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Asymptotics of the intersection intensity of two independent renewal processes: a note on Lemma 3c in [BGKS]

The aim of this little note is to clarify one more nice connection between the papers [BGKS] and [HS], beyond those that are already discussed in [BGKS]. This concerns one single (part of a) lemma in [BGKS], for whose proof the authors of [BGKS] refer to [HS], apparently with a typo in the number of the cited lemma. Recently we were able to find the desired proof "in and between the lines" of [HS], at least under a (generic) additional assumption, namely the ultimate monotonicity of the right hand side of (5). Already before this, we had found an independent proof that works under an even less restrictive assumption, see (16) below. While the proof based on [HS] works with arguments based on Fourier analysis and Tauberian arguments, our alternative proof, making use of a result of [D] on the asymptotics of (2), plugs this asymptotics in and proceeds fairly directly to the analysis of a Riemann sum.

In passing, we could repair misprints of the constants in (5), (6) and (16) as they appear in [BGKS], [HS] and [D], respectively. The fact that these constants in their various forms fit together so nicely is a consequence not least of Euler's beautiful reflection formula (14).

Here is the setting: Let $\alpha \in (0, \frac{1}{2})$, and let μ be a probability distribution on $\mathbb{N}$ obeying

$$\mu(\{n, n+1, \dots\} \sim n^{-\alpha} L(n) \tag{1}$$

for some slowly varying function L. A renewal process with interarrival time distribution μ will be called a *μ-renewal process* for short. Specifically, we consider an i.i.d. sequence $R_1, R_2, \dots$ of random variables with distribution μ and put

$$q_n := \mathbf{P}(\exists j \geq 0 : R_1 + \dots + R_j = n) \tag{2}$$

i.e. $q = (q_n)_{n \geq 0}$ is the (sequence of weights of the) intensity measure of the μ-renewal point process started in 0. We write

$$\mathcal{R} := \{n \in \mathbb{N} : R_1 + \dots + R_j = n \text{ for some } j \geq 0\} \tag{3}$$

for the support of this process, i.e. the set of renewal points. Let $\mathcal{R}^{(m)}$, $m \in \mathbb{Z}$, be i.i.d. copies of $\mathcal{R}$, and put $\mathcal{R}_m := m + \mathcal{R}^{(m)}$. Hence, $\mathcal{R}_m$ is the support of of a μ-renewal process started in m, with all these processes being independent.

For $m > 0$ and $k \in \mathbb{Z}$,

$$\mathbf{E}[|\mathcal{R}_k \cap \mathcal{R}_{k+m}|] = \mathbf{E}[|\mathcal{R}_0 \cap \mathcal{R}_m|] = \sum_{j=0}^{\infty} q_j q_{j-m} = \sum_{j=0}^{\infty} q_j q_{j+m}. \tag{4}$$

We refer to (4) as the *intersection intensity* of two independent μ-renewal processes started distance m apart.

Lemma 3c in [BGKS] states that

$$\sum_{j=0}^{\infty} q_j q_{j-m} \sim \frac{(1-\alpha)^2}{\Gamma(2-\alpha)^2\Gamma(2\alpha)} m^{2\alpha-1} L(m) \quad \text{as } m \to \infty \tag{5}$$

For the proof of this result, the authors refer to Lemma 5.1 in [HS]. While such a direct reference seems to be missing in the published version of [HS], a trace that comes close is the proof of Lemma 3.1 in [HS], in which the following asymptotics is proved:

$$\sum_{m=1}^{n} \sum_{j=0}^{\infty} q_j q_{j+m} \sim \frac{1}{4\alpha\cos(\pi\alpha)\Gamma(1-\alpha)^2\Gamma(2\alpha)} \frac{n^{2\alpha}}{L(n)^2} \tag{6}$$

Note that the prefactor that is stated in the formula in [HS], top of p. 706, has a 2α instead of 4α, and also has $\Gamma(1-2\alpha)^2$ instead of $\Gamma(1-\alpha)^2$. These typos are corrected in [I], where the proof from [HS] is reproduced in detail. Briefly stated, the proof of (6) in [HS] consists in recognizing

$$(|Q|^2)_m := \sum_{j=0}^{\infty} q_j q_{j+m} \tag{7}$$

as the coefficients in the Fourier series

$$|Q(t)|^2 = \sum_{m \in \mathbb{Z}} (|Q|^2)_m \, e^{imt}, \tag{8}$$

with

$$Q(t) := \sum_{k \in \mathbb{Z}} q_k \, e^{ikt}, \tag{9}$$

and $q_k := 0$ for $k < 0$. With

$$p_n := \mu(\{n\}), \quad n \in \mathbb{Z}, \tag{10}$$

a "first jump decomposition" for R translates into the convolution relation

$$q_m := \sum_{n=1}^{m} p_n q_{m-n} \tag{11}$$

and with

$$P(t) := \sum_{m\in\mathbb{Z}} p_m \, e^{imt} \tag{12}$$

the relation (11) together with $q_0 = 1$ turns into $Q(t) = (1 - P(t))^{-1}$. The asymptotics (6) then follows from known behaviour of the Fourier transform (12) near $t = 0$ (based on the assumption (1)) and a Tauberian argument ([BGT], Theorem 4.10.1 (a)).

