On a Lévy-driven Continuous-Time Garch Model

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Joint work with

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Modelling prices and volatility

The beginning:

Black and Scholes model (1973):

The log-price process $G_t = \log S_t$ is modelled by geometric Brownian motion

$$G_t = G_0 + \mu t + \sigma B_t \,, \quad t \geq 0 \,.$$

Equivalently, given S_0, the price process is the solution to the SDE

$$\frac{dS_t}{S_t} = (\mu + \frac{\sigma^2}{2})dt + \sigma dB_t \,, \quad t > 0 \,.$$

The stochastic integral used is the Itô integral.

Kiyosi Itô is the Winner of the Gauss Prize 2006,

awarded jointly by the DMV and the IMU.

Kiyosi Itô is one of the pioneers of probability theory, and the originator of Itô Calculus. First published in 1942 in Japanese, this epoch-making theory of stochastic differential equations describes nondeterministic and random evolutions. The so-called Itô formula has found applications in other branches of mathematics as well as in various other fields including, e.g., conformal field theory in physics, stochastic control theory in engineering, population genetics in biology, and most recently, **mathematical finance in economics**.

The citation from the National Academy of Science states:
If one disqualifies the Pythagorean Theorem from contention, it is hard to think of a mathematical result which is better known and more widely applied in the world today than "Ito's Lemma." This result holds the same position in stochastic analysis that Newton's fundamental theorem holds in classical analysis. That is, it is the sine qua non of the subject.

More than 40 years after Black and Scholes:

$$G_t = G_0 + \int_0^t \mu_s ds + \int_0^t \sigma_{s-} dL_s\,, \quad t \geq 0\,,$$

where L is a **Lévy process** and μ, σ are adapted càdlàg processes.

The **Lévy-Itô representation** states that for $t \geq 0$,

$$L_t = \gamma_L t + \tau_L B_t + \int_{(0,t]\times\{|x|>1\}} x J_L(ds\times dx) + \lim_{\varepsilon\downarrow 0} \int_{(0,t]\times\{|x|\in(\varepsilon,1]\}} x(J_L(ds\times dx) - ds\Pi_L(dx)),$$

where J_L is a Poisson random measure on $[0,t]\times\mathbb{R}$ with intensity $dt\Pi_L(dx)$.

Question: How to model the volatility $(\sigma_t)_{t\geq 0}$.

