Introduction to inverse problems
Pierre Simon Laplace (1814):

"An intellect which ... would know all forces ... and all positions of all items, if this intellect were also vast enough to submit these data to analysis ... for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes."
Computational Science:

*If we know all necessary parameters, then we can numerically predict the outcome of an experiment (by solving mathematical formulas).*

Goals:

- Prediction
- Optimization
- Inversion/Identification
Generic simulation problem:

Given input $x$ calculate outcome $y = F(x)$.

$x \in X$: parameters / input
$y \in Y$: outcome / measurements
$F : X \rightarrow Y$: functional relation / model

Goals:

- **Prediction**: Given $x$, calculate $y = F(x)$.
- **Optimization**: Find $x$, such that $F(x)$ is optimal.
- **Inversion/Identification**: Given $F(x)$, calculate $x$. 
Example: X-ray computerized tomography (CT)

Nobel Prize in Physiology or Medicine 1979:
Allan M. Cormack and Godfrey N. Hounsfield

(Photos: Copyright ©The Nobel Foundation)

Idea: Take x-ray images from several directions
Computerized tomography (CT)


Direct problem: Simulate/predict the measurements (from knowledge of the interior density distribution)

Given \( x \) calculate \( F(x) = y! \)

Inverse problem: Reconstruct/image the interior distribution (from taking x-ray measurements)

Given \( y \) solve \( F(x) = y! \)
Computerized tomography

- CT forward operator $F : x \mapsto y$ is linear
- Evaluation of $F$ is simple matrix vector multiplication
  (after discretizing image and measurements as long vectors)

Simple low resolution example:

Problem: Matrix $F$ invertible, but $\|F^{-1}\|$ very large.
Ill-posedness

- In the continuous case: $F^{-1}$ not continuous
- After discretization: $\|F^{-1}\|$ very large

Are stable reconstructions impossible?

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Ill-posedness

Generic linear ill-posed inverse problem

- $F : X \rightarrow Y$ bounded and linear, $X, Y$ Hilbert spaces,
- $F$ injective, $F^{-1}$ not continuous,
- True solution and noise-free measurements: $F\hat{x} = \hat{y}$,
- Real measurements: $y^\delta$ with $\|y^\delta - \hat{y}\| \leq \delta$

$$F^{-1}y^\delta \rightharpoonup F^{-1}\hat{y} = \hat{x} \quad \text{for} \quad \delta \rightarrow 0.$$ 

Even the smallest noise may corrupt the reconstructions.
Regularization

Generic linear Tikhonov regularization

\[ R_\alpha = (F^* F + \alpha I)^{-1} F^* \]

\( \sim R_\alpha \) continuous, \( x = R_\alpha y^\delta \) minimizes

\[ \|Fx - y^\delta\|^2 + \alpha \|x\|^2 \rightarrow \min! \]

**Theorem.** Choose \( \alpha := \delta \). Then for \( \delta \rightarrow 0 \),

\[ R_\delta y^\delta \rightarrow F^{-1}\hat{y}. \]
Regularization

Theorem. Choose $\alpha := \delta$. Then for $\delta \to 0$,

$$R_\delta y^\delta \to F^{-1}\hat{y}.$$ 

Proof. Show that $\|R_\alpha\| \leq \frac{1}{\sqrt{\alpha}}$ and apply

$$\|R_\alpha y^\delta - F^{-1}\hat{y}\| \leq \underbrace{\|R_\alpha (y^\delta - \hat{y})\|}_{\leq \|R_\alpha\| \delta} + \underbrace{\|R_\alpha \hat{y} - F^{-1}y\|}_{\to 0 \text{ for } \alpha \to 0}. $$

Inexact but continuous reconstruction (regularization)
+ Information on measurement noise (parameter choice rule)
= Convergence
Example ($\delta = 1\%$)
Electrical impedance tomography
Electrical impedance tomography (EIT)

- Apply electric currents on subject’s boundary
- Measure necessary voltages
  - Reconstruct conductivity inside subject.

