ - the intensity of the input Poisson flow. The customers arrive in batches of random size. The size of the $i$-th group is $\eta(i)$

$$g(s) = E s^\eta = \sum_{k=1}^{\infty} P(\eta = k) s^k.$$  

The first customer → to the server

$\nu(1)$ - the number of customers coming during the service time of the first customer.

$\nu(2)$ - the number of customers coming during the service time of all first $\nu(1)$ customers.

$\nu(j)$ - the number of customers coming during the service time of all $\nu(j-1)$ customers.

If NO customers arrive during the service time of a group of customers then we wait for the new batch and take all of them. We have

$$\nu(n + 1) = \xi_1^{(n)} + \ldots + \xi_{\nu(n)}^{(n)} + \eta(n) I \{ \nu(n) = 0 \} .$$

$$\xi_i^{(n)} \overset{d}{=} \xi, \text{ and iid.}$$

This is a BRANCHING PROCESS WITH IMMIGRATION AT ZERO. Clearly,

$$E s^\xi = \sum_{j=0}^{\infty} P(\xi = j) s^j = \sum_{k=0}^{\infty} \int_0^{\infty} e^{-\lambda u} \frac{(\lambda u)^k}{k!} g_k(s) dG(u)$$

$$= \int_0^{\infty} e^{-\lambda u(1-g(s))} dG(u) = f(s).$$

Direct calculations show that

$$A = E \xi = f'(1) = \lambda g'(1) \int_0^{\infty} u dG(u) = \lambda g'(1)m$$

where $m$ is the expected service time of a customer. Hence we can apply the previous theorem to study the queueing system under heavy traffic when $A = \lambda g'(1)m \not\sim 1$.

2 Continuous time Markov processes

A stochastic process $\{Z(t, \omega), t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, P)$ is called a continuous time Markov branching process if

1) the state-space - nonnegative integers;

2) stationary Markov Chain with respect to the $\sigma$-algebra $\mathcal{F}_t = \sigma\{Z(s, \omega), s \leq t\}$;
3) for all $t \geq 0, i = 0, 1, 2, \ldots$ and $|s| \leq 1$ the following branching property is valid:

$$\sum_{j=0}^{\infty} P_{ij}(t)s^j = \left( \sum_{j=0}^{\infty} P_{1j}(t)s^j \right)^i = (F(t,s))^i.$$ 

## 2.1 Construction

$P_{ij}(\tau, \tau + t) = P\{Z(\tau + t) = j|Z(\tau) = i\} = P_{ij}(t)$.

Now probabilistic interpretation: if there are $i$ particles at some moment then each of them has exponential remaining life-length with parameter, say, $\rho$, and then dies producing children in accordance with the pgf $f(s) = \sum_{k=0}^{\infty} P(\xi = k)s^k = \sum_{k=0}^{\infty} p_k s^k, 0 \leq s \leq 1,$ independently of other individuals.

Thus, for $j \geq i - 1, i \neq j$

$$P_{ij}(\Delta t) = \rho i p_{j-i+1} \Delta t + o(\Delta t),$$

$$P_{ii}(\Delta t) = 1 - \rho i \Delta t + o(\Delta t),$$

$$P_{ij}(\Delta t) = o(\Delta t), \ j < i - 1.$$ 

From here one can deduce the forward

$$\frac{d}{dt} P_{ij}(t) = -j \rho P_{ij}(t) + \rho \sum_{k=1}^{j+1} k P_{ik}(t)p_{j-k+1}$$

and backward Kolmogorov equations

$$\frac{d}{dt} P_{ij}(t) = -i \rho P_{ij}(t) + i \rho \sum_{k=i+1}^{\infty} p_{k-i+1} P_{kj}(t).$$

with boundary

$$P_{ij}(+0) = \delta_{ij}.$$ 

From here for $f^{(\rho)}(s) = \rho(f(s) - s)$ and $i = 1$ we have for

$$F(t,s) = E\left[s^{Z(t)} | Z(0) = 1 \right]$$

the following equations

$$\frac{\partial F(t; s)}{\partial t} = f^{(\rho)}(s) \frac{\partial F(t; s)}{\partial s}, \ F(0, s) = s,$$

and

$$\frac{\partial F(t; s)}{\partial t} = -\rho \sum_{j=0}^{\infty} P_{1j}(t)s^j + \rho \sum_{j=0}^{\infty} s^j \sum_{k=0}^{\infty} p_k P_{kj}(t)$$

$$= \rho (f(F(t,s)) - F(t,s)) = f^{(\rho)}(F(t,s)),$$

$$F(0, s) = s.$$  \hspace{1cm} (6)
2.2 Classification

Let

\[ A(t) = EZ(t). \]

Then

\[ \frac{\partial}{\partial s} \frac{\partial F(t; s)}{\partial t} = f^{(\nu)}(F(t, s)) \frac{\partial F(t, s)}{\partial s} \]

or, setting \( s = 1 \)

\[ \frac{dA(t)}{dt} = \rho(f'(1) - 1)A(t), \quad A(0) = 1. \]

Solving this equation we get

\[ A(t) = e^{at}, \quad a = \rho(f'(1) - 1). \]