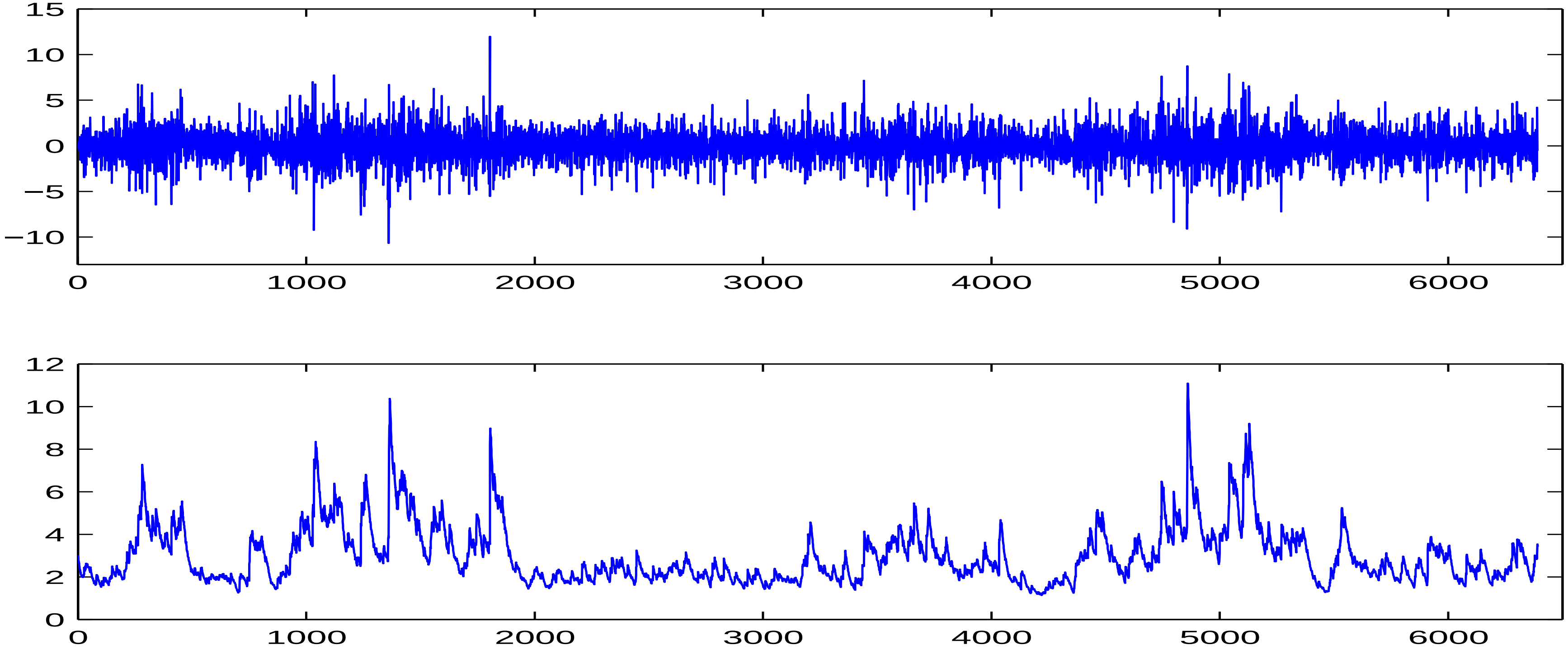


Figure 1: Deseasonalised 5 minutes log-returns of Intel (February 1 - May 31, 2002) and estimated volatility.

Stylized facts of volatility:

(1) volatility is not constant (smile effect);

(2) volatility is random;

(3) volatility has heavy-tailed marginals (higher moments do not exist);

(4) volatility has skewed marginals (leverage effect)

(5) volatility is a stochastic process with long-range dependence effect;

(6) volatility is a stochastic process with clusters in the extremes.

Recall discrete time GARCH(1,1) model

As an Euler approximation of the above Lévy log-price model we obtain for the martingale part

$$Y_n = \sigma_n Z_n \quad \text{i.i.d. } \textbf{innovations } (Z_n)_{n \in \mathbb{N}_0},$$

Volatility process: Define for σ^2 the random recurrence equation

$$\sigma_n^2 = \beta + \lambda Y_{n-1}^2 + \delta \sigma_{n-1}^2, \quad n \in \mathbb{N}.$$

Random recurrence equation models are successfully applied in many areas of stochastics; see Diaconis and Freedman (1999). For the introduction of such models 1982 into econometrics, and his contributions to their statistical analysis **Robert Engle** was awarded in 2003 the Nobel prize in Economics.

Reorganise and iterate the recurrence:

$$\begin{aligned}\sigma_n^2 &= \beta + \lambda Y_{n-1}^2 + \delta\sigma_{n-1}^2 = \beta + (\delta + \lambda Z_{n-1}^2)\sigma_{n-1}^2 \\ &\vdots \\ &= \beta \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} (\delta + \lambda Z_j^2) + \sigma_0^2 \prod_{j=0}^{n-1} (\delta + \lambda Z_j^2) \qquad (1)\end{aligned}$$

Under appropriate conditions

$$\sigma_n^2 \quad \xrightarrow{d} \quad \sigma_\infty^2 \stackrel{d}{=} \beta \sum_{i=0}^{\infty} \prod_{j=1}^{i} (\delta + \lambda Z_j^2).$$

Continuous time GARCH(1,1)

Idea: start with **(1)** and **replace** the sum by an integral

$$\Leftrightarrow \quad \sigma_n^2 = \left(\beta \int_0^n \exp\left(-\sum_{j=0}^{[s]} \log(\delta + \lambda Z_j^2)\, ds\right) + \sigma_0^2\right) \exp\left(\sum_{j=0}^{n-1} \log(\delta + \lambda Z_j^2)\right)$$

Replace $\boldsymbol{Z_j}$ by jumps of a Lévy process L and take $\beta, \eta = -\log\delta, \varphi = \lambda/\delta$.

