

Optimisation Problems in Non-Life Insurance

Hanspeter Schmidli

Universität zu Köln

Frankfurt, 6. Juli 2007

- 1 The de Finetti Problem
 - Introduction
 - The Optimal Strategy
 - De Finetti's Example
- 2 Minimal Ruin Probabilities
 - Introduction
 - The Hamilton-Jacobi-Bellman Equation
 - Two Examples
- 3 Optimal Dividends
 - Dividends in a Diffusion Approximation
 - Dividends in the Classical Model
 - Dividends with Reinvestment
- 4 Further Problems

- 1 The de Finetti Problem
 - Introduction
 - The Optimal Strategy
 - De Finetti's Example
- 2 Minimal Ruin Probabilities
 - Introduction
 - The Hamilton-Jacobi-Bellman Equation
 - Two Examples
- 3 Optimal Dividends
 - Dividends in a Diffusion Approximation
 - Dividends in the Classical Model
 - Dividends with Reinvestment
- 4 Further Problems

The Problem

Denote by Y_n the result of an insurance company in year n . We suppose $Y_n \in \mathbb{N}$ and the variables $\{Y_n\}$ are iid. Without any control the surplus of the insurer becomes

$$X_n^0 = x + \sum_{k=1}^n Y_k .$$

The insurer pays a dividend U_n at the end of year n . Then the pre-dividend process becomes

$$X_n^U = x + \sum_{k=1}^n (Y_k - U_{k-1}) .$$

An admissible dividend fulfils $0 \leq U_n \leq X_n^U$.

The Value Function

Let $\tau^U = \inf\{n : X_n^U < 0\}$ be the time of ruin. The value of a dividend strategy is

$$V^U(x) = \mathbb{E} \left[\sum_{k=0}^{\tau^U-1} U_k r^k \right]$$

for some discounting variable $r \in (0, 1)$.

The goal is to maximise the value

$$V(x) = \sup_U V^U(x),$$

and — if it exists — to find an optimal strategy.



Bellman's Equation

Suppose we know already the dividend $U_0 = u$ at time zero. This is the same as starting with initial capital $x - u$ and not paying a dividend at time zero.

At time 1 one has the capital $x - u + Y_1$. The optimal dividend value at time 1 is $V(x - u + Y_1)$. The value at time zero is $rV(x - u + Y_1)$. Thus the value at time zero is

$$u + r\mathbb{E}[V(x - u + Y_1)] .$$

We have the maximal value, thus

$$V(x) = \sup_{0 \leq u \leq x} u + r\mathbb{E}[V(x - u + Y_1)] .$$

Some Properties

The dividend strategy $U'_0 = x - \lfloor x - U_0 \rfloor$ and

$U'_n = \lfloor X_n \rfloor - \lfloor X_n - U_n \rfloor$ leads to the process $X'_n = \lfloor X_n \rfloor$.

Ruin occurs at the same time, but dividends are paid out earlier.

Thus $V^U(x) \leq V^{U'}(x)$. If $\{X_n^U : n \geq 1\} \neq \{X_n^{U'} : n \geq 1\}$, then $V^U(x) < V^{U'}(x)$.

Moreover, $V(x) = V(\lfloor x \rfloor) + (x - \lfloor x \rfloor)$. Thus it is enough to consider integer values only. In particular, there exists $u(x)$, such that

$$V(x) = \sup_{0 \leq u \leq x} u + r\mathbb{E}[V(x - u + Y_1)] = u(x) + r\mathbb{E}[V(x - u(x) + Y_1)].$$

A Lower Bound

Use the following strategy. Let $U_0 = x$ and $U_n = Y_n^+$. Then $X_n = Y_n$ and ruin occurs at the first time where $Y_n < 0$. The value of this strategy is

$$x + \frac{r\mathbb{E}[Y^+]}{1 - p_+ r} = V^U(x) \leq V(x),$$

where $p_+ = \mathbb{P}[Y_k \geq 0]$.