If $n^{2\alpha-1}L(n)$ is ultimately monotone, then (11) together with [F], Ch. XIII.5, Theorem 5, implies

$$\sum_{j=0}^{\infty} q_j q_{j+m} \sim \frac{1}{2\cos(\pi\alpha)\Gamma(1-\alpha)^2\Gamma(2\alpha)} \frac{m^{2\alpha-1}}{L(m)^2}. \tag{13}$$

By Euler's reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\sin(\pi x)}{x} \tag{14}$$

and the relation $2\cos(\pi\alpha) = \sin(2\pi\alpha)/\sin(\pi\alpha)$, the asymptotics (13) becomes

$$\sum_{j=0}^{\infty} q_j q_{j+m} \sim \frac{\Gamma(1-2\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)^3} \frac{m^{2\alpha-1}}{L(m)^2} \tag{15}$$

We are going to give an alternative proof of (15) which works without the assumption of ultimate monotonicity of $n^{-\alpha}L(n)$ but instead uses the weaker assumption

$$\sup_{n\geq 1} \frac{n\mathbf{P}(R_1 = n)}{\mathbf{P}(R_1 > n)} < \infty. \tag{16}$$

Indeed, Theorem B in [D] states that (1) together with the assumption (16) implies[1]

$$q_n \sim \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} n^{\alpha-1} \frac{1}{L(n)} \tag{17}$$

[1]Note that in [D] Theorem B the prefactor is erroneously printed as $\Gamma(1-\alpha)/\Gamma(\alpha)$, but a look e.g. into [GL] shows the correct prefactor $\frac{\pi}{\sin(\pi\alpha)} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$.

Proposition 0.1 *Let* $\alpha \in (0, \frac{1}{2})$ *and assume* (17). *Then, as* $m \to \infty$,

$$\sum_{j \geq 1} q_j q_{m+j} \sim C^2 \tilde{L}(m)^2 m^{2\alpha - 1} \int_0^\infty (1+x)^{\alpha-1} x^{\alpha-1} \mathrm{d}x \tag{18}$$

for $C := \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$ *and* $\tilde{L}(m) := (L(m))^{-1}$.

Before turning to the proof of Proposition 0.1, let us note that the right hand sides of (18) and (15) are equal. Indeed, the substitution $x = \frac{t}{1+t}$, $\mathrm{d}x = \frac{\mathrm{d}t}{(1-x)^2}$, gives

$$\int_0^\infty (1+x)^{\alpha-1} x^{\alpha-1} \mathrm{d}x = B(\alpha, 1-2\alpha) = \frac{\Gamma(\alpha)\Gamma(1-2\alpha)}{\Gamma(1-\alpha)}. \tag{19}$$

We will prove (18) first under a special assumption on the Karamata representation of the slowly varying function L.

Lemma 0.2 *Let* $\alpha \in (0, \frac{1}{2})$, *and assume*

$$r_n = n^{\alpha-1} K(n) \tag{20}$$

where $K(n)$ *is of the form*

$$K(n) = \exp\left(\int_B^n \frac{l(t)}{t} \, \mathrm{d}t \right) \tag{21}$$

with B *a positive constant and* $l(t)$, $t \geq B$, *a bounded measurable function converging to* 0 *as* $t \to \infty$. *Then* (r_n) *is ultimately decreasing, and*

$$\sum_{j \geq 1} r_j r_{i+j} \sim K(i)^2 i^{2\alpha-1} \int_0^\infty (1+x)^{\alpha-1} x^{\alpha-1} \mathrm{d}x. \tag{22}$$

Proof. a) The fact that (r_n) is ultimately decreasing follows from (20) together with the Karamata representation (21) of $K(n)$: We first set $\beta := 1 - \alpha$. Now since $l(t)$ tends to zero for $t \to \infty$ we know that there exists $n_0 \in \mathbb{N}$ such that for alle $t \geq n_0$ one has $l(t) < \beta$. This implies

$$\begin{aligned}
\frac{r_n}{r_{n+1}} &= \frac{n^{\alpha-1}}{(n+1)^{\alpha-1}} \cdot \frac{K(n)}{K(n+1)} \\
&= \frac{n^{\alpha-1}}{(n+1)^{\alpha-1}} \exp\left(- \int_n^{n+1} \frac{l(t)}{t} \mathrm{d}t \right) \\
&= \exp\left(\beta \left(\ln(n+1) - \ln(n) \right) - \int_n^{n+1} \frac{l(t)}{t} \mathrm{d}t \right) \\
&= \exp\left(\int_n^{n+1} \left(\frac{\beta}{t} - \frac{l(t)}{t} \right) \mathrm{d}t \right).
\end{aligned}$$

Since by assumption the integrand on the r.h.s. is strictly positive for $n \geq n_0$, we obtain that $(r_n)_n$ is decreasing for $n \geq n_0$.