Then for a finite r.v. σ_0^2 define the **volatility process**

$$\sigma_t^2 = \left(\beta \int_0^t e^{X_s} ds + \sigma_0^2\right) e^{-X_t} \quad t \geq 0.$$

with **auxiliary process**

$$X_t = t\eta - \sum_{0<s\leq t} \log(1 + \varphi(\Delta L_s)^2) \quad t \geq 0.$$

Recall: $(L_t)_{t\geq 0}$ is Lévy process if $Ee^{isL_t} = e^{t\psi_L(s)}$, $s \in \mathbb{R}$, with

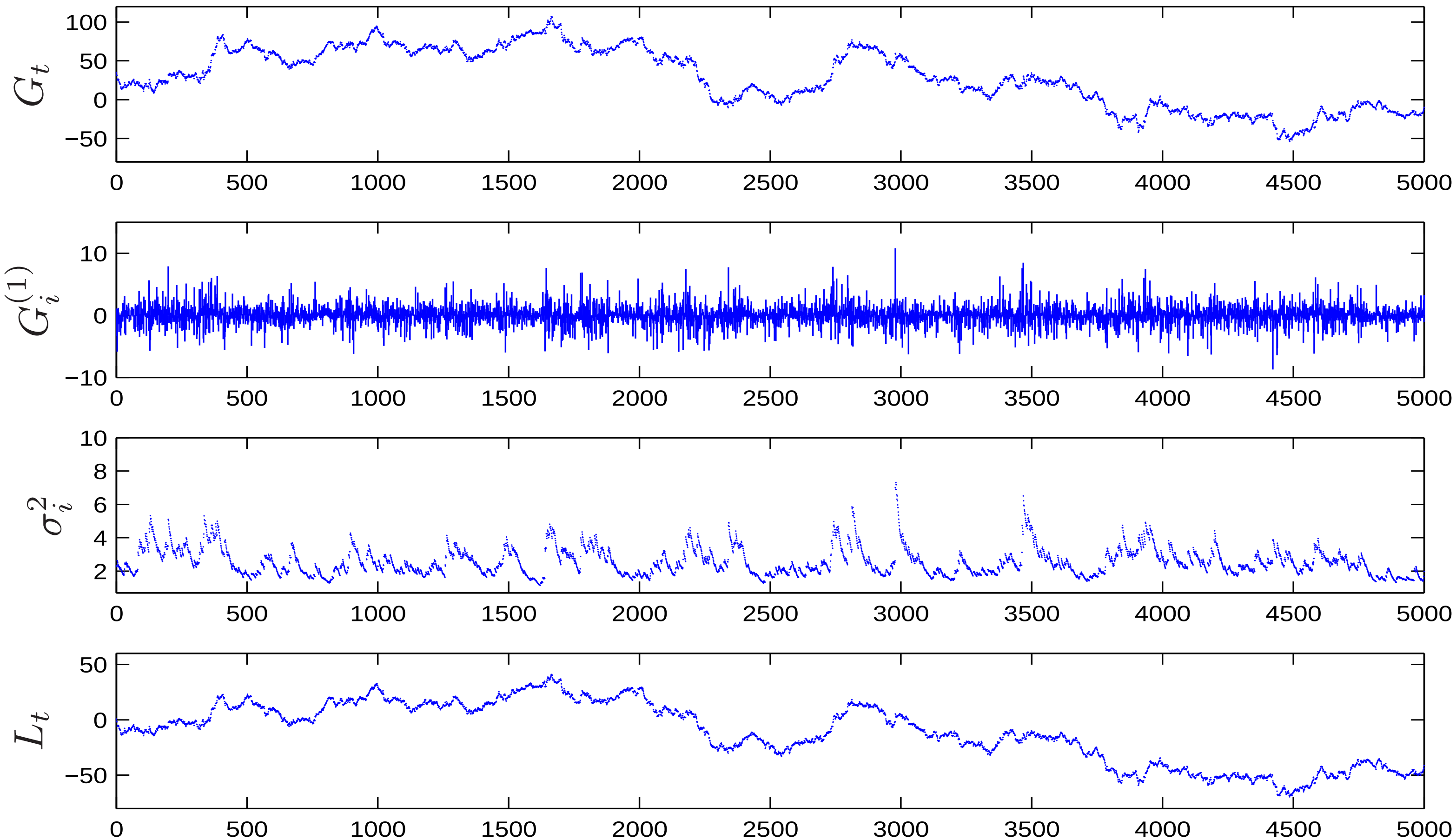
$$\psi_L(s) = i\gamma_L s - \tau_L^2 \frac{s^2}{2} + \int_{\mathbb{R}} (e^{isx} - 1 - isxI_{\{|x|<1\}})\Pi_L(dx), \quad s \in \mathbb{R}.$$