An Upper Bound

Suppose we use the dividend strategy $U_0 = x$ and $U_n = Y_n^+$, but we do not stop at ruin. The aggregate dividend payment of this strategy is an upper bound for the aggregate dividend payment of any admissible strategy. Hence any admissible strategy must have a smaller value. Thus we get the upper bound

$$V(x) \leq x + \frac{r\mathbb{E}[Y^+]}{1-r}.$$

- 1 The de Finetti Problem**
 - Introduction
 - **The Optimal Strategy**
 - De Finetti's Example
- 2 Minimal Ruin Probabilities**
 - Introduction
 - The Hamilton-Jacobi-Bellman Equation
 - Two Examples
- 3 Optimal Dividends**
 - Dividends in a Diffusion Approximation
 - Dividends in the Classical Model
 - Dividends with Reinvestment
- 4 Further Problems**



The Optimal Strategy

Consider the Strategy $U_n = u(X_n)$. Then the process

$$\left\{ V(X_n)r^n + \sum_{k=0}^{n-1} r^k U_k \right\}$$

is a martingale.

In particular,

$$V(x) = \lim_{n \rightarrow \infty} \mathbb{E} \left[V(X_{\tau \wedge n}) r^{\tau \wedge n} + \sum_{k=0}^{(\tau \wedge n) - 1} r^k U_k \right] = \mathbb{E} \left[\sum_{k=0}^{\tau - 1} r^k U_k \right].$$



The Optimal Strategy

The same argument shows that if U_n is not a maximiser of

$$u + r\mathbb{E}[V(X_n - u + Y_{n+1})]$$

then $\{U_n\}$ is not an optimal strategy.

If there are several maximisers u we choose $u(x)$ the largest value.

We can characterise the strategy in the following way

$$u(x) = \sup\{n : V(x) = V(x - n) + n\} .$$

Thus $u(x) = 0$ if $V(x) > V(x - 1) + 1$.



The Optimal Strategy

Suppose $u(x) = 0$. Then

$$\begin{aligned}
 x + \frac{r\mathbb{E}[Y^+]}{1-p+r} &\leq V(x) = r \sum_{j=-x}^{\infty} p_j V(x+j) \\
 &\leq r \sum_{j=-x}^{\infty} p_j \left(x+j + \frac{r\mathbb{E}[Y^+]}{1-r} \right) \\
 &\leq rx + r \left(\mathbb{E}[Y^+] + \frac{r\mathbb{E}[Y^+]}{1-r} \right) \\
 &= rx + \frac{r\mathbb{E}[Y^+]}{1-r}.
 \end{aligned}$$



The Optimal Strategy

Solving for x gives

$$x \leq \frac{r^2 \mathbb{E}[Y^+](1 - p_+)}{(1 - r)^2(1 - p_+ r)}.$$

Thus there is a barrier $x_0 = \sup\{x : u(x) = 0\}$. The post-dividend process fulfils $X_n - U_n \leq x_0$, thus ruin will occur almost surely.



- 1 The de Finetti Problem**
 - Introduction
 - The Optimal Strategy
 - De Finetti's Example
- 2 Minimal Ruin Probabilities**
 - Introduction
 - The Hamilton-Jacobi-Bellman Equation
 - Two Examples
- 3 Optimal Dividends**
 - Dividends in a Diffusion Approximation
 - Dividends in the Classical Model
 - Dividends with Reinvestment
- 4 Further Problems**



De Finetti's Example

Let $\mathbb{P}[Y = 1] = 1 - \mathbb{P}[Y = -1] = p$.

Consider first the case where all capital is paid out as dividend.