Images from BMBF-project on EIT

(Hanke, Kirsch, Kress, Hahn, Weller, Schilcher, 2007-2010)
Electric current strength: $5 - 500\,\text{mA}_{\text{rms}}$, 44 images/second, CE certified by Viasys Healthcare, approved for clinical research
Mathematical Model

- Electrical potential $u(x)$ solves
  \[ \nabla \cdot (\sigma(x) \nabla u(x)) = 0 \quad x \in \Omega \]  
  (EIT)

  $\Omega \subset \mathbb{R}^n$: imaged body, $n \geq 2$
  $\sigma(x)$: conductivity
  $u(x)$: electrical potential

- Idealistic model for boundary meas. (continuum model):
  \[ \sigma \frac{\partial u(x)}{\partial \Omega} : \text{applied electric current} \]
  \[ u(x)\big|_{\partial \Omega} : \text{measured boundary voltage (potential)} \]

- Neumann-to-Dirichlet-Operator:
  \[ \Lambda(\sigma) : L^2(\partial \Omega) \rightarrow L^2(\partial \Omega), \quad g \mapsto u\big|_{\partial \Omega} , \]

  where $u$ solves (EIT) with $\sigma \frac{\partial u}{\partial \Omega} \big|_{\partial \Omega} = g$. 

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Electrical impedance tomography

Inverse problem of EIT: Recover $\sigma$ from $\Lambda(\sigma)$

Challenges:
- **Uniqueness**
  - Is $\sigma$ uniquely determined from ”perfect data” $\Lambda(\sigma)$?
- **Non-linearity and ill-posedness**
  - Reconstruction algorithms to determine $\sigma$ from $\Lambda(\sigma)$?
  - Local/global convergence results for noisy data $\Lambda^\delta_{\text{meas}} \approx \Lambda(\sigma)$?
- **Realistic data**
  - What can we recover from real measurements? *(fixed number of electrodes, realistic electrode models, . . .)*
  - Measurement and modelling errors? Resolution?
Inversion of $\sigma \mapsto \Lambda(\sigma) = \Lambda_{\text{meas}}$?

Generic solvers for non-linear inverse problems:

- **Linearize and regularize:**
  \[
  \Lambda_{\text{meas}} = \Lambda(\sigma) \approx \Lambda(\sigma_0) + \Lambda'(\sigma_0)(\sigma - \sigma_0).
  \]
  $\sigma_0$: Initial guess or reference state (e.g. exhaled state)

- **Linear inverse problem for $\sigma$**
  (Solve, e.g., using linear Tikhonov regul., repeat for Newton-type algorithm.)

- **Regularize and linearize:**
  E.g., minimize non-linear Tikhonov functional
  \[
  \| \Lambda_{\text{meas}} - \Lambda(\sigma) \|^2 + \alpha \| \sigma - \sigma_0 \|^2 \to \text{min}!
  \]

Very flexible, but high comput. cost and convergence unclear

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Linearization and shape reconstruction


Let $\kappa$, $\sigma$, $\sigma_0$ pcw. analytic.

\[
\Lambda'(\sigma_0) \kappa = \Lambda(\sigma) - \Lambda(\sigma_0) \implies \text{supp}_\partial \Omega \kappa = \text{supp}_\partial \Omega (\sigma - \sigma_0)
\]

$\text{supp}_\partial \Omega$: outer support ( = supp + parts unreachable from $\partial \Omega$)

- Linearized EIT equation contains correct shape information
- For the shape reconstruction problem $\Lambda(\sigma) \mapsto \text{supp}_\partial \Omega (\sigma - \sigma_0)$ fast, rigorous and globally convergent method seem possible.
- **Goal**: Given $\Lambda^\delta_{\text{meas}} \approx \Lambda(\sigma) - \Lambda(\sigma_0)$, can we regularize
  \[
  \|\Lambda'(\sigma_0) \kappa - \Lambda^\delta_{\text{meas}}\| \to \min!
  \]
  so that $\text{supp}_\partial \Omega \kappa^\delta \to \text{supp}_\partial \Omega (\sigma - \sigma_0)$.  

