Unbiased shifts of Brownian motion

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Unbiased shifts

Setting

Let \( B = (B_t)_{t \in \mathbb{R}} \) be a two-sided standard Brownian motion defined as the identity on \((\Omega, \mathcal{A}, \mathbb{P}_0)\) where \( \Omega \) is the set of all continuous functions from \( \mathbb{R} \) to \( \mathbb{R} \), \( \mathcal{A} \) is the Kolmogorov product \( \sigma \)-algebra and \( \mathbb{P}_0 \) is the distribution of \( B \). In particular \( B_0 = 0 \) a.s.

Note that no external randomization is allowed.

Definition

An unbiased shift of \( B \) is a random time \( T \) in \( \mathbb{R} \) such that:
- \((B_{T+t} - B_T)_{t \in \mathbb{R}}\) is a standard Brownian motion,
- \((B_{T+t} - B_T)_{t \in \mathbb{R}}\) is independent of \( B_T \).
### Unbiased shifts

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#### Remark

i.e. \( T \) is an unbiased shift if \((B_{T+t})_{t \in \mathbb{R}}\) is a two-sided standard Brownian motion not necessarily taking the value 0 at time 0.
Example

If $T \geq 0$ is a stopping time, then $(B_{T+t} - B_T)_{t \geq 0}$ is a one-sided Brownian motion independent of $B_T$. However, the example

$$T := \inf\{ t \geq 0 : B_t = a \}$$

shows that $(B_{T+t} - B_T)_{t \in \mathbb{R}}$ need not be a two-sided Brownian motion.

Example

Consider a deterministic $T = t_0$. Then $\tilde{B} := (B_{t_0+t} - B_{t_0})_{t \in \mathbb{R}}$ is a two-sided Brownian motion. However, it is not independent of $B_{t_0}$ since $B_{t_0} = -\tilde{B}_{-t_0}$.

Remark

We will see later that an unbiased shift need not be a stopping time, even when it is nonnegative.
The case when $B_T = 0$ is of special interest.

**Example**

Let $\ell^0$ be the **local time** random measure of $B$ at 0. Let $(T_r)_{r \in \mathbb{R}}$ be the (generalized) inverse of the cumulative mass,

$$T_r := \begin{cases} \sup\{t \geq 0: \ell^0[0, t] = r\}, & r \geq 0, \\ \sup\{t < 0: \ell^0[t, 0] = -r\}, & r < 0. \end{cases}$$

Then each $T_r$ is an **unbiased shift**.
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Then each $T_r$ is an unbiased shift.

Since $B_{T_r} = 0$ this simply means that

$$(B_{T_r+t})_{t \in \mathbb{R}}$$

is a two-sided Brownian motion for all $r \in \mathbb{R}$. 
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Since $B_{T_r} = 0$ this simply means that $(B_{T_r+t})_{t \in \mathbb{R}}$ is a two-sided Brownian motion for all $r \in \mathbb{R}$.

Thus: when traveling in time according to the clock of local time we always see globally a two-sided Brownian motion.
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Example

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\[
T_r := \begin{cases} 
\sup \{ t \geq 0 : \ell^0[0, t] = r \}, & r \geq 0, \\
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\end{cases}
\]

Then each \( T_r \) is an unbiased shift.

Since \( B_{T_r} = 0 \) we have:

\[
( B_{T_r+t} )_{t \in \mathbb{R}} \text{ is a two-sided Brownian motion for all } r \in \mathbb{R}.
\]

Thus: when traveling in time according to the clock of local time at zero you always see globally a two-sided Brownian motion.

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We shall now sketch the theory that leads to this result, and to many more results on unbiased shifts.
The \(\sigma\)-finite stationary \(\mathbb{P}\)

Set \(\mathbb{P}_x := \mathbb{P}_0(B + x \in \cdot)\), \(x \in \mathbb{R}\), and define a \(\sigma\)-finite measure by

\[
\mathbb{P} := \int \mathbb{P}_x \, dx.
\]

The stationary increments yield that \(B\) is stationary under \(\mathbb{P}\):

\[
\mathbb{P}(\theta_t B \in \cdot) = \mathbb{P}, \quad t \in \mathbb{R},
\]

where \(\theta_t : \Omega \mapsto \Omega\) is defined by \((\theta_t \omega)_s := \omega_{t+s}\) for \(s \in \mathbb{R}\).