Plugging in the function $V(x) = x + pr/(1 - pr)$ into Bellman's equation

$$V(x) = \max\{r(pV(x+1) + (1-p)V(x-1)), V(x-1) + 1\}$$

yields that this is the solution whenever

$$1 \geq r \left(p + \sqrt{p(1-p)} \right) .$$



de Finetti's Example: Equations

Let $x_1 = \sup\{n : u(n) = 0\}$ and suppose $x_1 > 0$ and let $n \leq x_1$.
Then we have the equations

$$V(n+1) = \frac{1}{pr} V(n) - \frac{1-p}{p} V(n-1),$$

if $n \geq 1$,

$$V(1) = \frac{1}{pr} V(0),$$

and

$$V(x_1 + 1) = V(x_1) + 1.$$



de Finetti's Example: Solution

The solution is

$$V(n) = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} V(0).$$

where

$$\lambda_{1/2} = \frac{1 \pm \sqrt{1 - 4p(1-p)r^2}}{2pr} = \frac{r^{-1} \pm \sqrt{r^{-2} - 1 + (2p-1)^2}}{2p}.$$



de Finetti's Example: Solution

The equation

$$V(x_1 + 1) = V(x_1) + 1$$

yields $V(0)$. Maximising $V(0)$ gives the solution on $[0, x_1 + 1]$. It turns out that this solution solves Bellman's equation, thus the optimal strategy is a barrier strategy.

1 The de Finetti Problem

- Introduction
- The Optimal Strategy
- De Finetti's Example

2 Minimal Ruin Probabilities

- Introduction
- The Hamilton-Jacobi-Bellman Equation
- Two Examples

3 Optimal Dividends

- Dividends in a Diffusion Approximation
- Dividends in the Classical Model
- Dividends with Reinvestment

4 Further Problems

The Classical Risk Model

$$X_t^1 = x + ct - \sum_{i=1}^{N_t} Y_i$$

- x : initial capital
- c : premium rate
- $\{N_t\}$: Poisson process with rate λ
- $\{Y_i\}$: iid, independent of $\{N_t\}$
- $0 = T_0 < T_1 < T_2 < \dots$: claim times
- $G(y)$: distribution of Y_i , $G(0) = 0$
- $\mu_n = \mathbb{E}[Y_i^n]$, $\mu = \mu_1$

Proportional Reinsurance

The insurer can buy proportional reinsurance. Choosing retention level b the premium rate $c - c(b)$ has to be paid. For a strategy $\{b_t\}$ the surplus process becomes

$$X_t^b = x + \int_0^t c(b_s) ds - \sum_{k=1}^{N_t} b_{T_k} - Y_k .$$

We let $\tau^b = \inf\{t : X_t^b < 0\}$ denote the time of ruin and by $\psi^b(x) = \mathbb{P}[\tau^b < \infty]$ the ruin probability.

We define the value function

$$\delta(x) = 1 - \inf_b \psi^b(x) .$$



Proportional Reinsurance

We assume:

- $G(x)$ is continuous.
- $c(b)$ continuous.
- $c(b)$ increasing.
- $c(1) = c$.
- $c(0) < 0$.

- 1** The de Finetti Problem
 - Introduction
 - The Optimal Strategy
 - De Finetti's Example
- 2** Minimal Ruin Probabilities
 - Introduction
 - The Hamilton-Jacobi-Bellman Equation
 - Two Examples
- 3** Optimal Dividends
 - Dividends in a Diffusion Approximation
 - Dividends in the Classical Model
 - Dividends with Reinvestment
- 4** Further Problems

The Hamilton-Jacobi-Bellman Equation

If the function $\delta(x)$ is 'nice' it fulfils the Hamilton-Jacobi-Bellman equation

$$\sup_{b \in [0,1]} c(b) \delta'(x) + \lambda \left[\int_0^{x/b} \delta(x - by) dG(y) - \delta(x) \right] = 0 .$$

Reformulation yields

$$\delta'(x) = \lambda \inf_{b \in (0,1]} \frac{\delta(x) - \int_0^{x/b} \delta(x - by) dG(y)}{(c(b))^+} .$$

Continuity gives that there is a value $b(x)$ where the sup (inf) is taken.