A continuous time Markov branching process is called supercritical, critical, subcritical if, respectively \( f'(1) > 1, = 1, < 1. \)

Example 1. Let

\[ f^{(\nu)}(s) = a(s - 1) + \lambda(1 - s)^{1+\alpha}, \quad 0 < \alpha < 1, \lambda > \max\{a, 0\}. \]

Then the respective probability generating function \( F(t; s) \) solves the equation

\[ \frac{\partial F(t; s)}{\partial t} = a(F(t; s) - 1) + \lambda(1 - F(t; s))^{1+\alpha}, \quad F(0, s) = s. \]

Set \( v = \frac{1}{1 - F}. \)

Then

\[ \frac{dv}{dt} = -\alpha av + \lambda v, \quad v = \frac{1}{1 - s}. \]

Hence

\[ F(t, s) = 1 - \left[ \frac{\lambda}{a} \left( 1 - e^{-\alpha at} \right) + e^{-\alpha at}(1 - s)^{-\alpha} \right]^{-1/\alpha}, \quad a \neq 0, \]

and

\[ F(t, s) = 1 - \left[ \alpha \lambda t + (1 - s)^{-\alpha} \right]^{-1/\alpha}, \quad a = 0. \]

Example 2. Let

\[ f^{(\nu)}(s) = a(s - 1 - (1 - s)^{\alpha}), \quad 0 < \alpha < 1, a > 0. \]

Then

\[ \frac{\partial F(t; s)}{\partial t} = a(F(t; s) - 1 - (1 - F(t; s))^{\alpha}), \quad F(0, s) = s. \]
Set
\[ v = (1 - F)^{1-\alpha}. \]

Then we get a linear equation whose solution is
\[ F(t, s) = 1 - \left[ 1 - e^{-a(1-\alpha)t} + e^{-a(1-\alpha)t}(1 - s)^{1-\alpha} \right]^{\frac{1}{1-\alpha}}. \]

Observe that
\[ F(t, 1) = \lim_{s \to 1} F(t, s) = 1 - \left[ 1 - e^{-a(1-\alpha)t} \right]^{\frac{1}{1-\alpha}} < 1. \]

Thus, in this case we have the so-called explosion phenomena:
\[ F(t, 1) = \sum_{k=0}^{\infty} P(Z(t) = k) = P(Z(t) < \infty) \]

and \( 1 - F(t; 1) = 1 - P(Z(t) < \infty) = P(Z(t) = \infty) > 0 \) showing that within any finite time interval the number of individuals in the population becomes infinite with a positive probability!

### 2.2.1 Criterion

A Markov process does not explode if and only if for any \( \varepsilon \in (0, 1) \)
\[ \int_{1-\varepsilon}^{1} \frac{du}{1 - f(u)} = \infty. \]

We prove the criterion in a more general situation later on.

**Theorem.** If \( f'(1) < \infty \) then the equation
\[ \frac{\partial F(t; s)}{\partial t} = \rho(f(F(t, s)) - F(t, s)) = f^{(\rho)}(F(t, s)), F(0, s) = s \]

has a unique solution in the class of functions \( F(t, s) \) with \( F(t, 1) = 1 \).

**Proof.** Let \( G(t) = 1 - e^{-rt} \). Then
\[ F(t, s) = s (1 - G(t)) + \int_{0}^{t} f(F(t - u, s))dG(u). \]

If there are two solutions \( F_1(t, s) \) and \( F_2(t, s) \) then
\[ |F_1(t, s) - F_2(t, s)| \leq \int_{0}^{t} |f(F_1(t - u, s)) - f(F_2(t - u, s))|dG(u) \leq f'(1) \int_{0}^{t} |F_1(t - u, s) - F_2(t - u, s)|dG(u) \leq f'(1)G(t) \sup_{0 \leq v \leq t} |F_1(v, s) - F_2(v, s)|. \]
If $t_0 > 0$ is such that $f'(1)G(t_0) < 1$ then

$$\sup_{0 \leq v \leq t_0} |F_1(v, s) - F_2(v, s)| \leq f'(1)G(t_0) \sup_{0 \leq v \leq t_0} |F_1(v, s) - F_2(v, s)|.$$ 

Hence

$$F_1(t, s) = F_2(t, s), \ 0 \leq t \leq t_0.$$ 

Again

$$|F_1(t, s) - F_2(t, s)| \leq \int_0^t |f(F_1(t-u, s)) - f(F_2(t-u, s))|dG(u)$$

$$= \int_0^{t-t_0} |f(F_1(t-u, s)) - f(F_2(t-u, s))|dG(u)$$

$$\leq f'(1) \int_0^{t-t_0} |F_1(t-u, s) - F_2(t-u, s)|dG(u)$$

$$\leq f'(1)G(t - t_0) \sup_{0 \leq v \leq t-t_0} |F_1(t_0 + v, s) - F_2(t_0 + v, s)|$$

if $t - t_0 \leq t_0$ and this is for all $|s| \leq 1$.

One can check that

$$\lim_{t \to \infty} F(t, s) = f(\lim_{t \to \infty} F(t, s))$$

Hence all the properties related with the extinction of the Markov continuous time processes are similar to those for the Galton-Watson processes.