$(\gamma_L, \tau_L, \Pi_L)$ **characteristic tripel**, Π_L **Lévy measure**: $\int_{|x|<\varepsilon} x^2 \Pi_L(dx) < \infty$.

Define the **COGARCH(1,1)process** by

$$G_t = \int_{(0,t]} \sigma_{t-}\, dL_t \quad t \geq 0.$$

(Note: this defines the martingale part of the price process.)



First: Simulated VG driven COGARCH(1,1) process with $\beta = 0.04$, $\eta = 0.053$ and $\varphi = 0.038$;

second: differenced COGARCH process $(G_t^{(1)})$;

third: volatility process (σ_t);

last: VG process (L_t) with characteristic function $Ee^{iuL_1} = (1 + u^2/(2C))^{-C}$ and $C = 1$;

Properties

- G jumps at the same times as L with jump size $\Delta G_t = \sigma_t \Delta L_t$.
- $(X_t)_{t\geq 0}$ is spectrally negative, has drift η, no Gaussian part, Lévy measure

$$\Pi_X([0,\infty)) = 0 \quad \Pi_X((-\infty,-x]) = \Pi_L(\{|y| \geq \sqrt{(e^x-1)\varphi}\}) \text{ for } x>0.$$

- $d\sigma_t^2 = (\beta - \eta\sigma_{t-}^2)\, dt + \varphi\, \sigma_{t-}^2 d[L,L]_t^{(d)}$

where $[L,L]_t^{(d)} = \sum_{0<s\leq t}(\Delta L_s)^2$ and

$$\sigma_t^2 = \sigma_0^2 + \beta t - \eta \int_0^t \sigma_s^2 ds + \varphi \sum_{0<s\leq t} \sigma_{s-}^2 (\Delta L_s)^2 \quad t \geq 0. \qquad (2)$$

- $$\int_{\mathbb{R}} \log\left(1+\varphi x^2\right) \Pi_L(dx) < \eta \quad \Longleftrightarrow \quad EX_1 > 0$$

$$\Longleftrightarrow \quad \sigma_t^2 \xrightarrow{d} \sigma_\infty^2 \stackrel{d}{=} \beta \int_0^\infty e^{-X_t}\, dt.$$