Existence of a Solution

Denote by $\delta_1(x)$ the survival probability without reinsurance. Then $f_0(x) = c\delta_1(x)/(c - \lambda\mu)$ fulfils

$$f_0'(x) = \lambda \frac{f_0(x) - \int_0^x f_0(x-y) dG(y)}{c(1)}, \quad f_0(0) = 1.$$

Define recursively

$$f_{n+1}'(x) = \lambda \inf_{b \in (0,1]} \frac{f_n(x) - \int_0^{x/b} f_n(x-by) dG(y)}{(c(b))^+}, \quad f_{n+1}(0) = 1.$$

Then $f_n(x)$ is decreasing in n and converges to some function $f(x)$ fulfilling the Hamilton-Jacobi-Bellman equation. Thus there is a solution to the Hamilton-Jacobi-Bellman equation.

The Verification Theorem

Let $f(x)$ be an increasing solution to the Hamilton-Jacobi-Bellman equation. Then the process

$$f(X_t) - \int_0^t \left\{ c(b_s) f'(X_s) + \lambda \left[\int_0^{X_s/b_s} f(X_s - b_s y) dG(y) - f(X_s) \right] \right\} ds$$

is a martingale.

Because $f(x)$ solves the Hamilton-Jacobi-Bellman equation the process $\{f(X_t)\}$ is a supermartingale.



The Verification Theorem

We get

$$f(x) = f(X_0^b) \geq \mathbb{E}[f(X_t^b)] \rightarrow f(\infty)\delta^b(x).$$

Thus $f(x)$ is bounded.

Let $b_t^* = b(X_t^*)$. Then $f(X_s^*)$ is a martingale and

$$f(x) = \mathbb{E}[f(X_t^*)] \rightarrow f(\infty)\delta^*(x) = f(\infty)\delta(x).$$

Thus $\delta(x)$ really solves the Hamilton-Jacobi-Bellman equation and $\{b_t^*\}$ is an optimal strategy.



- 1 The de Finetti Problem**
 - Introduction
 - The Optimal Strategy
 - De Finetti's Example
- 2 Minimal Ruin Probabilities**
 - Introduction
 - The Hamilton-Jacobi-Bellman Equation
 - Two Examples
- 3 Optimal Dividends**
 - Dividends in a Diffusion Approximation
 - Dividends in the Classical Model
 - Dividends with Reinvestment
- 4 Further Problems**

Exponentially Distributed Claim Sizes

In the following two examples we choose $\lambda = \mu = 1$ and $c(b) = 1.7b - 0.2$.

Example

Exponentially distributed claim sizes:

$$G(y) = 1 - e^{-y}.$$

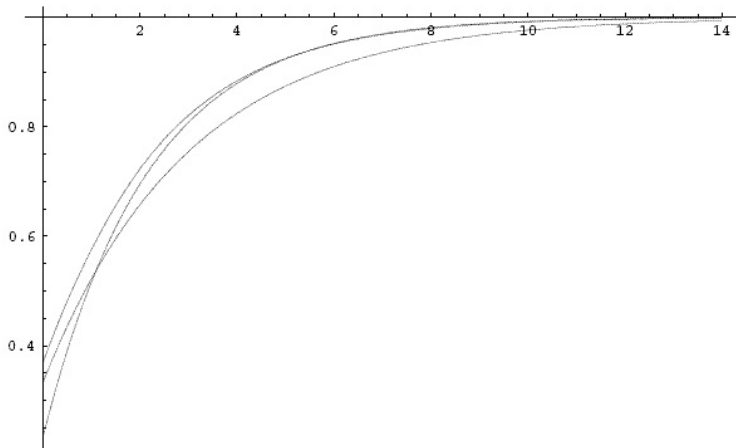
$\delta_1(x) = 1 - 2e^{-x/3}/3$. From Waters (1983)

$$b^R = \left(1 - \frac{1.5}{1.7}\right) \left(1 + \frac{1}{\sqrt{1.7}}\right) = 0.504847.$$

$\delta^R(x)$ known explicitly.



Exponentially Distributed Claim Sizes: $\delta(u)$





Exponentially Distributed Claim Sizes: $b^*(u)$



Pareto Distributed Claim Sizes

Example

Pareto distributed claim sizes:

$$G(y) = 1 - (1 + y)^{-2}.$$

Asymptotically optimal strategy?