b) In view of (20), the claimed asymptotics (22) is equivalent to

$$\frac{1}{i} \sum_{j \geq 1} \frac{r_j}{r_i} \frac{r_{i+j}}{r_i} \to \int_0^\infty (1+x)^{\alpha-1} x^{\alpha-1} \mathrm{d}x. \tag{23}$$

We now set out to prove (23). To this purpose we show first that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all sufficiently large i

$$\frac{1}{i} \sum_{j \geq Ni} \frac{r_j}{r_i} \frac{r_{i+j}}{r_i} < \varepsilon. \tag{24}$$

Since (r_n) is ultimately decreasing, (24) will follow if we can show that exists an $N \in \mathbb{N}$ such that for all sufficiently large i

$$\frac{1}{i} \sum_{j \geq Ni} \left(\frac{r_j}{r_i} \right)^2 < \varepsilon. \tag{25}$$

Again because of the ultimate monotonicity of (r_n), the l.h.s. of (25) is for sufficiently large i bounded from above by

$$\sum_{m=N}^{\infty} \left(\frac{r_{mi}}{r_i} \right)^2 = \sum_{m=N}^{\infty} m^{2\alpha-2} \left(\frac{K(mi)}{K(i)} \right)^2. \tag{26}$$

Using (21) one obtains that for any $\delta > 0$ and i so large that $l(t) < \delta$ for all $t \geq i$,

$$\frac{K(mi)}{K(i)} = \exp\left(\int_i^{mi} \frac{l(t)}{t} \mathrm{d}t \right) \leq \exp\left(\delta \left(\ln(mi) - \ln(i) \right) \right) \leq m^\delta, \tag{27}$$

which implies by dominated convergence that for sufficiently large N the r.h.s. of (26) is smaller than ε for all sufficienly large i. We have thus proved (24).

Next we show that for all $\varepsilon > 0$ there exists an $\eta \in \mathbb{N}$ such that for all sufficiently large i

$$\frac{1}{i} \sum_{j \leq \eta i} \frac{r_j}{r_i} \frac{r_{i+j}}{r_i} < \varepsilon. \tag{28}$$

Again by ultimate monotonicity of (r_n) which gives us $\frac{r_{i+j}}{r_i} \leq 1$ for i large enough, for this it suffices to show that for all $\varepsilon > 0$ there exists an $\eta > 0 \in \mathbb{N}$

such that for all sufficiently large i

$$\frac{1}{i}\sum_{j\le \eta i}\frac{r_j}{r_i} < \varepsilon. \tag{29}$$

From Feller Vol 2, Theorem 5 on p. 447 we obtain that

$$\sum_{j\le \eta i} r_j \sim \frac{1}{\alpha}(\eta i)^\alpha K(\lfloor \eta i\rfloor), \tag{30}$$

and hence

$$\frac{1}{i}\sum_{j\le \eta i}\frac{r_j}{r_i} \sim \frac{1}{\alpha}\eta^\alpha \frac{K(\lfloor \eta i\rfloor)}{K(i)} \sim \frac{\eta^\alpha}{\alpha}, \tag{31}$$

which proves (29), and hence also (28).(The last asymptotic is by the fact that K is slowly varying.)

In view of (20), (24) and (28), for proving (22) it remains to show that

$$\frac{1}{i}\sum_{\eta i\le j\le Ni}\frac{K(j)}{K(i)}\frac{K(i+j)}{K(i)}\left(1+\frac{j}{i}\right)^{\alpha-1}\left(\frac{j}{i}\right)^{\alpha-1} \to \int_\eta^N (1+x)^{\alpha-1}x^{\alpha-1}\mathrm{d}x. \tag{32}$$

From (21) one derives that

$$\lim_{i\to\infty}\sup_{\eta i\le j\le (N+1)i}\left|\frac{K(j)}{K(i)}-1\right| = 0. \tag{33}$$

Hence (32) boils down to a convergence of Riemann sums to its integral limit. □

Let us now complete the proof of the Proposition. The assumption (17) can be rewritten as

$$q_n = C_n n^{\alpha-1}\tilde{L}(n) \tag{34}$$

where $\tilde{L}(n)$ is a slowly varying function and $C_n \to C > 0$. As in the proof of the Lemma, it suffices to show that

$$\frac{1}{i}\sum_{j\ge 1}\frac{q_j}{q_i}\frac{q_{i+j}}{q_i} \to \int_0^\infty (1+x)^{\alpha-1}x^{\alpha-1}\mathrm{d}x. \tag{35}$$

Because of the Karamata representation theorem (see e.g. Theorem 1.3.1 in [BGT]) there exists a $K(n)$ satisfying (21) and a sequence D_n converging to a positive constant D such that

$$\tilde{L}(n) = D_n K(n), \quad n = 1, 2, \ldots \tag{36}$$

Defining r_n as in (20) we have

$$q_n = D_n C_n r_n, \quad n = 1, 2, \ldots \tag{37}$$

Since the asymptotics of neither the l.h.s. of (23) nor that of the l.h.s. of (35) reacts to the omission of a fixed finite number of summands, we see that (23) carries over to (35). $\square$

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