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Monotonicity method (for simple test example)


\( \Omega \setminus \overline{D} \) connected. \( \sigma = 1 + \chi_D \).

\[ B \subseteq D \iff \Lambda(1 + \chi_B) \geq \Lambda(\sigma). \]

For faster implementation:

\[ B \subseteq D \iff \Lambda(1) + \frac{1}{2} \Lambda'(1) \chi_B \geq \Lambda(\sigma). \]

Shape can be reconstructed by linearized monotonicity tests.

\(~\) fast, rigorous, allows globally convergent implementation
Sketch of proof

Theorem $\Omega \setminus \overline{D}$ connected, $B$ open.

$$B \subseteq D \iff \Lambda(1 + \chi_B) \geq \Lambda(1 + \chi_D).$$

"$\Rightarrow$": follows from monotonicity inequality:

$$\int_\Omega (\sigma_1 - \sigma_0)|\nabla u_0|^2 \geq \int_{\partial \Omega} g (\Lambda(\sigma_0) - \Lambda(\sigma_1)) g \geq \int_\Omega \frac{\sigma_0}{\sigma_1} (\sigma_1 - \sigma_0)|\nabla u_0|^2$$

"$\Leftarrow$": follows from using localized potentials in monot. inequality.

If $B \not\subseteq D$ then there exist solutions $u_0^{(k)}$, $k \in \mathbb{N}$ with

$$\int_B |\nabla u_0^{(k)}|^2 \, dx \to \infty \quad \text{and} \quad \int_D |\nabla u_0^{(k)}|^2 \, dx \to 0.$$
Improving residuum-based methods

Let $\Omega \setminus \overline{D}$ connected. $\sigma = 1 + \chi_D$.

- Pixel partition $\Omega = \bigcup_{k=1}^{m} P_k$
- Regularized monotonicity tests

$$\beta^\delta_k \in [0, \infty] \text{ max. values s.t. } \beta^\delta_k \Lambda'(1) \chi_{P_k} \geq \Lambda^\delta_{\text{meas}} - \delta I$$

- Monotonicity-constrained residuum minimization

$$\|\Lambda'(1) \kappa^\delta - \Lambda^\delta_{\text{meas}}\|_F \to \min!$$

such that $\kappa^\delta|_{P_k} = \text{const.}, \ 0 \leq \kappa^\delta|_{P_k} \leq \min\{\frac{1}{2}, \beta^\delta_k\}$

($\| \cdot \|_F$: Frobenius norm of Galerkin projektion to finite-dimensional space)

Theorem (H./Minh, submitted)

- For $\delta = 0$, there exists unique minimizer $\kappa$ and

$$P_k \subseteq \text{supp } \kappa \iff P_k \subseteq \text{supp}(\sigma - 1).$$

- For noisy data, minimizers $\kappa^\delta$ exist and $\kappa^\delta \to \kappa$ pointwise.
### Phantom experiment

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<tr>
<th>Standard method</th>
<th>With monotonicity constraints</th>
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Enhancing standard methods by (heuristic) monotonicity constraints

(Zhou/H./Seo, submitted)

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Benchmark example

Rigorous monotonicity-constrained method vs. community standard

(H./Minh)

- EIT community standard: GREIT in EIDORS
- EIDORS: http://eidors3d.sourceforge.net (Adler/Lionheart)
- GREIT: Graz consensus Reconstruction algorithm for EIT (Adler et al.)
- Dataset: iirc_data_2006 (Woo et al.): 2cm inclusion in 20cm tank
  - using interpolated data on active electrodes (H., Inverse Problems 2015)

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Conclusions

Computational science and inverse problems

- Computational science is the core of many new advances.
- Inverse problems is the core of new medical imaging systems.

For ill-posed inverse problems

- Regularization is required for convergent algorithms.
- Regularization can also incorporate additional information (e.g., total variation penalization, stochastic priors, etc.)

For the non-linear ill-posed inverse problem of EIT

- Convergence of standard regularization is still unclear.
- Monotonicity-based regularization allows fast, rigorous, and globally convergent reconstruction of shape information.