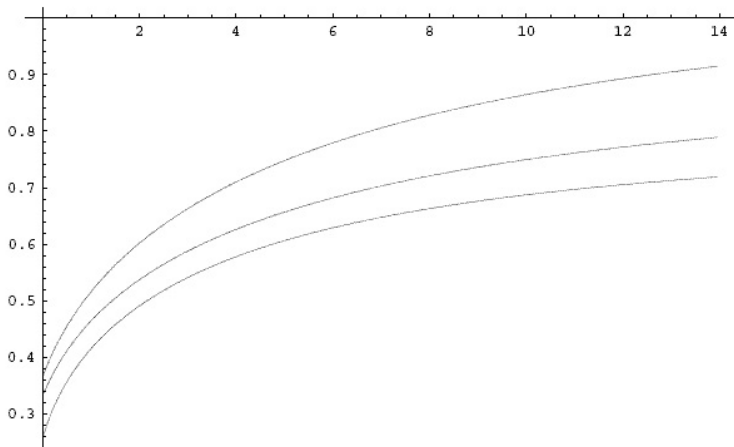
$$\psi^b(x) \approx \frac{1}{0.7b - 0.2} \frac{b}{1 + x/b} = \frac{1}{(0.7 - 0.2/b)(1 + x/b)}.$$

Minimised for $b = 0.4x / (0.7x - 0.2)$.

$$b^R = 4/7.$$



Pareto Distributed Claim Sizes: $\delta(u)$



○○○○○○
○○○○
○○○○○○○○
○○○○
○○○○○●○○○○
○○○○
○○○○

○○○○○○○○

Pareto Distributed Claim Sizes: $b^*(u)$



1 The de Finetti Problem

- Introduction
- The Optimal Strategy
- De Finetti's Example

2 Minimal Ruin Probabilities

- Introduction
- The Hamilton-Jacobi-Bellman Equation
- Two Examples

3 Optimal Dividends

- Dividends in a Diffusion Approximation
- Dividends in the Classical Model
- Dividends with Reinvestment

4 Further Problems



Optimal Dividends in a Diffusion Approximation

Consider the Diffusion Approximation

$$X_t^0 = x + \sigma W_t + mt$$

where $\{W_t\}$ is a standard Brownian motion.

A dividend strategy is an increasing process $\{D_t\}$ with $D_0 \geq 0$.

The surplus becomes $X_t^D = X_t^0 - D_t$.

The value of a strategy is

$$V^D(x) = \mathbb{E} \left[\int_{0-}^{\tau-} e^{-\delta t} dD_t \right].$$

We let $V(x) = \sup_D V^D(x)$.



The Hamilton-Jacobi-Bellman Equation

The value function then fulfils

$$\max\{1 - V'(x), \frac{1}{2}\sigma^2 V''(x) + mV'(x) - \delta V(x)\} = 0.$$

We know that $V(0) = 0$.

Optimal Strategy: Do not pay dividend if $V'(x) > 1$. At a point where $V'(x) = 1$ reflect the process.



The Value Function

Intuition: Barrier strategy close to zero.

Solve $\frac{1}{2}\sigma^2 f''(x) + mf'(x) - \delta f(x) = 0$ with $f(0) = 0$.

$$f(x) = e^{\theta_1 x} - e^{\theta_2 x}, \quad V(x) = Cf(x).$$

Barrier in a yields $V'(a) = 1$. Solution is maximal if

$$C = \frac{1}{f'(a)}$$

is maximal, i.e. $f''(a) = 0$. Note that $V(a) = m/\delta$.