**Local time \(\ell^x\)**

The local time random measures \(\ell^x\) at \(x \in \mathbb{R}\) can be defined such that \(\mathbb{P}\)-a.e. \(\ell^x\) is diffuse for all \(x\) and

\[
\ell^x(\theta_t B, C - t) = \ell^x(C), \quad C \in \mathcal{B}, t \in \mathbb{R}, \quad (\ell^x \text{ is invariant})
\]

\[
\ell^x(B, \cdot) = \ell^0(B - x, \cdot).
\]
Palm measures and local time

Recall that the Palm measure $Q_\xi$ of an invariant random measure $\xi$ with respect to a stationary $\mathbb{P}$ is defined by

$$Q_\xi(A) := \mathbb{E} \int 1_{[0,1]}(s)1_A(\theta_s B) \xi(ds), \quad A \in \mathcal{A}.$$ 

Theorem (Geman and Horowitz ’73)

$\mathbb{P}_x$ is the Palm probability measure of the local time $\ell^x$

Let $\nu$ be a probability measure and put

$$\mathbb{P}_\nu = \int \mathbb{P}_x \nu(dx) \quad \text{and} \quad \ell^\nu = \int \ell^x \nu(dx).$$

Corollary

$\mathbb{P}_\nu$ is the Palm probability measure of $\ell^\nu$
General Palm measures and allocation rules

On this slide we can allow a general setting with a stationary $\mathbb{P}$. In fact $B$ can be a spatial random field, or acted on by a group.

Define the allocation rule associated with a random time $T$ by

$$\tau_T : s \rightarrow T \circ \theta_s + s.$$ 

An allocation rule $\tau$ balances two random measures $\xi$ and $\eta$ if

$$\xi \{ s : \tau(s) \in \cdot \} = \eta \quad \mathbb{P}\text{-a.e.}$$

**Theorem (Last and Thorisson '09 – general stationary setting)**

Let $\xi$ and $\eta$ be invariant random measures with positive and finite intensities. Then $\xi$ and $\eta$ have the same intensity and

$$\mathbb{Q}_{\xi}(\theta_T B \in \cdot) = \mathbb{Q}_{\eta} \quad \text{(shift-coupling of Palm versions)}$$

if and only if $\tau_T$ balances $\xi$ and $\eta$. 

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**Hermann Thorisson**

Unbiased shifts of Brownian motion
Consider again the Brownian motion $B$ under $P_0$.

**Definition (Skorokhod embedding problem)**

A random time $T$ *embeds* $\nu$ if $B_T$ has distribution $\nu$.

Until now this has only been considered in one-sided time and usually $T$ is assumed to be a stopping time.

Here is a key characterization result in two-sided time:

**Theorem**

$T$ is an unbiased shift embedding $\nu$ $\iff$ $\tau_T$ balances $\ell^0$ and $\ell^\nu$.

**Proof:** $T$ is unbiased shift embedding $\nu$ $\iff$ $P_0(\theta_T B \in \cdot) = P_\nu$. Since $P_0$ and $P_\nu$ are Palm probability measures of $\ell^0$ and $\ell^\nu$, the shift-coupling theorem (previous slide) yields the theorem.
Existence of unbiased shifts

**Theorem**

*For each* $r \in \mathbb{R}$, *the time* $T_r$ *imbeds* $\delta_0$ *and is an unbiased shift.*

**Proof:** For $r \geq 0$, the allocation rule associated with $T_r$,

$$\tau_{T_r}(s) = \sup \{ t > s : \ell^0([s, t]) = r \}, \quad s \in \mathbb{R},$$

balances $\ell^0$ and $\ell^0$ for $r > 0$. Similarly for $r < 0$.

**Theorem**

*Let* $\nu$ *be a probability measure on* $\mathbb{R}$ *with* $\nu\{0\} = 0$. *Then*

$$T^\nu := \inf \{ t > 0 : \ell^0([0, t]) = \ell^\nu([0, t]) \}$$

*embeds* $\nu$ *and is an unbiased shift.*

**Remark**

This stopping time $T^\nu$ was introduced in Bertoin and Le Jan ’92 as a solution of the Skorokhod embedding problem. The unbiasedness seems to be a new observation.
The crucial ingredient in the proof

This theorem:

Theorem (from previous slide)

Let \( \nu \) be a probability measure on \( \mathbb{R} \) with \( \nu\{0\} = 0 \). Then

\[
T^\nu := \inf\{ t > 0 : \ell^0([0, t]) = \ell^\nu([0, t]) \}
\]

embeds \( \nu \) and is an unbiased shift.

follows from the next one by applying joint ergodicity of the \( \mathbb{P}_x \) and the balancing characterization of unbiased shifts.