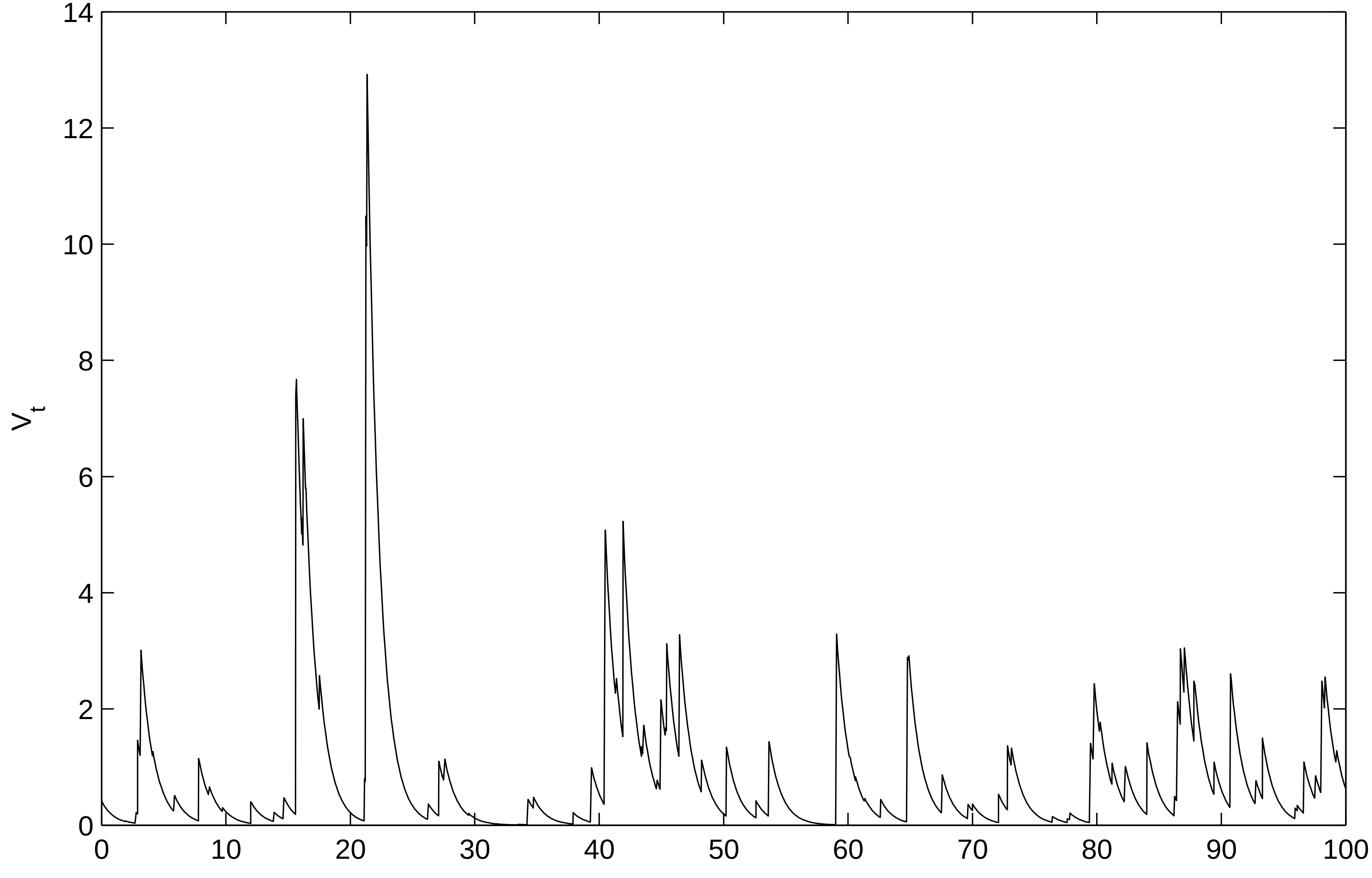
Sample path behaviour

- From (2) we know that σ_t^2 has only upwards jumps.