The optimal solution is

$$V(x) = \begin{cases} f(x)/f'(a), & \text{if } x \leq a, \\ f(a) + x - a, & \text{otherwise.} \end{cases}$$

The Verification Theorem

By Itô's formula

$$\begin{aligned}
 0 &= V(X_\tau)e^{-\delta\tau} = V(X_0) + \int_0^\tau \sigma V'(X_s)e^{-\delta s} dW_s \\
 &\quad + \int_0^\tau \left[\frac{1}{2}\sigma^2 V''(X_s) + mV'(X_s) - \delta V(X_s) \right] e^{-\delta s} ds \\
 &\quad - \int_0^\tau e^{-\delta s} V'(X_s) dD_s \\
 &\leq V(X_0) + \int_0^\tau \sigma V'(X_s)e^{-\delta s} dW_s - \int_0^\tau e^{-\delta s} dD_s .
 \end{aligned}$$

Taking expected values yields the result. Using the optimal strategy equality holds.

1 The de Finetti Problem

- Introduction
- The Optimal Strategy
- De Finetti's Example

2 Minimal Ruin Probabilities

- Introduction
- The Hamilton-Jacobi-Bellman Equation
- Two Examples

3 Optimal Dividends

- Dividends in a Diffusion Approximation
- Dividends in the Classical Model
- Dividends with Reinvestment

4 Further Problems



Dividends in the Classical Model

Consider the classical model

$$X_t^0 = x + ct - \sum_{k=1}^{N_t} Y_k .$$

With dividend payments we have $X_t^D = X_t^0 - D_t$.

Here we get the equation

$$\max \left\{ 1 - V'(x), \right. \\ \left. cV'(x) - \delta V(x) + \lambda \left[\int_0^x V(x-y) dG(y) - V(x) \right] \right\} = 0 .$$



The Optimal Strategy

It turns out: The solution is not unique and not necessarily differentiable. But the jumps of $V'(x)$ are all upwards with $V'(x-) = 1$. The correct solution is the minimal solution.

The optimal strategy is the following:

No dividend if $V'(x-) > 1$.

Dividends at rate c if x is an upper boundary point of the set where no dividend is paid.

A dividend of size $\sup\{y : V(x) = V(x - y) + y\}$.

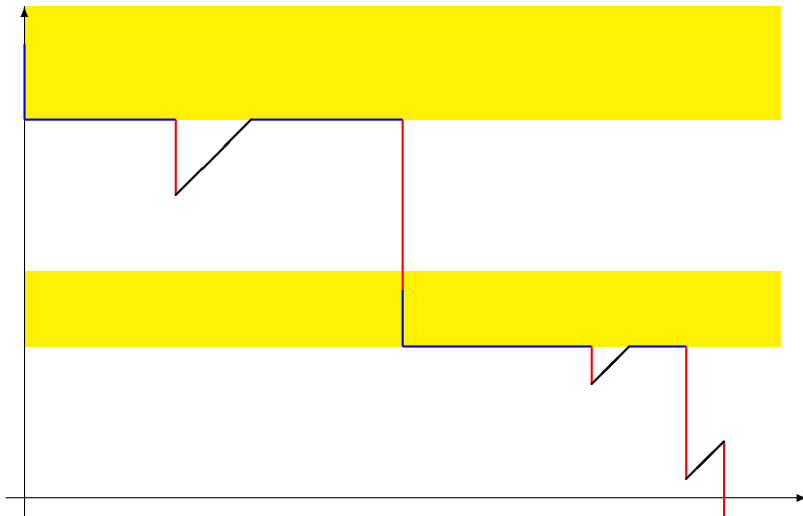
○○○○○○○
 ○○○○○
 ○○○○○

○○○○
 ○○○○○
 ○○○○○○○

○○○○○
 ○○○●
 ○○○○○

○○○○○○○○○

Dividends in the Classical Model



- 1 The de Finetti Problem**
 - Introduction
 - The Optimal Strategy
 - De Finetti's Example
- 2 Minimal Ruin Probabilities**
 - Introduction
 - The Hamilton-Jacobi-Bellman Equation
 - Two Examples
- 3 Optimal Dividends**
 - Dividends in a Diffusion Approximation
 - Dividends in the Classical Model
 - **Dividends with Reinvestment**
- 4 Further Problems**

