New stability results for inverse coefficient problems

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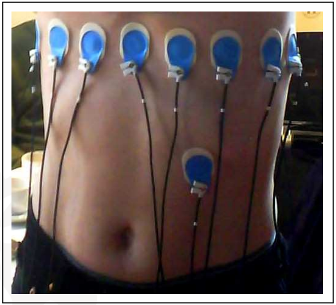
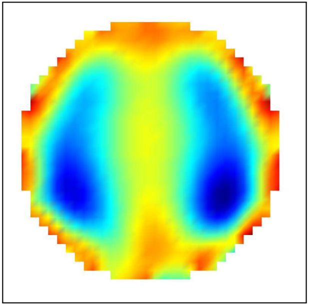
Electrical impedance tomography





- Apply electric currents on subject's boundary
- Measure necessary voltages

⤳ Reconstruct conductivity inside subject

Calderón problem

Can we recover $\sigma \in L^\infty_+(\Omega)$ in

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega \subset \mathbb{R}^d \qquad (1)$$

from all possible Dirichlet and Neumann boundary values

$$\{(u|_{\partial\Omega}, \sigma \partial_\nu u|_{\partial\Omega}) \quad : \quad u \text{ solves } (1)\} \,?$$

Equivalent: Recover σ from **Neumann-to-Dirichlet-Operator**

$$\Lambda(\sigma): \; L^2_\diamond(\partial\Omega) \to L^2_\diamond(\partial\Omega), \quad g \mapsto u|_{\partial\Omega},$$

where u solves (1) with $\sigma \partial_\nu u|_{\partial\Omega} = g$.

Challenges in idealized EIT

Mathematical idealization of EIT $\rightsquigarrow$ Calderón problem

- infinitely many unknowns $\sigma \in L^\infty_+(\Omega)$
- infinitely many measurements $\Lambda(\sigma) \in \mathcal{L}(L^2_\diamond(\partial\Omega))$
- nonlinear forward map $\sigma \mapsto \Lambda(\sigma)$

Mathematical challenges

- Uniqueness? Does $\Lambda(\sigma)$ determine σ?
- Stability? $\Lambda^{-1}: \Lambda(\sigma) \mapsto \sigma$ continuous?
- Convergence (local/global)? How to determine σ from $\Lambda(\sigma)$?

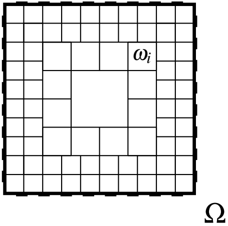
Consequences for practical EIT?

EIT in practice

In practice

- finitely many unknowns, σ pcw. const. on given resolution $\Omega = \bigcup_{i=1}^{m} \omega_i$
- finitely many measurements

⇝ Finite-dimensional inverse problem



Model for finitely many measurements:

- Galerkin projection $P_{G_n} \Lambda(\sigma) P_{G_n}$. P_{G_n}: orthoprojection to

$$G_1 \subseteq G_2 \subseteq \ldots \subseteq L^2_\diamond(\partial\Omega), \quad \overline{\bigcup_{n\in\mathbb{N}} G_n} = L^2_\diamond(\partial\Omega)$$

- Better: use realistic electrode model

Theoretical challenges in practical EIT

For a fixed desired resolution:

- Do finitely many measurements uniquely determine σ?

  (...and how many measurements/electrodes do we need?)
- Is the resulting finite-dimensional inverse problem stable?

  (...and how large is the stability constant / noise amplification?)
- Do the results hold for realistic electrode models?

  (...and how can we derive globally convergent reconstruction algorithms?)

This talk: Affirmative answer to these challenges

(...and some handwaving comments)

Uniqueness and stability

Given desired resolution $\Omega = \bigcup_{i=1}^{m} \omega_i$, $b > a > 0$

$$\mathcal{F}_{[a,b]} = \{\sigma \in L_+^\infty(\Omega) : \ a \leq \sigma(x) \leq b, \ \sigma \text{ pcw. const. on } \Omega\}$$

and subspaces

$$G_1 \subseteq G_2 \subseteq \ldots \subseteq L^2_\diamond(\partial\Omega), \quad \overline{\bigcup_{n\in\mathbb{N}} G_n} = L^2_\diamond(\partial\Omega).$$