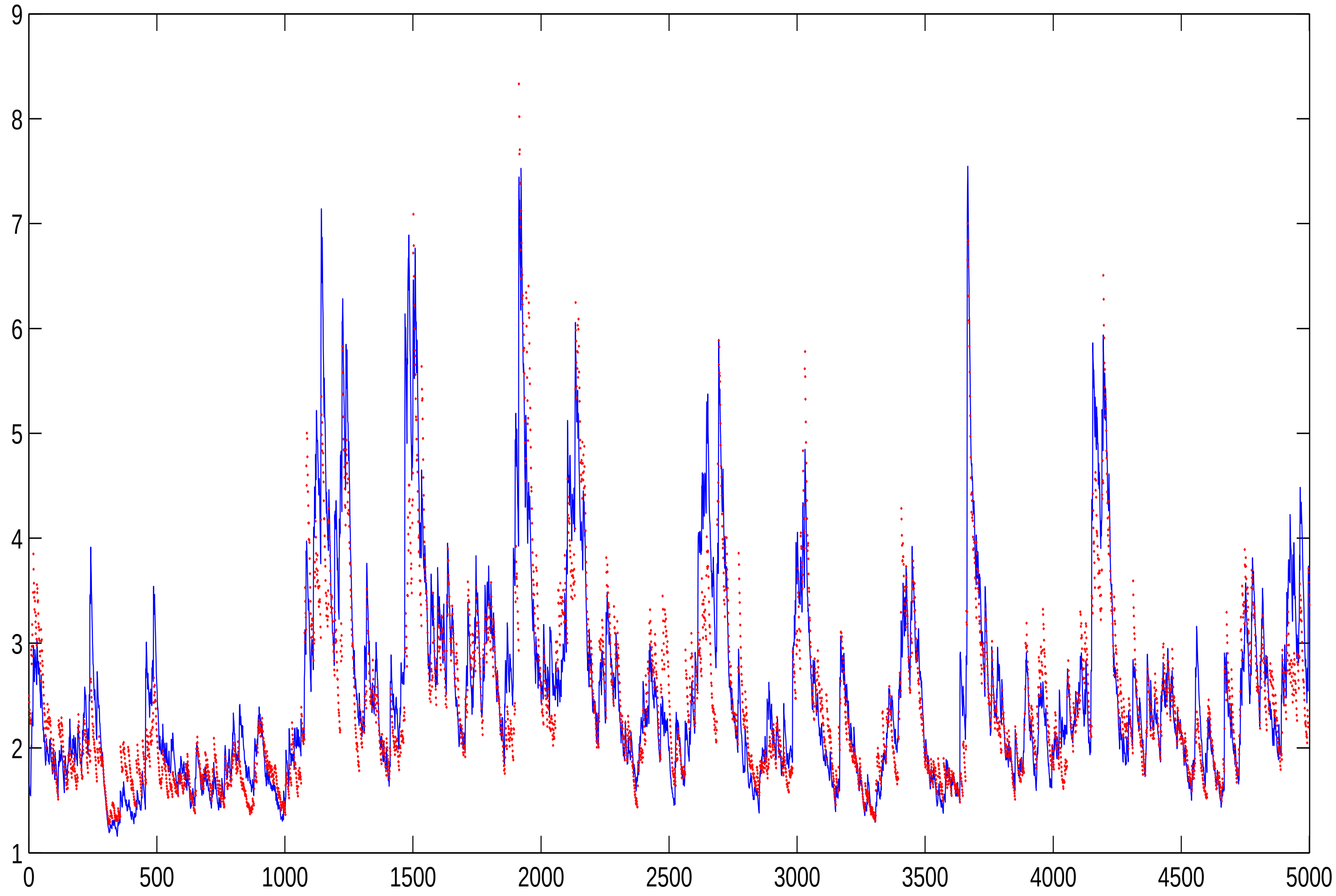
- If $(L_t)_{t\geq 0}$ is compound Poisson with jump times $0 = T_0 < T_1 < \ldots$,

$$\sigma_t^2 = \frac{\beta}{\eta} + \left(\sigma_{T_j}^2 - \frac{\beta}{\eta}\right) e^{-(t-T_j)\eta}, \quad t \in (T_j, T_{j+1}).$$

- For the stationary process, we have $\sigma_\infty^2 \geq \dfrac{\beta}{\eta}$ a.s.



Volatility process for L compound Poisson with Weibull distributed jump sizes.



Sample paths of σ_t^2 (solid line) and $\widehat{\sigma}_t^2$ (dotted line) of one simulation.