Dividends with Reinvestment

We now consider the process $X_t^D = X_t^0 - D_t + A_t$ where A_t is the minimal increasing process such that $X_t^D \geq 0$ for all t . That is, the process is reflected in 0 and A_t is the cumulative investments until time t . The value of a strategy $\{D_t\}$ is

$$V^D(x) = \mathbb{E} \left[\int_0^\infty e^{-\delta t} dD_t - \theta \int_0^\infty e^{-\delta t} dA_t \right].$$

The value function is $V(x) = \sup_D V^D(x)$

If $\theta < 1$ then $V(x) = \infty$.

If $\theta = 1$ then the optimal process fulfils $X_t^* = 0$.

We assume $\theta > 1$.



Concavity of $V(x)$

Let $x \neq y$ and $\alpha \in (0, 1)$, $\beta = 1 - \alpha$. Let D_t^y and A_t^y be a strategy for initial capital y . We let $z = \alpha x + \beta y$, $D_t = \alpha D_t^x + \beta D_t^y$, $B_t = \alpha A_t^x + \beta A_t^y$.

For initial capital z we get the process

$$X_t^z = \alpha X_t^x + \beta X_t^y \geq 0.$$

We conclude that $A_t^z \leq B_t^z$.

Thus $V(z) \geq V^z(z) \geq \alpha V^x(x) + \beta V^y(y)$. Taking the supremum we find

$$V(\alpha x + \beta y) \geq \alpha V(x) + \beta V(y).$$

The Diffusion Approximation

The value function fulfils

$$\max\{1 - V'(x), \frac{1}{2}\sigma^2 V''(x) + mV'(x) - \delta V(x)\} = 0 .$$

Moreover, $V'(0) \leq \theta$. The strategy becomes a barrier strategy at a barrier a with $V'(a) = 1$.

The solution is of the form

$$V(x) = C_1 e^{\theta_1 x} + C_2 e^{\theta_2 x} .$$

The constants are determined such that $V'(0) = \theta$, $V'(a) = 1$ and $V(0)$ becomes maximal.

The Classical Model

The value function fulfils

$$\max \left\{ 1 - V'(x), cV'(x) - \delta V(x) + \lambda \left[\int_0^x V(x-y) dG(y) + \int_x^\infty (V(0) - \theta(y-x)) dG(y) - V(x) \right] \right\} = 0.$$

The optimal strategy is a barrier strategy with a barrier at a .

Problem: $1 \leq V'(0) < \theta$. Thus no condition at zero.

Calculate the value of a barrier strategy, and maximise $V(0)$.

- 1 The de Finetti Problem
 - Introduction
 - The Optimal Strategy
 - De Finetti's Example
- 2 Minimal Ruin Probabilities
 - Introduction
 - The Hamilton-Jacobi-Bellman Equation
 - Two Examples
- 3 Optimal Dividends
 - Dividends in a Diffusion Approximation
 - Dividends in the Classical Model
 - Dividends with Reinvestment
- 4 Further Problems

Optimal Investment

Let $Z_t = \exp\left\{\left(r - \frac{1}{2}\sigma_I^2\right)t + \sigma_I W_t^I\right\}$ be the value of some risky asset or some portfolio. The insurer invests the amount A_t into the risky asset. Then the surplus fulfils

$$dX_t^A = dX_t^0 + rA_t dt + \sigma_I A_t dW_t^I .$$

We want to minimise the ruin probability.
The corresponding HJB equation becomes

$$\sup_{A \in \mathbb{R}} \mathfrak{A}\delta(x) + rA\delta'(x) + \frac{1}{2}\sigma_I^2 A^2 \delta''(x) = 0 .$$

Here \mathfrak{A} is the generator of the uncontrolled process $\{X_t^0\}$.