Theorem. (**H.**, IP 2019) There exists $N \in \mathbb{N}$ and $c > 0$:

$$\|P_{G_n}(\Lambda(\sigma_1) - \Lambda(\sigma_2))P_{G_n}\| \geq c\,\|\sigma_1 - \sigma_2\| \quad \forall \sigma_1, \sigma_2 \in \mathcal{F}_{[a,b]}, \ n \geq N.$$

Finitely many measurement uniquely determine σ at a given resolution if enough measurements are being used

Main tools for proof

Monotonicity lemma. (Kang/Seo/Sheen 1997, Ikehata 1998)

For all $\sigma_1, \sigma_2 \in L^\infty_+(\Omega)$, $g \in L^2_\diamond(\partial\Omega)$

$$\int_{\partial\Omega} g(\Lambda(\sigma_1) - \Lambda(\sigma_2))g \, \mathrm{d}s \geq \int_{\partial\Omega} g\Lambda'(\sigma_2)(\sigma_1 - \sigma_2)g \, \mathrm{d}s$$
$$= \int_\Omega (\sigma_2 - \sigma_1)|\nabla u^g_{\sigma_2}|^2 \, \mathrm{d}x$$

Localized potentials lemma. (**H.** 2008, **H.**/Ullrich 2013)

For pcw. anal. $\sigma \in L^\infty_+(\Omega)$, measurable $D_1, D_2 \subseteq \overline{\Omega}$, $\operatorname{int} D_1 \not\subseteq \operatorname{out}_{\partial\Omega} D_2$

$$\exists (g_k)_{k\in\mathbb{N}} \in L^2_\diamond(\partial\Omega): \int_{D_1} |\nabla u^{g_k}_\sigma|^2 \, \mathrm{d}x \to \infty, \quad \int_{D_2} |\nabla u^{g_k}_\sigma|^2 \, \mathrm{d}x \to 0.$$

(Closed outer hull $\operatorname{out}_{\partial\Omega} D_2$: complement of all open sets connected to $\partial\Omega$ not intersecting D_2)

Sketch of proof 1/2

By self-adjointness

$$\|\Lambda(\sigma_1)-\Lambda(\sigma_2)\| = \sup_{\|g\|=1} \left| \int_{\partial\Omega} g(\Lambda(\sigma_1)-\Lambda(\sigma_2))g \, \mathrm{d}s \right|$$

By monotonicity

$$\begin{aligned}
&\left| \int_{\partial\Omega} g(\Lambda(\sigma_1)-\Lambda(\sigma_2))g \, \mathrm{d}s \right| \\
&= \max\left\{ \int_{\partial\Omega} g(\Lambda(\sigma_1)-\Lambda(\sigma_2))g \, \mathrm{d}s, \ \int_{\partial\Omega} g(\Lambda(\sigma_2)-\Lambda(\sigma_1))g \, \mathrm{d}s \right\} \\
&\geq \max\left\{ \int_{\partial\Omega} g\Lambda'(\sigma_2)(\sigma_1-\sigma_2)g \, \mathrm{d}s, \ \int_{\partial\Omega} g\Lambda'(\sigma_1)(\sigma_2-\sigma_1)g \, \mathrm{d}s \right\} \\
&= \|\sigma_1-\sigma_2\| \, f\left(\sigma_1, \sigma_2, \frac{\sigma_1-\sigma_2}{\|\sigma_1-\sigma_2\|}, g\right)
\end{aligned}$$

with $f(\tau_1,\tau_2,\kappa,g) := \max\left\{ \int_{\partial\Omega} g\Lambda'(\tau_1)\kappa g \, \mathrm{d}s, \ -\int_{\partial\Omega} g\Lambda'(\tau_2)(\kappa)g \, \mathrm{d}s \right\}$