Theorem Suppose that $EL_1 = 0$, $\mathrm{var}(L_1) = 1$. Define $Ee^{-sX_t} = e^{t\Psi_X(s)}$. Assume that the volatility process is stationary, and define $G_i^{(1)} := \int_{i-r}^{i} \sigma_{s-} dL_s$. If $\Psi_X(1) < 0$, then

$$EG_i^{(1)} = 0,\ E(G_i^{(1)})^2 = \frac{r\beta}{-\Psi_X(1)} EL_1^2 \text{ and } \mathrm{corr}(G_i^{(1)}, G_{i+h}^{(1)}) = 0.$$

If $EL_1^4 < \infty$, $\Psi_X(2) < 0$ and $\int_{\mathbb{R}} x^3 \nu_L(dx) = 0$, then for $k, p > 0$

$$\mathrm{corr}((G_i^{(1)})^2, (G_{i+h}^{(1)})^2) = ke^{-hp}, \quad h \in \mathbb{N}. \qquad \square$$

Theorem Assume that L_1 is symmetric and that there exists $\kappa > 0$ such that

$$|L_1|^\kappa \log^+ |L_1| < \infty \quad \text{and} \quad \Psi_X(\kappa/2) = 1.$$

Then a stationary version of the volatility process exists with

$$P(\sigma_t > x) \sim cx^{-\kappa/2}, \quad x \to \infty. \qquad \square$$

Stylized facts of volatility:

(1) volatility is not constant (smile effect);

(2) volatility is random;

(3) volatility has heavy-tailed marginals (higher moments do not exist: K., Lindner and Maller (2004), Fasen, K., Lindner (2004));

(4) volatility has skewed marginals (leverage effect introduced in Haug et al.)

(5) volatility is a stochastic process with long-range dependence effect (acf decreases geometrically: K., Lindner and Maller (2004));

(6) volatility is a stochastic process with clusters in the extremes: Fasen: Extremes of genOU processes (2006,2007).

- $(\sigma_t^2)_{t\geq 0}$ is a Markov process.
- $(\sigma_t^2, G_t)_{t\geq 0}$ is a bivariate Markov process.
- The infinitesimal generator of $(\sigma_t^2, G_t)_{t\geq 0}$ can be calculated:

Kallsen and Vesenmayer (2006): can be used for derivatives pricing.

Question: Can we find a discrete time skeleton, which approximates the COGARCH(1,1) process, and is a GARCH(1,1) process.

The following approximation, called **first jump approximation** shows that (under some technical conditions) the solution of a Lévy-driven SDE can be approximated arbitrarily close, by replacing the Lévy process with its first jump approximation.

Theorem [Szimayer and Maller (2007), Haug and Stelzer (2007)]
Let L be a Lévy process in $\mathbb{R}^d$, which has no Brownian part, drift γ_L and Lévy measure Π_L and satisfies $EL^2(1) = 1$.
For $n \in \mathbb{N}$ let $1 > \varepsilon^{(n)} \downarrow 0$ and $0 = t_0^{(n)} < t_1^{(n)} < t_2^{(n)} \cdots \uparrow \infty$.
Set $\delta^{(n)} := \sup_{i \in \mathbb{N}}(t_i^{(n)} - t_{i-1}^{(n)})$ and assume that $\lim_{n\to\infty} \delta^{(n)} = 0$. Assume that

$$\lim_{n\to\infty} \delta^{(n)}(\Pi(\{x \in \mathbb{R}^d : |x| > \varepsilon^{(n)}\})^2 = 0\,. \qquad (3)$$

Define for all $n \in \mathbb{N}$

$$\begin{aligned}
\gamma^{(n)} &:= \gamma_L - \int_{\varepsilon^{(n)}<|x|\le 1} x\Pi_L(dx) \\
\tau_i^{(n)} &:= \inf\{t \,:\, t_{i-1}^{(n)} < t \le t_i^{(n)},\, |\Delta L_t| > \varepsilon^{(n)}\} \quad \forall i \in \mathbb{N} \\
\widetilde{L}_t^{(n)} &:= \gamma^{(n)} t + \sum_{\{i\in\mathbb{N}:\tau_i^{(n)}\le t\}} \Delta L_{\tau_i^{(n)}} \quad \forall t \ge 0 \\
\overline{L}_t^{(n)} &:= \widetilde{L}^{(n)}_{t_{i-1}^{(n)}}.
\end{aligned}$$

