Invariant transports of stationary random measures

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1. The Monge-Kantorovich problem

Setting

Let $\xi$ and $\eta$ be measures on $\mathbb{R}^d$ such that

$$0 < \xi(\mathbb{R}^d) = \eta(\mathbb{R}^d) < \infty.$$ 

Let $c(x, y)$ be the cost of transporting one unit of mass from $x \in \mathbb{R}^d$ to $y \in \mathbb{R}^d$. 

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Problem (Monge 1781)

Minimize

\[ \int c(x, \tau(x)) \xi(dx) \]

among all transport maps \( \tau : \mathbb{R}^d \to \mathbb{R}^d \) satisfying \( \tau^*(\xi) = \eta \), that is

\[ \int 1\{\tau(x) \in B\} \xi(dx) = \eta(B), \quad B \subset \mathbb{R}^d. \]

Such a \( \tau \) is called admissible.
Remark
If $\xi$ and $\eta$ have the same number of atoms of equal size, the Monge Problem corresponds to optimal matching.

Remark
Admissible transports need not exist, for instance if $\xi$ and $\eta$ have atoms of different sizes.

Remark
If $\xi$ and $\eta$ are absolutely continuous and $c(x, y) = \|x - y\|^p$ for some $p > 1$ then (under moment assumptions on $\xi$ and $\eta$) there is a unique solution of the Monge problem.
**Definition (Coupling)**

Let $\Pi(\xi, \eta)$ denote the set of all (finite) measures $\pi$ on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\pi(\cdot \times \mathbb{R}^d) = \xi$ and $\pi(\mathbb{R}^d \times \cdot) = \eta$. Any such $\pi$ is called a **coupling** of $\xi$ and $\eta$.

**Problem (Kantorovich 1940)**

Minimize

$$\int c(x, y) \pi(d(x, y))$$

among all $\pi \in \Pi(\xi, \eta)$. 
Remark

Any $\pi \in \Pi(\xi, \eta)$ can be identified with a stochastic kernel $T(x, dy)$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that

$$\int T(x, B)\xi(dx) = \eta(B), \quad B \subset \mathbb{R}^d.$$  

Such a $T$ is called transport kernel.

Remark

If the costs are finite for some transport kernel, then there exists a solution to the Monge-Kantorovich problem.
2. Invariant random measures

Setting

$G$ denotes a (multiplicative) LCSC group with Borel $\sigma$-field $\mathcal{G}$, neutral element $e$, Haar measure $\lambda$, and modular function $\Delta$.

Definition

(i) Let $\mathcal{M}$ denote the space of all locally finite measures on $G$.

(ii) The $\sigma$-field $\mathcal{M}$ is the smallest $\sigma$-field of subsets of $\mathcal{M}$ making the mappings $\mu \mapsto \mu(B)$ for all Borel sets $B \in \mathcal{G}$ measurable.

(iii) A random measure $\xi$ on $G$ is a measurable mapping $\xi : \Omega \to \mathcal{M}$, where $(\Omega, \mathcal{A}, \mathbb{P})$ is a given $\sigma$-finite measure space.
Consider measurable mappings $\theta_g : \Omega \to \Omega$, $g \in G$, satisfying $\theta_e = \text{id}_\Omega$ and the flow property

$$
\theta_g \circ \theta_h = \theta_{gh}, \quad g, h \in G
$$

The mapping $(\omega, g) \mapsto \theta_g \omega$ is assumed measurable. The measure $\mathbb{P}$ is assumed stationary under the flow, that is

$$
\mathbb{P} \circ \theta_g = \mathbb{P}, \quad g \in G.
$$
Definition

A random measure $\xi$ is **invariant** if

$$\xi(\theta_g \omega, gB) = \xi(\omega, B), \quad \omega \in \Omega, \ g \in G, \ B \in \mathcal{G}.$$ 

Definition

Let $w : G \to \mathbb{R}_+$ be a measurable with $\int w(g) \lambda(dg) = 1$. Let $\xi$ be an invariant random measure on $G$. The measure

$$\mathbb{P}_\xi(A) := \mathbb{E}_{\mathbb{P}} \int 1\{\theta_g^{-1} \in A\} w(g) \xi(dg), \quad A \in \mathcal{A},$$

is called the **Palm measure** of $\xi$. 
3. Transport properties of Palm measures

Definition

A measurable mapping $\tau : \Omega \times G \to G$ is called allocation if

$$\tau(\theta_g \omega, gh) = g \tau(\omega, h), \quad \omega \in \Omega, \; g, h \in G.$$ 

Definition

An allocation balances two random measures $\xi$ and $\eta$ if $\mathbb{P}$-a.e.

$$\int 1\{\tau(g) \in \cdot\} \xi(dg) = \eta(\cdot)$$
Theorem (Mecke ’75, Holroyd and Peres ’05, L. and Thorisson ’09, L. 10)

Consider two invariant random measures $\xi$ and $\eta$ and let $\tau$ be an allocation. Then $\tau$ balances $\xi$ and $\eta$ iff

$$
\mathbb{E}_{\mathbb{P}_\xi} f(\theta^{-1}_{\tau(e)})\Delta(\tau(e)^{-1}) = \mathbb{E}_{\mathbb{P}_\eta} f,
$$

for all measurable $f : \Omega \to \mathbb{R}^+$. In particular, if $G$ is unimodular, this is equivalent with

$$
\mathbb{P}_\xi (\theta^{-1}_{\tau(e)} \in A) = \mathbb{P}_\eta (A), \quad A \in \mathcal{A}.
$$
Definition