Sketch of proof 2/2

- By last slide

$$\frac{\|\Lambda(\sigma_1)-\Lambda(\sigma_2)\|}{\|\sigma_1-\sigma_2\|} \geq \sup_{\|g\|=1} f\left(\sigma_1,\sigma_2,\frac{\sigma_1-\sigma_2}{\|\sigma_1-\sigma_2\|},g\right)$$
$$\geq \inf_{\tau_1,\tau_2,\kappa} \sup_{\|g\|=1} f(\tau_1,\tau_2,\kappa,g)$$

with infimum taken over compact set of all

τ_1,τ_2,κ pcw. const., $\quad \tau_1(x),\tau_2(x)\in[a,b], \quad \|\kappa\|=1$

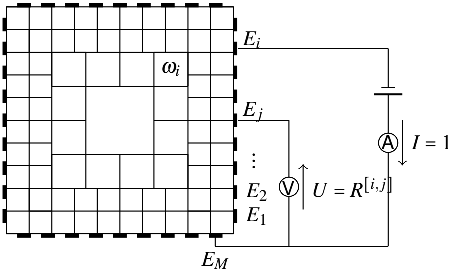
- f continuous $\rightsquigarrow$ $\sup f$ l.s.c. $\rightsquigarrow$ infimum is attained
- Localized potentials $\rightsquigarrow$ $\forall \tau_1,\tau_2,\kappa:\ \exists g:\ f(\tau_1,\tau_2,\kappa,g)>0$

$\rightsquigarrow$ $\exists c>0:\ \|\Lambda(\sigma_1)-\Lambda(\sigma_2)\|\geq c\|\sigma_1-\sigma_2\|.$

(and Galerkin projection can be treated by another compactness argument) $\square$

Complete Electrode Model

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega$$
$$u|_{E_m} + z\sigma \partial_\nu u|_{E_m} = \text{const.} =: U_m$$
$$\int_{E_m} \sigma \partial_\nu u|_{E_m} \, \mathrm{d}s = J_m$$
$$\sigma \partial_\nu u = 0 \quad \text{else}$$



Current-to-Voltage operator

$$R_M(\sigma) : \mathbb{R}^M_\diamond \to \mathbb{R}^M_\diamond, \quad (J_1, \ldots, J_M) \mapsto (U_1, \ldots, U_M).$$

Uniqueness and stability (for enough electrodes)?

Uniqueness and Lipschitz-stability for fixed resolution

Assumptions:

- Increasing number of electrodes fulfills Hyvönen conditions
- $\mathcal{F}$: finite-dimensional subset of pcw.-analytic functions
  (e.g., pcw. constant on fixed a-priori known partition)
- Known background conductivity:
  $\exists U$ nbr.hood of $\partial\Omega$, $\sigma_0 \in C^\infty$, so that $\sigma|_U = \sigma_0|_U$ for all $\sigma \in \mathcal{F}$
- A-prior known bounds
  $$\mathcal{F}_{[a,b]} := \{\sigma \in \mathcal{F} : \ a \leq \sigma(x) \leq b \text{ for all } x \in \Omega\}$$

Theorem. (**H**, IP 2019) $\exists N \in \mathbb{N},\ c > 0$:

$$\|R_M(\sigma_1) - R_M(\sigma_2)\|_{\mathcal{L}(\mathbb{R}^M_\diamond)} \geq c\, \|\sigma_1 - \sigma_2\|_{L^\infty(\Omega)} \quad \forall \sigma_1, \sigma_2 \in \mathcal{F}_{[a,b]}, M \geq N.$$

Conclusions and Outlook (on the application)

EIT with fixed resolution is uniquely and stably solvable if enough electrodes are being used.

- EIT's ill-posedness due to inf.-dimens., not due to non-linearity
- Stability gets worse (exponentially) for finer resolution

  (For full NtD: *Alessandrini/Vessella 2005, Rondi 2006*)

Open questions:

- How many electrodes are required for a desired resolution?
- How good is the stability (error-amplification) in a given setting?
- Globally convergent solvers the discretized non-linear problem?
- Consequences of conductivity discretization?

Conclusions and Outlook (on the method)

Stability can be proven by monotonicity and localized potentials

Advantages:

- Simple: No analytic construction of special solutions required.
- Flexible: Method already applied to show stability for
  - Robin coefficient problem (***H.****/Meftahi, SIAP 2019*)
  - Deep learning approach to EIT (*Seo/Kim/Jargal/Lee/****H.*** *SIIMS, to appear)*
  - Fractional Calderón problem (***H.****/Lin, arXiv:1903.08771*)
- Constructive (possibly):
  - In Robin coeff. problem, Lipschitz constant for given resolution can be calculated by solving finitely many well-posed PDEs
  - Identifying necessary meas. for desired resol. seems in reach