Then

$$\widetilde{L}^{(n)} \to L \quad \text{in ucp as } n \to \infty \quad \text{and} \quad d_S(\overline{L}^{(n)}, L) \xrightarrow{P} 0 \quad n \to \infty.$$

$\square$

Remark (i) Whenever one of the sequences $(\delta^{(n)})$ or $(\varepsilon^{(n)})$ are given, one can always choose the other such that (3) holds.

(ii) Note that the time grid is not necessarily equidistant. The construction allows for discrete sampling of a continuous-time Lévy-driven model. This is useful for high-frequency data.

(iii) The construction allows also the embedding of a discrete-time jump model into a continuous-time one. □

Example [COGARCH(1,1) and its GARCH(1,1) approximation, Maller, Müller and Szimayer (2007)]

We use the notation as in the theorem and assume that all assumptions hold. For $n \in \mathbb{N}$ set $\Delta t_i(n) := t_i^{(n)} - t_{i-1}^{(n)}$ and define $\Delta L_{\tau_i^{(n)}}$ as the first jump of size larger than $\varepsilon^{(n)}$ in $(t_{i-1}^{(n)}, t_i^{(n)}]$. Define

$$Z_{i,n} = \frac{\mathbf{1}_{\{\tau_i^{(n)}\}<\infty} \Delta L_{\tau_i^{(n)}} - \nu_i^{(n)}}{\xi_i^{(n)}}, \quad i \in \mathbb{N}.$$

By the strong Markov property $(\mathbf{1}_{\{\tau_i^{(n)}<\infty\}} \Delta L_{\tau_i^{(n)}})_{i\in\mathbb{N}}$ is an iid sequence with distribution

$$\frac{\Pi(dx)\mathbf{1}_{\{|x|>\varepsilon^{(n)}\}}}{\Pi(\{x \in \mathbb{R}^d : |x| > \varepsilon^{(n)}\}} \Big(1 - e^{-\eta \Delta t_i(n) \Pi(\{x\in\mathbb{R}^d : |x|>\varepsilon^{(n)}\})}\Big), \quad x \in \mathbb{R} \setminus \{0\}.$$

Then $(Z_{i,n})_{i\in\mathbb{N}}$ is an iid sequence with mean 0 and variance 1.

Now recall

$$d\sigma_t^2 = (\beta - \eta\sigma_{t-}^2)\,dt + \varphi\,\sigma_{t-}^2 d[L,L]_t^{(d)} \quad \text{and} \quad G_t = \int_{(0,t]} \sigma_{t-} dL_t \quad t > 0.$$

We discretise as follows: for $G_{0,n} = G_0 = 0$ set

$$G_{i,n} - G_{i-1,n} = \sigma_{i-1,n}\sqrt{\Delta t_i(n)} Z_{i,n}, \quad i \in \mathbb{N},$$

and

$$\sigma_{i,n}^2 = \beta\Delta t_i(n) + \big(1 + \varphi\Delta t_i(n) Z_{i,n}^2\big) e^{-\eta\Delta t_i(n)} \sigma_{i-1,n}^2, \quad i \in \mathbb{N}.$$

This defines a discrete time GARCH(1,1) random recurrence equation; cf. p. 8.

Follow the construction as before and introduce continuous-time versions (piecewise constant) of the auxiliary process $X_{i,n}$, $\sigma^2_{i,n}$ and $G_{i,n}$. Then with the usual technical efforts, it is shown that

$$d_S((G_n, \sigma_n^2), (G, \sigma^2)) \xrightarrow{P} 0 \quad n \to \infty. \qquad \square$$

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