A transport-kernel is a kernel $T$ from $\Omega \times G$ to $G$ such that $T(\omega, x, \cdot)$ is a locally finite measure for all $(\omega, g) \in \Omega \times G$ which is invariant, that is

$$T(\theta_g \omega, gh, gB) = T(\omega, h, B), \quad g, h \in G, \omega \in \Omega, B \in \mathcal{B}(G).$$

Definition

Let $\xi$ and $\eta$ be random measures. A transport kernel balances $\xi$ and $\eta$ if

$$\int T(\omega, x, \cdot) \xi(\omega, dx) = \eta(\omega, \cdot) \quad \mathbb{P}\text{-a.e. } \omega \in \Omega.$$
Consider two invariant random measures $\xi$ and $\eta$ and let $T$ be a transport kernel. Then $T$ is balances $\xi$ and $\eta$ iff

$$
\mathbb{E}_P \xi \int f(\theta_g^{-1}) \Delta(g^{-1}) T(e, dg) = \mathbb{E}_P \eta f,
$$

for all measurable $f : \Omega \rightarrow \mathbb{R}_+$.  


Theorem (L. and Thorisson '09)

Consider two invariant random measures $\xi$ and $\eta$ and let $T$ and $T^*$ be transport-kernels satisfying

$$
\int \int \mathbf{1}\{(g, h) \in \cdot\} T(g, dh) \xi(dg) = \int \int \mathbf{1}\{(g, h) \in \cdot\} T^*(h, dg) \eta(dh)
$$

$\mathbb{P}$-a.e. Then we have for any measurable function $f : \Omega \times G \rightarrow \mathbb{R}_+$ that

$$
\mathbb{E}_{\mathbb{P}_\xi} \int f(\theta_g^{-1}, g^{-1}) \Delta(g^{-1}) T(e, dg) = \mathbb{E}_{\mathbb{P}_\eta} \int f(\theta_e, g) T^*(e, dg).
$$
Corollary (Neveu ’77)

Let $\xi, \eta$ be invariant random measure on $G$. Then we have for any measurable function $f : \Omega \times G \to \mathbb{R}_+$ that

$$
\mathbb{E}_{P_\xi} \int f(\theta_{g^{-1}}, g^{-1}) \Delta(g^{-1}) \eta(dg) = \mathbb{E}_{P_\eta} \int f(\theta_e, g) \xi(dg).
$$

Corollary (mass transport principle)

Let $t : \Omega \times G \times G \to \mathbb{R}_+$ be measurable and invariant. Then

$$
\mathbb{E} \iint 1\{g \in B\} t(h, g) \Delta(g^{-1}) \Delta(h) \eta(dh) \xi(dg)
$$

$$
= \mathbb{E} \iint 1\{g \in B\} t(g, h) \eta(dg) \xi(dh),
$$

for any $B \in \mathcal{G}$ with positive and finite Haar measure.
Definition

The **intensity** of an invariant random measure $\xi$ is the number

$$\mathbb{E} \int w(g) \xi(dg),$$

where $\int w \, d\lambda = 1$.

Definition

The **invariant** $\sigma$-field $\mathcal{I} \subset \mathcal{A}$ is the class of all sets $A \in \mathcal{A}$ satisfying $\theta_g A = A$ for all $g \in G$. 
Theorem

Suppose that $\xi$ and $\eta$ are invariant random measures with positive and finite intensities. Then there exists a transport-kernel balancing $\xi$ and $\eta$ and satisfying

$$\int \Delta(g^{-1}) T(e, dg) = 1$$

iff

$$\mathbb{E}[\xi(B) | \mathcal{I}] = \mathbb{E}[\eta(B) | \mathcal{I}] \quad \mathbb{P}\text{-a.e.}$$

for some $B \in \mathcal{B}(G)$ satisfying $0 < \lambda(B) < \infty$. 
6. The Mecke characterization

Theorem (Mecke ’67, Rother and Zähle ’90)

Let \( \xi \) be an invariant random measure and \( \mathcal{Q} \) a measure on \( (\Omega, \mathcal{A}) \). The measure \( \mathcal{Q} \) is a Palm measure of \( \xi \) with respect to some \( \sigma \)-finite stationary measure iff \( \mathcal{Q} \) is \( \sigma \)-finite,

\[
\mathcal{Q}\{\xi(G) = 0\} = 0,
\]

and

\[
\mathbb{E}_\mathcal{Q} \int f(\theta_g^{-1}, g^{-1}) \Delta(g^{-1}) \xi(dg) = \mathbb{E}_\mathcal{Q} \int f(\theta_e, g) \xi(dg)
\]

holds for all measurable \( f : \Omega \times G \to \mathbb{R}_+ \).
Remark

Many of the previous results can be extended to the case of invariant random measures on some state spaces $S$ and $T$, on which acts $G$ in a proper way. See Kallenberg (2007,2011) and Gentner and Last (2011).
8. References