Theorem (holds in a general stationary setting on \( \mathbb{R} \))

Let \( \xi \) and \( \eta \) be invariant diffuse and orthogonal random measures on \( \mathbb{R} \) with the same conditional intensity given the invariant \( \sigma \)-algebra. Then the allocation rule \( \tau \) defined by

\[
\tau(s) := \inf\{ t > s : \xi([s, t]) = \eta([s, t]) \}, \quad s \in \mathbb{R},
\]

balances \( \xi \) and \( \eta \).
Existence of unbiased shifts – one more theorem

Theorem

For each $r \in \mathbb{R}$, the time $T_r$ imbeds $\delta_0$ and is an unbiased shift.

Theorem

Let $\nu$ be a probability measure on $\mathbb{R}$ with $\nu\{0\} = 0$. Then

$$T^\nu := \inf\{ t > 0 : \ell^0([0, t]) = \ell^\nu([0, t]) \}$$

embeds $\nu$ and is an unbiased shift.

Theorem

$\forall \nu \exists$ stopping time $T \geq 0$ which is unbiased shift embedding $\nu$.

$\nu\{0\} < 1 \Rightarrow$ all unbiased $T$ embedding $\nu$ satisfy $\mathbb{P}_0(T = 0) = 0$.

$\nu = \delta_0 \Rightarrow \forall p \in [0, 1] \exists$ an unbiased shift $T \geq 0$ embedding $\nu$
and satisfying $\mathbb{P}_0(T = 0) = p$.
Moment properties

**Theorem**

Suppose $\nu\{0\} = 0$ and the stopping time $T \geq 0$ is an unbiased shift embedding $\nu$. Then

$$\mathbb{E}_0 T^{1/4} = \infty.$$ 

If additionally $\int |x| \nu(dx) < \infty$ and

$$T = T^\nu = \inf\{t > 0 : \ell^0([0, t]) = \ell^x([0, t])\}.$$

Then

$$\forall \beta \in [0, 1/4) : \mathbb{E}_0 T^\beta < \infty.$$ 

**Theorem**

If $T$ is an unbiased shift embedding $\nu \neq \delta_0$, then $\mathbb{E}_0 \sqrt{|T|} = \infty$.

Consider the one-sided stable matching $\tau$ between independent Poisson processes $\xi$ and $\eta$. Then $\mathbb{E}_{\mathbb{Q}_\xi} \sqrt{\tau(0)} = \infty$. The stable marriage of Lebesgue and Poisson (Holroyd and Peres ’05) has also this property.
Theorem

Suppose $\nu\{0\} = 0$ and the stopping time $T \geq 0$ is an unbiased shift embedding $\nu$. Then

$$\mathbb{E}_0 T^{1/4} = \infty.$$  

If additionally $\int |x| \nu(dx) < \infty$ and

$$T = T^\nu = \inf\{ t > 0 : \ell^0([0, t]) = \ell^x([0, t])\}.$$  

Then

$$\forall \beta \in [0, 1/4) : \quad \mathbb{E}_0 T^\beta < \infty.$$  

Theorem

If $T$ is an unbiased shift embedding $\nu \neq \delta_0$, then $\mathbb{E}_0 \sqrt{|T|} = \infty$.

Theorem

Let $T$ be a nontrivial unbiased shift embedding $\delta_0$. Then it is possible that $\mathbb{E} e^{\lambda |T|} < \infty$ for some $\lambda > 0$.

But if $T \geq 0$ and $\mathbb{P}_0(T > 0) > 0$ then $\mathbb{E}_0 T = \infty$. 

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Unbiased shifts of Brownian motion
### Definition

An unbiased shift $T \geq 0$ embedding $\nu$ is called **minimal** if for each unbiased shift $S$ embedding $\nu$ and satisfying $\mathbb{P}_0(0 \leq S \leq T) = 1$ we have that $\mathbb{P}_0(S = T) = 1$.

### Theorem

*If* $\nu\{0\} = 0$ *then the Bertoin-Le Jan stopping time $T^\nu$ is a minimal unbiased shift.*
Levy processes

The central results extend to recurrent Levy processes under mild regularity conditions:

**Theorem**

$T$ is an unbiased shift embedding $\nu \iff \tau_T$ balances $\ell^0$ and $\ell^\nu$

**Theorem**

For each $r \in \mathbb{R}$, the time $T_r$ imbeds $\delta_0$ and is an unbiased shift.

**Theorem**

Let $\nu$ be a probability measure on $\mathbb{R}$ with $\nu\{0\} = 0$. Then

$$T^\nu := \inf\{t > 0 : \ell^0([0, t]) = \ell^\nu([0, t])\}$$

embeds $\nu$ and is an unbiased shift.
References