Optimal New Business

Let $\{Y_t\}$ be another independent surplus process. The insurer can buy part of the risk. Then the surplus fulfils

$$X_t^b = X_t^0 + \int_0^t b_{s-} dY_s ,$$

where $b_t \in [0, 1]$. We again minimise the ruin probability. The corresponding HJB equation is

$$\sup_{b \in [0, 1]} \mathfrak{A}_X \delta(x) + bc_Y \delta'(x) + \lambda_Y \left[\int_0^{x/b} \delta(x-by) dG_Y(y) - \delta(x) \right] = 0 .$$

Maximising the Surplus

Consider a diffusion approximation with reinsurance. We only allow 'cheap' reinsurance. We want to maximise

$$\mathbb{E} \left[\int_0^\tau e^{-\delta s} X_s^b ds \right].$$

The HJB equation is

$$x + \sup_b mbV'(x) + \frac{1}{2}\sigma^2 b^2 V''(x) - \delta V(x) = 0.$$

It turns out that ruin will never occur ($b(x) \rightarrow 0$ as $x \rightarrow 0$)

Maximising the Surplus

For a classical model the HJB equation is

$$x + \sup_b c(b) V'(x) - \delta V(x) + \lambda \left[\int_0^{x/b} V(x - by) dG(y) - V(x) \right].$$

Usually, there is a positive probability for ruin.



Minimal Reinvestment

Consider the process with reinsurance $\{X_t^b\}$. Suppose the insurer has to reinvest a possible deficit. The surplus becomes $Y_t^b = X_t^b + A_t^b$. The value of the strategy is

$$V^b(x) = \mathbb{E} \left[\int_0^\infty e^{-\delta t} dA_t \right].$$

The value function is $V(x) = \inf_b V^b(x)$.

$\delta > 0$: Future reinvestment preferred to present reinvestment.

$\delta = 0$: Indifferent between future and present reinvestment.

$\delta < 0$: Present reinvestment preferred to future reinvestment.

Minimal Reinvestment


$V(x)$ fulfils for $x \geq 0$ the HJB equation


$$\inf_b c(b) V'(x) + \lambda \left[\int_0^\infty V(x - by) dG(y) - V(x) \right] - \delta V(x) = 0 ,$$


with $V(x) = V(0) - x$ for $x < 0$.


The solution we look for is the maximal decreasing solution.

References

- 





Azcue, P. and **Muler, N.** (2005). Optimal reinsurance and dividend distribution policies in the Cramér-Lundberg model. *Math. Finance* **15**, 261–308.
- 

de Finetti, B. (1957). Su un' impostazione alternativa della teoria collettiva del rischio. *Transactions of the XVth International Congress of Actuaries* **2**, 433–443.
- 




Gerber, H.U. (1969). Entscheidungskriterien für den zusammengesetzten Poisson-Prozess. *Schweiz. Verein. Versicherungsmath. Mitt.* **69**, 185–228.
- 

Hipp, C. and **Plum, M.** (2000). Optimal investment for insurers. *Insurance Math. Econom.* **27**, 215–228.

References

-  **Hipp, C.** and **Schmidli, H.** (2004). Asymptotics of ruin probabilities for controlled risk processes in the small claims case. *Scand. Actuarial J.*, 321–335.
-  **Hipp, C.** and **Taksar, M.** (2000). Stochastic control for optimal new business. *Insurance Math. Econom.* **26**, 185–192.
-  **Højgaard, B.** and **Taksar, M.** (1997). Optimal proportional reinsurance policies for diffusion models. *Scand. Actuarial J.*, 166–180.
-  **Schmidli, H.** (2001). Optimal proportional reinsurance policies in a dynamic setting. *Scand. Actuarial J.*, 55–68.

References

-  **Schmidli, H.** (2002). On minimising the ruin probability by investment and reinsurance. *Ann. Appl. Probab.* **12**, 890–907.
-  **Schmidli, H.** (2007). *Stochastic Control in Insurance*. Springer-Verlag, London, forthcoming.
-  **Shreve, S.E., Lehocsky, J.P. and Gaver, D.P.** (1984). Optimal consumption for general diffusions with absorbing and reflecting barriers. *SIAM J. Control Optim.* **22**, 55–75.